Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Event-Triggered Tracking Control: a Discrete-Time Approach ⋆

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Abstract: In this work, the problem of designing event-triggered control strategies for disturbance rejection and reference tracking for discrete-time linear systems is addressed. Based on the Lyapunov theory, LMI-based conditions for the guarantee of perfect reference tracking/disturbance rejection under the proposed event-trigger strategy are derived. Furthermore, to avoid that the ETC strategy degenerates to a periodic one (in the case of non constant signals), a practical tracking/rejection solution considering a trade-off between the reduction of the control updates and the tolerance to a small error in steady state is proposed. The conditions are then casted in LMI-based optimization problems to compute the triggering functions aiming at reducing the control updates while ensuring the perfect or the practical tracking.

Keywords: event-triggered control; tracking; regulation; disturbance observer; discrete-time systems.

1. INTRODUCTION

Aperiodic control strategies, especially event-triggered control (ETC), are very important to deal with bandwidth limitations in shared networks and energy consumption in embedded and wireless systems (Akyildiz et al., 2002). Event-triggered strategies update the control signal according to a trigger condition. A systematic design analysis of an ETC was presented in Tabuada (2007). In this work, the trigger function was defined in terms of the control error, and provides conditions for a lower bound on the inter-event time. It should be noted that a large number of ETC strategies have been developed in a continuous-time framework (Peng and Li, 2018), (Heng et al., 2015). However, the final controller implementation will be in discrete-time. This means that after design of the continuous time controller a discretization will be required, leading to an approximated implementation of the continuous ETC.

An early work addressing the problem of reducing the bandwidth utilization in discrete-time networked control systems by means of full state observers is described in Yook et al. (2002). Lately, output-feedback methods for discrete-time ETCs (also called periodic event-triggered control, PETC) have been proposed in Heemels et al. (2013). This work provides stability conditions based on an hybrid system modeling, which allows to consider the continuous-time behavior of the plant and the impulsive updating of the control. A model-based output-feedback is proposed in Heemels and Donkers (2013), with guaranteed stability based on LMI conditions. The model is used to estimate the plant states even when not being updated by the sensor. More recently, an output-based event-triggered control solution with performance guarantee, while reducing the communication load has been presented in Khanhooei et al. (2017). This work considers a performance expressed as an average quadratic cost and a plant disturbed by Gaussian process and measurement noises. In Eqtami et al. (2010) both nonlinear and linear discrete time systems were considered. Sufficient conditions for inter-execution times were derived and extended to a self-triggered formulation. This framework was also expanded to consider a Model Predictive Control. A state feedback controller design strategy for discrete-time linear systems subject to bounded disturbances is presented in Wu et al. (2016). Criteria to design the controller in order to guarantee that the states are uniformly ultimately bounded in an ellipsoidal-positive invariant set are proposed. The minimization of this set is achieved by solving a set of LMIs for the feedback control gain and the parameters of the event-triggering condition.

An observer-based output-feedback control is proposed in Jetto and Orsini (2011). The proposed trigger function is based on the value of a functional at each instant, requiring a number of periodic samples to be taken before a certain threshold is reached and the system starts to operate in an event-triggered fashion. In Groff et al. (2016) an

⋆ This study was financed in part by CAPES - Finance Code 001 (PROEX, SticAmSud), CNPQ (grants 422992/2016-0, 307449/2019-0) IFSUL (Project PD00190519/011), Brazil; and by CONYCIT (grants STIC-AmSud 18-STIC-01 and Fondap 15110019), Chile.

event-triggered strategy for an observer-based controller is considered. In particular, conditions for the stability of the closed-loop system in terms of linear matrix inequalities (LMIs), which are cast in optimization problems to tune the trigger functions aiming at a reduction of the number of events, are proposed.

Regarding the problem of reference tracking by an event-triggered controller we can cite Ma et al. (2018), where a continuous-discrete 2-D model is adopted to analyze the tracking controller, and Postoyan et al. (2015), where the stabilization of time-varying trajectories for unicycle mobile robots is investigated.

In this work, an observer-based ETC strategy for tracking and disturbance rejection is proposed in a discrete-time framework. Considering a linear plant and a deterministic model for both the reference and disturbance (i.e. generated by exosystems), we suppose that a discrete-time observer and an observed state feedback control law have been computed to ensure the stability of the closed-loop system under a periodic sampling strategy. Moreover, to achieve perfect tracking/rejection in steady state we consider an observer for the disturbance and feedforward gains, satisfying the regulator equations. From this setup, instead of periodic control updating, an ETC strategy is proposed. Based on Lyapunov theory, a first formal result is provided to ensure that the tracking error converges to zero under the proposed ETC. It is however observed that the ETC degenerates to a periodic control updating strategy in steady state for non constant disturbance and/or reference signals. Thus a more practical approach is proposed to achieve a trade off between reducing the control updating in steady state and tolerating a given maximum tracking error. In this sense, a formal result providing conditions to ensure that the trajectories of the tracking error are ultimately bounded under a relaxed triggering function is proposed. From the derived conditions, LMI-based optimization problems to compute the parameters of the triggering functions aiming at reducing the control updates are formulated.

