

Flatness based ADRC Control of Lagrangian Systems: A moving crane.

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Abstract: A procedure is described for direct tangent linearization, around a given equilibrium point, of non-linear multivariable Lagrangian systems, in terms of second order variational expansions of the Lagrangian function. When the linearized model is controllable (i.e., it exhibits the flatness property), we present an Active Disturbance Rejection Control (ADRC) scheme, valid for stabilization and flat output reference trajectory tracking tasks designed on the basis of the incremental system. The linear approach requires only generalized incremental position measurements, with no explicit need for incremental velocity observers. The ADRC controller is cast in terms of equivalent classical linear compensation networks. A moving crane example is presented which illustrates, through digital computer simulations, the effectiveness of the proposed control scheme.

Keywords: Active Disturbance rejection control, Classical feedback control, Flatness, Moving crane, Flexible manipulators.

1. INTRODUCTION

In this article, a direct tangent linearization procedure is presented for multi-variable controlled Lagrangian systems on the basis of a second order expansion, around a given equilibrium point, of the Lagrangian function. The procedure is based on second order expansions of the Lagrangian via the associated Hessian matrices. Thanks to a suitable equivalence of the Active Disturbance Rejection Control method and classical compensation networks, the controllability of the linearized system is shown to imply that the underlying nonlinear system can be controlled using only measurements of the incremental position variables, without need of generalized velocity observers. This feature is specially convenient for flat output reference trajectory tracking problems, including large excursions from the equilibrium point.

Second order variational expansions of Lagrangian functions, for small signal model derivations, were suggested by V.I. Arnold in his seminal Classical Mechanics book (1) (See also Crouch and van der Schaft (2) for closely related use of adjoint variational systems). Here, a detailed linearization procedure is extended for controlled multi-input Lagrangian systems. The controllability (flatness) of the linearized model is shown to lead to flat outputs involving only linear combinations of incremental generalized position variables. The relations between ADRC and classical compensation networks, as developed in (5), is invoked for proposing a classical, frequency domain, linear solution of the underlying nonlinear tracking problem, whereby the excited nonlinearities, activated by the system (fast) tracking maneuvers, are substantially attenuated. This design route is equivalent to approximate extended

state observer-based endogenous disturbance estimation and ulterior feedback cancellation (See Han and Gao(6)).

Section II explains in detail the tangent linearization of Lagrangian multi-variable controlled systems and examines the controllability of the linearized plant in general terms. The incremental flat outputs, when they exist, are always a linear combination of the incremental generalized position variables. The incremental inputs-to-incremental flat outputs simplified dynamics is taken as a basis for ADRC controller design. Section III presents the example of an underactuated moving crane with a flexible joint, for which a set of linear ADRC controllers is designed using suitable feedback transfer functions. This section also presents the simulation results obtained for rest-to-rest maneuvers which exhibit diminished vertical oscillations of the tip of the underactuated link, using the inverse kinematics arising from the nonlinear plant in initial and final equilibria. Section IV contains the conclusions and suggestions for further work.

2. TANGENT LINEARIZATION OF CONTROLLED LAGRANGIAN SYSTEMS

Define the scalar function, $\mathcal{L}(q, \dot{q}, u)$, to be the Lagrangian of a controlled system, $q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, given by:

$$\mathcal{L}(q, \dot{q}, u) = \mathcal{K}(q, \dot{q}) - (\mathcal{V}(q) - q^T B u), \quad (1)$$

where $\mathcal{K}(q, \dot{q})$ is the kinetic energy associated with the motions of the masses and the rotations of inertias. The term, $\mathcal{V}(q) - q^T B u$, represents the total potential energy comprising a) the potential energy \mathcal{V} , present by elastic and gravitational effects, and b) the term: $-q^T B u$, representing the works, exercised by the control inputs, u , on the system, through the input channels represented by the

constant vectors $[b_1, \dots, b_m] = B$, with $m \leq n$ and B full rank.