Notation. For a given matrix M, M' denotes its transpose. For symmetric matrices M and N, $M > 0$ means that M is positive definite, and $M > N$ means that $M - N > 0$. $tr(M)$ stands for the trace of matrix M. I_* and 0_* are an identity matrix and a null matrix of appropriate dimensions. In partitioned matrices, $*$ stands for a symmetric block. $\text{diag}(M_1, ..., M_n)$ denotes the block-diagonal matrix whose diagonal blocks are $M_1, \ldots, M_n$.

2. DISCRETE-TIME SETUP

Consider the following linear discrete-time system:

$$\begin{aligned} x_s[k+1] &= Ax_s[k] + Bu[k] + Bd[k], \\ y_s[k] &= Cx_s[k], \\ e[k] &= r[k] - y_s[k], \end{aligned} \tag{1}$$

where $x_s \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y_s \in \mathbb{R}$, $d \in \mathbb{R}$ and $r \in \mathbb{R}$ are the system states, input, output, disturbance and reference respectively, $k \in \mathbb{N}$. The pairs (A, B) and (C, A) are supposed controllable and observable, respectively.

The disturbance and reference are generated by the following dynamical models (exosystems):

- Disturbance:
$$\begin{aligned} x_d[k+1] &= A_d x_d[k], \\ d[k] &= C_d x_d[k], \end{aligned} \tag{2}$$
where $x_d \in \mathbb{R}^{n_d}$ and $d \in \mathbb{R}$ are the states of the disturbance dynamics and the disturbance signal respectively.
- Reference:
$$\begin{aligned} x_r[k+1] &= A_r x_r[k], \\ r[k] &= C_r x_r[k], \end{aligned} \tag{3}$$
where $x_r \in \mathbb{R}^{n_r}$ and $r \in \mathbb{R}$ are the states of the reference dynamics and the reference signal respectively.

For the extended system considering (1) and (2), the following discrete-time observer is considered:

$$\begin{aligned} \begin{bmatrix} x_o[k+1] \\ x_{do}[k+1] \end{bmatrix} &= \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x_o[k] \\ x_{do}[k] \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u[k] \\ &\quad - \begin{bmatrix} K_{o1} \\ K_{o2} \end{bmatrix} (y_s[k] - y_o[k]) \\ y_o[k] &= Cx_o[k] \\ d_o[k] &= C_d x_{do}[k] \end{aligned} \tag{4}$$

where $x_o \in \mathbb{R}^n$, $x_{do} \in \mathbb{R}^{n_d}$, $u \in \mathbb{R}$ and $y_o \in \mathbb{R}$, $d_o \in \mathbb{R}$ are the observer states, input, estimated output and estimated disturbance, respectively. $K_{o1} \in \mathbb{R}^n$ and $K_{o2} \in \mathbb{R}^{n_d}$ are the observer gains. Then, consider an observed state feedback control law given by

$$u[k] = K_c(x_o[k] - K_{r1}x_r[k]) + K_{r2}x_r[k] - C_d x_{do}[k], \tag{5}$$

where $K_c' \in \mathbb{R}^n$ is the feedback gain designed to stabilize the system and $K_{r1} \in \mathbb{R}^{n \times n_r}$ and $K_{r2} \in \mathbb{R}^{1 \times n_r}$ are feedforward gains.

Defining now the observation error as

$$\tilde{x}[k] = \begin{bmatrix} x_s[k] - x_o[k] \\ x_d[k] - x_{do}[k] \end{bmatrix} \tag{6}$$

the closed-loop system obtained from the connection of (1), (4) and (5) can be represented by the following equations:

$$\tilde{x}[k+1] = (A_o + K_o C_o)\tilde{x}[k] \tag{7}$$

$$x_o[k+1] = (A + BK_c)x_o[k] + BK_r x_r[k] - K_{o1}C_o\tilde{x}[k] \tag{8}$$

where

$$A_0 = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}, \; K_o = \begin{bmatrix} K_{o1} \\ K_{o2} \end{bmatrix}, \; C_0 = [\, C \;\; 0 \,], \tag{9}$$

and $K_r = K_{r2} - K_c K_{r1}$.

We suppose that gains K_o, K_c, K_{r1} and K_{r2} have been designed such that the following assumptions are satisfied.

Assumption 1. K_o and K_c are such that $(A_o + K_o C_o)$ and $(A + BK_c)$ are Schur stable.

Assumption 2. K_{r1} and K_{r2} verify the following equations:

$$K_{r1}A_r = AK_{r1} + BK_{r2}, \tag{10}$$

$$0 = C_r - CK_{r1} \tag{11}$$

Equations (10) and (11) are the so-called regulator equations (Saberi et al., 2003). We show now that, under Assumptions 1 and 2, the perfect reference tracking is asymptotically achieved, i.e. $e[k] \to 0$ as $k \to \infty$. With this aim consider the following coordinate transformation (Saberi et al., 2003):

$$\overline{x}_o[k] = x_o[k] - K_{r1}x_r[k] \tag{12}$$

From (3), (8) and using equation (10), we have that:

$$\overline{x}_o[k+1] = (A + BK_c)\overline{x}_o[k] - K_{o1}C_o\tilde{x}[k]. \tag{13}$$

As K_o and K_c are such that $(A_o + K_oC_o)$ and $(A + BK_c)$ are Schur stable, from (7) and (13) we can conclude that $\tilde{x}[k] \to 0$, $\overline{x}_o[k] \to 0$ and thus $x_o[k] \to x_s[k]$ as $k \to \infty$. Then, taking into account (11), for $k \to \infty$ the tracking error satisfies:

$$\begin{aligned} e[k] = r[k] - Cx_o[k] &= C_rx[k] - C(\overline{x}_o[k] + K_{r1}x_r[k]) \\ &= -C\overline{x}_o[k] \end{aligned}$$

and we conclude that $\lim_{k\to\infty} e[k] = 0$.

3. EVENT-TRIGGERED CONTROL PROBLEM

In this section, we propose an event-triggered implementation of the observer-based tracking controller defined by (4) and (5). We consider that K_o, K_c, K_{r1} and K_{r2} have been designed to ensure prefect tracking and disturbance rejection under a periodic control updating strategy.

Using an ETC strategy, the update of the control signal is transmitted to the actuator only at the time instants n_i, $i = 0, 1, 2...$, leading to the following observer-based state feedback control law $\forall k \in [n_i,\ n_{i+1})$:

$$u[k] = u[n_i] = K_c\overline{x}_o[n_i] + K_{r2}x_r[n_i] - d_o[n_i]. \tag{14}$$

Then, at each sampling instant k, a decision on updating or not the control signal applied to the plant should be made. This decision will be based on an event that is associated to the evolution of a trigger function.

In this paper, following the ideas proposed in Tabuada (2007) and Heemels et al. (2013), we define

$$\begin{aligned} \delta_x[k] = \bar{x}_o[n_i] - \bar{x}_o[k], \quad \delta_d[k] = d_o[n_i] - d_o[k], \\ \delta_r[k] = x_r[n_i] - x_r[k], \end{aligned} \tag{15}$$

i. e., $\delta_j[k]$, with $j = x, d, r$ are a measure of the difference between the variable used for the last control update and the current observed one, and we consider a trigger function $f(\delta_x[k], \delta_d[k], \delta_r[k], \overline{x}_o[k])$. Thus, the control update instants are given by the rule:

$$n_{i+1} = \min\{k > n_i : f(\delta_x[k], \delta_d[k], \delta_r[k], \overline{x}_o[k]) > 0\} \tag{16}$$

In other words, if at any time-instant $k \in \mathbb{N}$ the event-trigger function $f(\delta_x[k], \delta_d[k], \delta_r[k], \overline{x}_o[k])$ is positive, a new control signal is transmitted to the actuator device and the plant control input is effectively updated.

The challenge consists in designing the function $f(\delta_x[k], \delta_d[k], \delta_r[k], \overline{x}_o[k])$ such that the closed-loop system with the control law (14) is asymptotically stable. Moreover, we aim at choosing $f(\delta_x[k], \delta_d[k], \delta_r[k], \overline{x}_o[k])$ such that less events are generated, or equivalently, the control updates are reduced. These problems are formally addressed in the next sections.

4. STABILITY CONDITIONS

Consider (7), (13), (14) and the definition of $\delta_j[k], j = x, d, r$ given in (15). Then, the closed-loop system behavior is described by the following equations:

$$\begin{aligned} \overline{x}_o[k+1] = (A + BK_c)\overline{x}_o[k] + BK_c\delta_x[k] - B\delta_d[k] \\ + BK_{r2}\delta_r[k] - K_{o1}C_o\tilde{x}[k] \end{aligned} \tag{17}$$

$$\tilde{x}[k+1] = (A_o + K_oC_o)\tilde{x}[k], \tag{18}$$

In this case, the following theorem regarding the stability of the closed-loop system can be stated.