The vectors: $q \in \mathbb{R}^n$, $\dot{q} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, are, respectively, the generalized position coordinates, the generalized velocities and the vector of control input functions. The dynamical controlled system is described by:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q} \quad (2)$$

In correspondence with a *simple* system, the Lagrangian of the system is specifically written as:

$$\mathcal{L}(q, \dot{q}, u) = \frac{1}{2} \dot{q}^T \mathcal{M}(q) \dot{q} - (\mathcal{V}(q) - q^T B u). \quad (3)$$

The vector of canonical momenta is defined from:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \mathcal{M}(q) \dot{q}, \quad \dot{q} = \mathcal{M}^{-1}(q) p, \quad (4)$$

where it is assumed that the matrix $\mathcal{M}(q) \in \mathbb{R}^{n \times n}$ is an invertible, symmetric, matrix.

Suppose the equilibrium condition on the variables: (q, \dot{q}, u) , is given by, $(\bar{q}, \mathbf{0}, \bar{u})$.

Define the incremental variables: q_δ , \dot{q}_δ , u_δ , as the variational deviations from the equilibrium point of the position, velocity and control input vectors.

$$q_\delta = q - \bar{q}, \quad \dot{q}_\delta = \dot{q}, \quad u_\delta = u - \bar{u}. \quad (5)$$

We have, up to second order variations:

$$\begin{aligned} \mathcal{L}(\bar{q} + q_\delta, \dot{q}_\delta, \bar{u} + u_\delta) &= \mathcal{L}(\bar{q}, \dot{q}, \bar{u}) + \\ &\frac{\partial \mathcal{L}}{\partial q^T} \Big|_{(\bar{q}, 0, \bar{u})} q_\delta + \frac{\partial \mathcal{L}}{\partial \dot{q}^T} \Big|_{(\bar{q}, 0, \bar{u})} \dot{q}_\delta + \frac{\partial \mathcal{L}}{\partial u^T} \Big|_{(\bar{q}, 0, \bar{u})} u_\delta \\ &+ \frac{1}{2} [q_\delta \quad \dot{q}_\delta \quad u_\delta] \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial q \partial q^T} & \frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}^T} & \frac{\partial^2 \mathcal{L}}{\partial q \partial u^T} \\ \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q^T} & \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial \dot{q}^T} & \frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial u^T} \\ \frac{\partial^2 \mathcal{L}}{\partial u \partial q^T} & \frac{\partial^2 \mathcal{L}}{\partial u \partial \dot{q}^T} & \frac{\partial^2 \mathcal{L}}{\partial u \partial u^T} \end{pmatrix} \Big|_{(\bar{q}, 0, \bar{u})} \begin{bmatrix} q_\delta \\ \dot{q}_\delta \\ u_\delta \end{bmatrix} \end{aligned} \quad (6)$$

From the general conditions for equilibrium, it follows that, for a controlled Euler-Lagrange system, the following must hold valid:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \Big|_{(\bar{q}, 0, \bar{u})} &= \mathbf{0}, \quad \bar{p} = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \Big|_{(\bar{q}, 0, \bar{u})} = \mathbf{0}, \\ \frac{\partial \mathcal{L}}{\partial q} \Big|_{(\bar{q}, 0, \bar{u})} &= \mathbf{0} \end{aligned} \quad (7)$$

From the last equilibrium relation in (7), we have:

$$\left(\frac{\partial \mathcal{V}(q)}{\partial q} - B u \right) \Big|_{(\bar{q}, 0, \bar{u})} = \mathbf{0}$$

This implies that all linear terms, in the expansion (6), disappear at the equilibrium, except for that associated with the incremental control input vector u_δ . In equilibrium, the elastic forces and the gravitational forces are in the image of the constant map B and they are counteracted by the equilibrium control inputs \bar{u}

The tangent linearization procedure and the determination of the approximate linearized dynamics, in terms of the incremental Lagrangian, follows from the following identities:

$$\begin{aligned} \dot{q} &= \frac{d}{dt} (\bar{q} + q_\delta) = \dot{q}_\delta, \\ \frac{\partial \mathcal{L}}{\partial \dot{q}} &= \frac{\partial (\bar{\mathcal{L}} + \mathcal{L}_\delta)}{\partial \dot{q}} \\ &= \left(\frac{\partial \mathcal{L}_\delta}{\partial \dot{q}_\delta} \right) \left(\frac{\partial \dot{q}_\delta^T}{\partial \dot{q}} \right) = \left(\frac{\partial \mathcal{L}_\delta}{\partial \dot{q}_\delta} \right) \frac{\partial (\dot{q} - \bar{q})^T}{\partial \dot{q}} = \frac{\partial \mathcal{L}_\delta}{\partial \dot{q}_\delta} \end{aligned}$$