Theorem 1. Consider that Assumptions 1 and 2 are satisfied. If there exist symmetric positive definite matrices P_1, P_2, $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d}, Q_{δ_r} and a matrix Q_o of appropriate dimensions such that

$$(A_o + K_oC_o)'P_1(A_o + K_oC_o) - P_1 + Q_o < 0 \tag{19}$$

$$\Phi - \mathrm{diag}(P_2, Q_{\delta_x}, Q_o, Q_{\delta_d}, Q_{\delta_r}, \bar{Q}_\sigma) < 0 \tag{20}$$

is verified, with

$$\Phi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & I \\ * & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} & 0 \\ * & * & \psi_{33} & \psi_{34} & \psi_{35} & 0 \\ * & * & * & \psi_{44} & \psi_{45} & 0 \\ * & * & * & * & \psi_{55} & 0 \\ I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{21}$$

$\psi_{11} = (A + BK_c)'P_2(A + BK_c)$, $\psi_{12} = (A + BK_c)'P_2BK_c$, $\psi_{13} = -(A + BK_c)'P_2K_{o1}C_o$, $\psi_{14} = -(A + BK_c)'P_2B$, $\psi_{15} = (A + BK_c)'P_2BK_{r2}$, $\psi_{22} = (BK_c)'P_2BK_c$, $\psi_{23} = -(BK_c)'P_2K_{o1}C_o$, $\psi_{24} = -(BK_c)'P_2B$, $\psi_{25} = (BK_c)'P_2BK_{r2}$, $\psi_{33} = C_o'K_{o1}'P_2K_{o1}C_o$, $\psi_{34} = C_o'K_{o1}'P_2B$, $\psi_{35} = -C_o'K_{o1}'P_2BK_{r2}$, $\psi_{44} = B'P_2B$, $\psi_{45} = -B'P_2BK_{r2}$, $\psi_{55} = (BK_{r2})'P_2(BK_{r2})$,

then the trajectories of the closed-loop system (17)-(18), under the event-triggered strategy proposed in (16), with the trigger function given by

$$\begin{aligned} f(\delta_x[k], \delta_d[k], \delta_r[k], \bar{x}_o[k]) = \delta_x[k]'Q_{\delta_x}\delta_x[k] + \delta_d[k]'Q_{\delta_d}\delta_d[k] \\ + \delta_r[k]'Q_{\delta_r}\delta_r[k] - \overline{x}_o[k]'Q_\sigma\overline{x}_o[k]. \end{aligned} \tag{22}$$

where $Q_\sigma = \bar{Q}_\sigma^{-1}$, converge asymptotically to the origin, which implies that the tracking error $e[k] \to 0$ as $k \to \infty$.

Proof: Let $V(\overline{x}_o[k], \tilde{x}[k]) = V_1(\tilde{x}[k]) + V_2(\overline{x}_o[k])$ be a Lyapunov function candidate for the composite system (17), (18), with $V_1(\tilde{x}[k]) = \tilde{x}[k]'P_1\tilde{x}[k]$ being a quadratic function associated to the sub-system (18) and $V_2(\overline{x}_o[k]) = \overline{x}_o[k]'P_2\overline{x}_o[k]$ being a quadratic function associated to the sub-system (17). Defining

$$\Delta V_1(\tilde{x}[k]) = V_1(\tilde{x}[k+1]) - V_1(\tilde{x}[k]) \tag{23}$$

$$\Delta V_2(\overline{x}_o[k]) = V_2(\overline{x}_o[k+1]) - V_2(\overline{x}_o[k]) \tag{24}$$

and omitting the dependency on k, $\Delta V_1(\tilde{x})$ and $\Delta V_2(\overline{x}_o)$ are given by:

$$\Delta V_1(\tilde{x}) = \tilde{x}'\big((A_o + K_oC_o)'P_1(A_o + K_oC_o) - P_1\big)\tilde{x} \tag{25}$$

$$\begin{aligned} \Delta V_2(\overline{x}_o) = \overline{x}_o'\big((A + BK_c)'P_2(A + BK_c) - P_2\big)\overline{x}_o \\ + \delta_x'(BK_c)'P_2BK_c\delta_x - 2\overline{x}_o'(A + BK_c)'P_2B\delta_d \\ + \delta_r'(BK_{r2})'P_2BK_{r2}\delta_r + \tilde{x}'C_o'K_{o1}'P_2K_{o1}C_o\tilde{x} \\ + 2\overline{x}_o'(A + BK_c)'P_2BK_c\delta_x + \delta_d'B'P_2B\delta_d \\ + 2\overline{x}_o'(A + BK_c)'P_2BK_{r2}\delta_r - 2\delta_d'B'P_2BK_{r2}\delta_r \\ - 2\overline{x}_o'(A + BK_c)'P_2K_{o1}C_o\tilde{x} - 2\delta_x'(BK_c)'P_2B\delta_d \\ + 2\delta_x'(BK_c)'P_2BK_{r2}\delta_r - 2\delta_x'(BK_c)'P_2K_{o1}C_o\tilde{x} \\ + 2\delta_d'B'P_2K_{o1}C_o\tilde{x} - 2\delta_r'(BK_{r2})'P_2BK_{o1}C_o\tilde{x} \end{aligned} \tag{26}$$

Provided that condition (19) is satisfied, it follows that $\Delta V_1(\tilde{x})$ is upper bounded as follows:

$$\Delta V_1(\tilde{x}) < -\tilde{x} Q_o \tilde{x} \tag{27}$$

Moreover, if (20) is satisfied, from Schur complement and considering that $Q_\sigma = \bar{Q}_\sigma^{-1}$, it follows that $\Delta V_2(\overline{x}_o)$ is upper bounded as follows:

$$\begin{aligned}\Delta V_2(\overline{x}_o) < -\overline{x}_o' Q_\sigma \overline{x}_o + \delta_x' Q_{\delta_x} \delta_x + \delta_d' Q_{\delta_d} \delta_d \\ + \delta_r' Q_{\delta_r} \delta_r + \tilde{x}' Q_o \tilde{x}.\end{aligned} \tag{28}$$

Now, looking at the Lyapunov function candidate difference equation we have:

$$\Delta V(\overline{x}_o, \tilde{x}) = \Delta V_1(\tilde{x}) + \Delta V_2(\overline{x}_o). \tag{29}$$

Hence, from (27) and (28), it follows that:

$$\Delta V(\overline{x}_o, \tilde{x}) < \delta_x' Q_{\delta_x} \delta_x + \delta_d' Q_{\delta_d} \delta_d + \delta_r' Q_{\delta_r} \delta_r - \overline{x}_o' Q_\sigma \overline{x}_o. \tag{30}$$

Suppose now that $n_{i+1} - n_i > 1$ and $k \in (n_i, n_{i+1})$. In this case, from (16), it means that $f(\delta_x, \delta_d, \delta_r, \overline{x}_o) \leq 0$, i. e.

$$\delta_x' Q_x \delta_x + \delta_d' Q_{\delta_d} \delta_d + \delta_r' Q_{\delta_r} \delta_r - \overline{x}_o' Q_\sigma \overline{x}_o \leq 0 \tag{31}$$

and it follows that $\Delta V(\overline{x}_o, \tilde{x}) < 0$, $\forall k \in (n_i, n_{i+1})$.

On the other hand, suppose that at a given instant k, $f(\delta_x, \delta_d, \delta_r, \overline{x}_o) > 0$ (note that this situation includes the case $n_{i+1} - n_i = 1$). Then an event is triggered and, from the triggering rule (16), we have that $n_i = k$ and it follows that $\delta_x[k] = 0$, $\delta_d[k] = 0$ and $\delta_r[k] = 0$, which also from (30) implies that $\Delta V(\overline{x}_o, \tilde{x}) < 0$, $\forall k = n_i$, $i = 1, 2, ...$ Hence, we conclude that $\Delta V(\overline{x}_o, \tilde{x}) < 0, \forall k \geq 0$. □

Theorem 1 provides conditions for perfect tracking and rejection. However, if the disturbance or reference signals are not constant signals, the event-trigger mechanism will not be effective in steady state. Indeed, in this case, the event-trigger strategy will degenerate to the periodic one. This can be clearly seen in the case of sinusoidal signals. For perfect tracking and/rejection in this case, as the system is linear, the control input in steady state must be a perfect sinusoidal signal in discrete-time with the same frequency of the one to be tracked or rejected. Hence, it can be practical to consider a trade-off between obtaining less control updates and tolerating a small tracking error in steady state. This solution is proposed in the next section by considering a relaxed triggering criterion.

5. TRACKING ACCURACY AND CONTROL EVENTS

In this section, we analyze the relaxation of the triggering conditions in order to decrease the control events at the expense of an increase in the tracking error. A threshold is applied to the triggering function; i.e. we consider a real constant $\gamma > 0$ so that the triggering condition will be given by $f_\gamma > 0$ with $f_\gamma = f - \gamma$.

In this case, instead of the convergence of the states of system (17)-(18) to the origin, we will be interested in guaranteeing that the trajectories are ultimately bounded in an invariant ellipsoidal set

$$\mathcal{E}(P, \eta) = \{x \in R^{2n} \ , \ x' P x \leq \eta\} \tag{32}$$

with $x = [\tilde{x}' \ \ \overline{x}_o']'$ and

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{33}$$

As a consequence, we have that the tracking error, which is given by $-C\bar{x}_o[k]$ will be ultimately bounded as well.

This idea is formalized in the following Theorem.

Theorem 2. Consider that Assumptions 1 and 2 are satisfied. If there exist positive scalars τ and $\bar{\gamma}$, symmetric positive definite matrices P_1, P_2, $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d}, Q_{δ_r} and a matrix Q_o of appropriate dimensions such that

$$(A_o + K_o C_o)' P_1 (A_o + K_o C_o) - (1 - \tau) P_1 + Q_o < 0 \tag{34}$$

$$\Phi - \text{diag}\left((1 - \tau) P_2, Q_{\delta_x}, Q_o, Q_{\delta_d}, Q_{\delta_r}, \bar{Q}_\sigma\right) < 0 \tag{35}$$

$$\begin{bmatrix} -\tau\eta & 1 \\ 1 & -\bar{\gamma} \end{bmatrix} < 0 \tag{36}$$