from where:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{d}{dt} \left(\frac{\partial \mathcal{L}_\delta}{\partial \dot{q}_\delta} \right)$$

and, also,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q} &= \frac{\partial (\bar{\mathcal{L}} + \mathcal{L}_\delta)}{\partial q} = \left(\frac{\partial \mathcal{L}_\delta}{\partial q_\delta} \right) \left(\frac{\partial q_\delta^T}{\partial q} \right) \\ &= \left(\frac{\partial \mathcal{L}_\delta}{\partial q_\delta} \right) \frac{\partial (q - \bar{q})^T}{\partial q} = \frac{\partial \mathcal{L}_\delta}{\partial q_\delta} \end{aligned}$$

The incremental, linearized, Euler-Lagrange equations are obtained directly from the expression isomorphic to the traditional formalism,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_\delta}{\partial \dot{q}_\delta} \right) = \frac{\partial \mathcal{L}_\delta}{\partial q_\delta} \quad (8)$$

2.1 Example

Consider the case of a *simple*, controlled, Lagrangian system, defined by (3). In equilibrium: $(\bar{q}, 0, \bar{u})$, the Lagrangian is reduced to,

$$\bar{\mathcal{L}} = \bar{q}^T B \bar{u} - \mathcal{V}(\bar{q}). \quad (9)$$

An expansion of the Lagrangian, up to second order, results in:

$$\begin{aligned} \mathcal{L}_\delta &= \bar{q}^T B u_\delta \\ &+ \frac{1}{2} (q_\delta^T, \dot{q}_\delta^T, u_\delta^T) \begin{bmatrix} -\frac{\partial^2 \mathcal{V}(q)}{\partial q \partial q^T} & 0 & B \\ 0 & \mathcal{M}(q) & 0 \\ B^T & 0 & 0 \end{bmatrix} \Big|_{(\bar{q}, 0, \bar{u})} \begin{bmatrix} q_\delta \\ \dot{q}_\delta \\ u_\delta \end{bmatrix}. \end{aligned} \quad (10)$$

The first summand: $\bar{q}^T B u_\delta$, can be eliminated because it is neither a function of q , nor of \dot{q} .

For this case, the incremental Lagrangian is given by:

$$\begin{aligned} \mathcal{L}_\delta(q_\delta, \dot{q}_\delta, u_\delta) &= \frac{1}{2} \dot{q}_\delta^T \mathcal{M}(\bar{q}) \dot{q}_\delta \\ &- \frac{1}{2} q_\delta^T \left(\frac{\partial^2 \mathcal{V}}{\partial q \partial q^T} \Big|_{(\bar{q}, 0, \bar{u})} \right) q_\delta + q_\delta^T B u_\delta \end{aligned} \quad (11)$$

Let $\mathcal{M}(\bar{q}) =: \mathcal{M}$, denote the constant value of, $\mathcal{M}(q)$, at the equilibrium point, \bar{q} . The simple, controlled, Lagrangian system exhibits the following tangent linearization:

$$\mathcal{M} \frac{d^2 q_\delta}{dt^2} = -\mathcal{K} q_\delta + B u_\delta, \quad (12)$$

where the matrix \mathcal{K} is symmetric and given by,

$$\mathcal{K} := \frac{\partial^2 \mathcal{V}(q)}{\partial q \partial q^T} \Big|_{(\bar{q}, 0, \bar{u})} \quad (13)$$

2.2 Controllability of the incremental lagrangian system

The state space model of the linearized system, (12), is written as:

$$\frac{d}{dt} \begin{bmatrix} q_\delta \\ \vartheta_\delta \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathcal{M}^{-1}\mathcal{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} q_\delta \\ \vartheta_\delta \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathcal{M}^{-1}B \end{bmatrix} u_\delta. \quad (14)$$