with Φ defined as in (21), then the trajectories of the closed-loop system (17)-(18), under the event-triggered strategy proposed in (16) with the trigger function given by

$$\begin{aligned} f_\gamma(\delta_x[k], \delta_d[k], \delta_r[k], \bar{x}_o[k]) = \delta_x[k]' Q_{\delta_x} \delta_x[k] \\ + \delta_d[k]' Q_{\delta_d} \delta_d[k] + \delta_r[k]' Q_{\delta_r} \delta_r[k] \\ - \bar{x}_o[k]' Q_\sigma \bar{x}_o[k] - \gamma, \end{aligned} \tag{37}$$

with $Q_\sigma = \bar{Q}_\sigma^{-1}$ and $\gamma = \bar{\gamma}^{-1}$, converge in finite time to the set $\mathcal{E}(P, \eta)$. Moreover, this set is positively invariant.

Proof: In order to show that the trajectories converge to $\mathcal{E}(P, \eta)$ in finite time it suffices to ensure that

$$\Delta V(x) = \Delta V(\tilde{x}[k], \overline{x}_o[k]) < -\epsilon ||x[k]||^2 \tag{38}$$

with some positive scalar ϵ, for all $x \notin \mathcal{E}(P, \eta)$. Indeed, these conditions ensure that $V(x) = x' P x$ is exponentially decreasing along the closed-loop trajectories $\forall x \notin \mathcal{E}(P, \eta)$.

A sufficient condition to ensure (38) for all $x \notin \mathcal{E}(P, \eta)$ is the existence of a positive scalar τ such that

$$\Delta V(x) + \tau x[k]' P x[k] - \tau\eta - f_\gamma(\cdot) < -\epsilon ||x[k]||^2 \tag{39}$$

Taking into account that $\gamma = \bar{\gamma}^{-1}$, applying Schur complement to (36) ensures that:

$$-\tau\eta + \gamma < 0 \tag{40}$$

Now considering $f_\gamma(\cdot) = f(\cdot) - \gamma$, with $f(\cdot)$ as in (22), from the developments done in the proof of Theorem 1 and (40), it is straightforward to see that (39) is satisfied for some $\epsilon > 0$ if matrix inequalities (34), (35) and (36) are verified.

We need now to show that $\mathcal{E}(P, \eta)$ is a positively invariant set with respect to the closed-loop system. If (39) is satisfied we have that

$$\begin{aligned} x[k+1]' P x[k+1] - x[k]' P x[k] - f(\cdot) + \\ \tau x[k]' P x[k] + \gamma - \tau\eta < 0 \end{aligned} \tag{41}$$

set $\tau_1 = (1 - \tau)$ and replacing $\tau = 1 - \tau_1$ in (41) we have

$$\begin{aligned} x[k+1]' P x[k+1] - \tau_1 x[k]' P x[k] - f(\cdot) \\ + \gamma + \tau_1 \eta - \eta < 0 \end{aligned} \tag{42}$$

As $f_\gamma = f - \gamma \leq 0$, $\forall k > 0$, it follows that

$$x[k+1]' P x[k+1] - \eta - \tau_1 (x[k]' P x[k] - \eta) < 0 \tag{43}$$

and therefore $x[k+1]' P x[k+1] - \eta < \tau_1 (x[k]' P x[k] - \eta)$

Note now that for $x[k] \in \mathcal{E}(P, \eta)$ one has $x[k]' P x[k] - \eta \leq 0$. Thus we conclude that if $x[k] \in \mathcal{E}(P, \eta)$ then $x[k+1]' P x[k+1] - \eta < 0$ and therefore $x[k+1] \in \mathcal{E}(P, \eta)$, which proves the positive invariance of $\mathcal{E}(P, \eta)$. □

6. OPTIMIZATION PROBLEMS

Matrices $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d} and Q_{δ_r} are free variables in the LMIs (19) and (20). The idea is therefore to "tune" $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d} and Q_{δ_r} numerically. In order to have less control updates the idea, from the function f defined in (22), consists in "minimizing" the weighting matrices Q_{δ_x}, Q_{δ_r}, Q_{δ_d} and "maximizing" $Q_\sigma = \bar{Q}_\sigma^{-1}$. Thus a suitable criterion aiming at reducing the number of generated events can be the following (Moreira et al., 2019):

$$\text{minimize} \quad \text{tr}(Q_{\delta_x}) + \text{tr}(Q_{\delta_d}) + \text{tr}(Q_{\delta_r}) + \text{tr}(\bar{Q}_\sigma)$$

In this case, the following optimization problem can be formulated to compute the criterion matrices aiming at reducing the events generation:

$$\begin{array}{rl} \text{minimize} & \text{tr}(Q_{\delta_x}) + \text{tr}(Q_{\delta_d}) + \text{tr}(Q_{\delta_r}) + \text{tr}(\bar{Q}_\sigma) \\ \text{subject to:} & (19),\ (20) \end{array} \tag{44}$$

with Q_{δ_x}, Q_{δ_r}, Q_{δ_d} and $\bar{Q}_\sigma$ free positive definite matrices of appropriate dimensions. Note that problem (44) is convex, since (19) and (20) are LMIs on the decision variables.

On the other hand, for the relaxed triggering condition (37), in addition to matrices $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d} and Q_{δ_r}, parameter γ has to be appropriately chosen. Recall that in this case we ensure only that the trajectories of (17) converge to a positively invariant set $\mathcal{E}(P,\eta)$, or equivalently, the tracking error will not converge to zero, but it will be ultimately bounded. A trade off between a relatively small tracking error and a reduction of the control updating should therefore be managed. With this aim, consider the following constraint:

$$P - \eta \begin{bmatrix} 0 & 0 \\ 0 & \sigma C'C \end{bmatrix} \geq 0 \tag{45}$$

This constraint ensures that $\mathcal{E}(P,\eta)$ is included in the set $\mathcal{S}_\sigma = \{x \in \mathbb{R}^{2n}\ ;\ \overline{x}_o C'C\overline{x}_o \leq \sigma^{-1}\}$, which implies that the tracking error is confined in a ball with radius $\sqrt{\sigma^{-1}}$. Hence, the tolerated tracking error in stead state can be regulated by the value of σ.

The idea is therefore to take into account in the criterion, in addition to the "minimization" of matrices $\bar{Q}_\sigma$, Q_{δ_x}, Q_{δ_d} and Q_{δ_r}, the maximization of γ, or equivalently, the minimization of $\bar{\gamma}$, ensuring that constraint (45) is verified for some given tracking error tolerance σ. For this, we propose the following optimization problem:

$$\begin{array}{rl} \text{minimize} & \text{tr}(Q_{\delta_x}) + \text{tr}(Q_{\delta_d}) + \text{tr}(Q_{\delta_r}) + \text{tr}(\bar{Q}_\sigma) + \bar{\gamma} \\ \text{subject to:} & (34),\ (35),\ (36),\ (45) \end{array} \tag{46}$$

Problem (46) is not convex due to the products τP and $\tau\eta$. Nonetheless, as τ is a scalar, the optimal solution can be iteratively obtained through the solution of LMI problems by gridding the scalar τ.

7. NUMERICAL EXAMPLE

Consider system (1), with the following matrices:

$$A = \begin{bmatrix} 1.0000 & 0.0975 \\ 0 & 0.9512 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0025 \\ 0.0488 \end{bmatrix}, \quad C = [1\ \ 0]$$

We consider the tracking of a constant reference and the rejection of a sinusoidal disturbance of frequency $\omega = 0.25\pi$. In this case it follows that:

$$A_r = C_r = 1, \quad A_d = \begin{bmatrix} 0.9969 & 0.0999 \\ -0.0628 & 0.9969 \end{bmatrix}, \quad C_d = [1\ \ 0].$$

The state feedback and observer gains satisfying Assumption 1 are given by:

$$K_c = \begin{bmatrix} 9.2269 \\ 7.7617 \end{bmatrix}', \quad K_{o1} = \begin{bmatrix} 1.3449 \\ 5.7430 \end{bmatrix}, \quad K_{o2} = \begin{bmatrix} 28.1418 \\ 15.6801 \end{bmatrix},$$

The feedforward gains K_{r1} and K_{r2}, satisfying the regulator equations (10) are $K_{r1} = [1\ \ 0]'$, $K_{r2} = 0$.

We consider first the conditions given in Theorem 1, which ensures the perfect tracking and rejection. In this case, to compute the triggering matrices we solve the optimization problem (44), which leads to:

$$Q_\sigma = \begin{bmatrix} 1.03 & -0.000224 \\ -0.000224 & 0.7 \end{bmatrix}, \quad Q_{\delta_x} = \begin{bmatrix} 1.3 & 1.09 \\ 1.09 & 0.918 \end{bmatrix}$$
$$Q_{\delta_d} = 0.184, \quad Q_{\delta_r} = 0.000588$$

The simulation of the closed-loop system with the obtained trigger function is depicted in Figure 1, considering the initial conditions $x_s[0] = [0\ \ 0]'$, $x_o[0] = [10\ \ 0]'$ and the application of a reference step with amplitude equal to 10 at the initial instant. Moreover, a sinusoidal disturbance with constant frequency $\omega = 0.25\pi$ and amplitude 5 is applied at instant $k = 120$. The bars in the last plot corresponds to the trigger instants. The amplitude of each bar denotes the number of sampling instants elapsed from the last event. The horizontal dashed line corresponds to an amplitude of 1 sampling. We can notice that the event-trigger strategy is quite efficient for reducing the control update considering the constant reference tracking in the absence of disturbance. However, when the disturbance is added, although the perfect rejection is achieved (as it is depicted in the third plot), the event-trigger strategy degenerates to a periodic one, as expected.