We study the controllability (i.e., the flatness) of the linearized Lagrangian system in general terms (See Brockett (3)). Recall that a linear, time-invariant, multivariable system is represented as,

$$\dot{x} = Fx + Gu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad m < n, \quad (15)$$

with F and G being constant matrices. We assume that, $G = [g_1, \dots, g_m]$, and of rank m . The system is referred to, as the pair (F, G) . The pair (F, G) is controllable, if and only if, the following, $n \times (nm)$, matrix satisfies:

$$\text{rank}[G, FG, \dots, F^{n-1}G] = n. \quad (16)$$

It is assumed that the following standard requirement is valid: There exists strictly positive integers, n_1, n_2, \dots, n_m , such that, $\sum n_j = n$, and that, the matrix,

$$K = [g_1, Fg_1, \dots, F^{n_1-1}g_1, g_2, Fg_2, \dots, F^{n_2-1}g_2, \dots, g_m, \dots, F^{n_m-1}g_m], \quad (17)$$

is full rank n .

The set of integers $\{n_1, \dots, n_m\}$, are known as, the Kronecker controllability indices. The controllable system, (F, G) , is flat with m flat outputs, given by (See (4)):

$$y = \alpha C_{Kr,n}^T K^{-1} Mx, \quad (18)$$

where, $\alpha = \text{diag}[\alpha_1, \dots, \alpha_m]$, is a diagonal matrix of arbitrary scalar, nonzero, parameters, and,

$$C_{Kr,n}^T = \begin{bmatrix} 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{m \times (n_1+n_2+\dots+n_m)} = \mathbb{R}^{m \times n}. \quad (19)$$

The i th row of $C_{Kr,n}^T$, consists of m ordered sub-vectors of dimensions: n_1, \dots, n_m , all of them zero vectors, except for the i th, which is of dimension, n_i , and has the form: $[0, 0, \dots, 0, 1]$. The matrix $C_{Kr,n}^T$ is full (row) rank m .

The linearized lagrangian system is controllable, if and only if, the product of the following composite matrices is full rank $2n$,

$$K_F = \begin{bmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \mathcal{M}^{-1} \end{bmatrix} \times \begin{bmatrix} 0 & B & 0 & -(\mathcal{K}\mathcal{M}^{-1})B & 0 & \dots \\ B & 0 & -(\mathcal{K}\mathcal{M}^{-1})B & 0 & (\mathcal{K}\mathcal{M}^{-1})^2 B & \dots \\ \dots & (-1)^{n-2} (\mathcal{K}\mathcal{M}^{-1})^{n-2} B & 0 & & & \\ \dots & 0 & (-1)^{n-1} (\mathcal{K}\mathcal{M}^{-1})^{n-1} B & & & \\ (-1)^{n-1} (\mathcal{K}\mathcal{M}^{-1})^{n-1} B & & & & & \\ & & & & & 0 \end{bmatrix} \quad (20)$$

The first matrix is full rank. It is evident, then, that the upper row of matrix blocks in the second matrix, as well as the lower row of blocks, (which has the same rank as the first row), are full rank. Therefore, the pair: $(-\mathcal{K}\mathcal{M}^{-1}, B)$, is necessarily controllable, i.e.,

$$\begin{bmatrix} B, (-\mathcal{K}\mathcal{M}^{-1})B, (\mathcal{K}\mathcal{M}^{-1})^2 B, \dots \\ \dots, (-1)^{n-1} (\mathcal{K}\mathcal{M}^{-1})^{n-1} B \end{bmatrix} \quad (21)$$

is full rank n . It is clear that if the pair, $(\mathcal{K}\mathcal{M}^{-1}, B)$, is controllable, the composite matrix is full rank, $2n$, and, hence, the overall system is controllable.