Let us now consider the result stated in Theorem 2, with the relaxed trigger function (37). In this case, for $\sigma = 1$, the parameters of the trigger function obtained from the solution of optimization problem (46) are the following:

$$Q_\sigma = \begin{bmatrix} 0.871 & -0.0462 \\ -0.0462 & 0.841 \end{bmatrix}, \quad Q_{\delta_x} = \begin{bmatrix} 1.99 & 1.67 \\ 1.67 & 1.41 \end{bmatrix}$$
$$Q_{\delta_d} = 0.282, \quad Q_{\delta_r} = 0.000604, \quad \gamma = 0.749$$

For the same initial conditions, reference and disturbance signals of the previous case, Figure 2 depicts the simulation results considering the relaxed trigger function (37) with the parameters above computed. We can observe now that a small tracking error appears after the disturbance is applied. On the other hand, the event-trigger mechanism is now effective and much less control updates are needed. This clearly illustrates the trade-off between precision and the reduction of control updates for the tracking/rejection of non-constant signals.

8. CONCLUDING REMARKS

In this paper we addressed the problem of reference tracking and disturbance rejection under event-triggered control. An observer-based control strategy has been devised

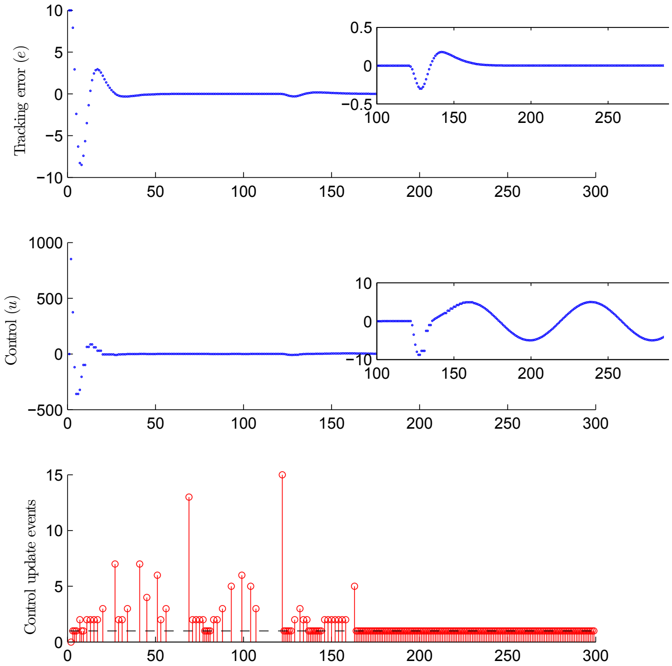

Fig. 1. Simulation with the trigger function $f(\cdot)$

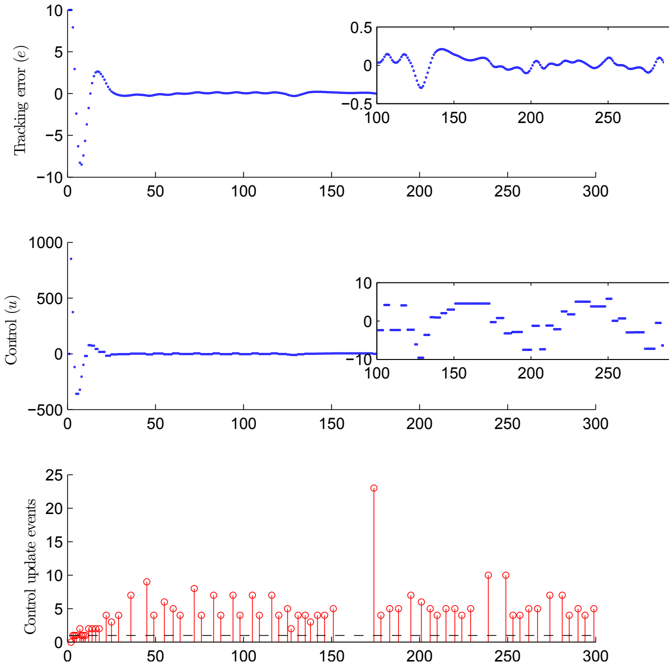

Fig. 2. Simulation with the trigger function $f_\gamma(\cdot)$

to deal with time varying references and disturbances considering the regulator equations. In general, the ETC strategy degenerates to a periodic one for non constant signals. Thus a practical tracking/rejection solution considering a relaxed triggering condition was proposed. The idea was to establish a trade-off between the reduction of the control updates and the tolerance to a small steady state error. The stability conditions were expressed as a set of LMIs. Optimization problems were therefore proposed to compute the triggering functions aiming at reducing the control updates while ensuring the perfect or the practical tracking. Simulation examples have illustrated the effectiveness of the proposed ETC strategies. On going work regards the extension of the approach to deal with the multivariable case and parametric uncertainties.

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