The minuses signs, in front of the matrices: $(\mathcal{K}\mathcal{M}^{-1})^j$, do not affect the rank of (21). We have, under the above assumptions, that the controllability condition, for the overall system, may be stated in terms of the full rank, n , of the matrix, K , given by:

$$K = \begin{bmatrix} b_1, \dots, (\mathcal{K}\mathcal{M}^{-1})^{n_1-1} b_1, b_2, \\ \dots, (\mathcal{K}\mathcal{M}^{-1})^{n_2-1} b_2, \dots, b_m, \dots, (\mathcal{K}\mathcal{M}^{-1})^{n_m-1} b_m \end{bmatrix} \quad (22)$$

Therefore, the controllability matrix (20) can be written as the $2n$ full rank matrix product,

$$K^R = \begin{bmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \mathcal{M}^{-1} \end{bmatrix} \times \begin{bmatrix} 0 & b_1 & 0 & \dots & (\mathcal{K}\mathcal{M}^{-1})^{n_1-1} b_1 & 0 & b_2 & \dots \\ b_1 & 0 & \dots & (\mathcal{K}\mathcal{M}^{-1})^{n_1-1} b_1 & 0 & b_2 & 0 & \dots \\ \dots & b_m & 0 & (\mathcal{K}\mathcal{M}^{-1})b_m & \dots & & & \\ b_m & 0 & (\mathcal{K}\mathcal{M}^{-1})b_m & 0 & \dots & & & \\ \dots & & 0 & (\mathcal{K}\mathcal{M}^{-1})^{n_m-1} b_m & & & & \\ \dots & (\mathcal{K}\mathcal{M}^{-1})^{n_m-1} b_m & & 0 & & & & \end{bmatrix} \quad (23)$$

An output vector of the linearized Lagrangian system, given by: $y_\delta = [C_q^T \ C_\vartheta^T] \begin{bmatrix} q_\delta \\ \vartheta_\delta \end{bmatrix} = C_q^T q_\delta + C_\vartheta^T \vartheta_\delta \in \mathbb{R}^m$, is said to be a vector of flat outputs, if and only if, y_δ is vector relative degree, $[n_1, \dots, n_m]$. Therefore, there exists a diagonal matrix of nonzero scalars, $\alpha = \text{diag}[\alpha_1, \dots, \alpha_m]$, such that the following system of linear equations, has a unique solution for the matrices, C_q^T y C_ϑ^T :

$$\begin{aligned} [C_q^T \ C_\vartheta^T] K^R &= \alpha C_{Kr,2n}^T = \\ &= \alpha \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (24) \end{aligned}$$

Let e_j be the m -dimensional unit column vector in the j -th cartesian coordinate. The matrices C_q^T and C_ϑ^T in $\mathbb{R}^{m \times n}$

must, independently, satisfy the following conditions for $j = 1, 2, \dots, m$:

$$C_q^T \mathcal{M}^{-1} [b_j, (\mathcal{K}\mathcal{M})b_j, \dots, (\mathcal{K}\mathcal{M})^{n_j-1}b_j] \\ = \alpha [\mathbf{0}, \mathbf{0}, \dots, e_j] \in \mathbb{R}^{m \times n_j}$$

$$C_\vartheta^T \mathcal{M}^{-1} [b_j, (\mathcal{K}\mathcal{M})b_j, \dots, (\mathcal{K}\mathcal{M})^{n_j-1}b_j] \\ = \alpha [\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}] \in \mathbb{R}^{m \times n_j}$$

If the matrix \mathbf{K} is full rank, the equality $C_\vartheta^T \mathcal{M}^{-1} \mathbf{K} = \mathbf{0}^T$, is satisfied, if and only if, $C_\vartheta^T = \mathbf{0}^T$, while the system of equations for the C_q^T components, given by: $C_q^T \mathcal{M}^{-1} \mathbf{K} = \alpha C_{Kr,n}^T$, has as a unique solution:

$$C_q^T = \alpha C_{Kr,n}^T \mathbf{K}^{-1} \mathcal{M} \in \mathbb{R}^{m \times n} \quad (25)$$

and, clearly, the components of the incremental flat output y_δ depend only on the position coordinates: q_δ :

$$y_\delta = [C_q^T \ \mathbf{0}^T] \begin{bmatrix} q_\delta \\ \vartheta_\delta \end{bmatrix} = C_q^T q_\delta \quad (26)$$

It is clear that, in general, the even order time derivatives of the components of the incremental flat output vector, y_δ , depend only on the components of the vector of incremental generalized positions, q_δ , while the odd order time derivatives of, y_δ , depend only on the components of the incremental generalized velocities, ϑ_δ .

The incremental inputs to incremental flat outputs, linear, relation is of the form:

$$\begin{bmatrix} y_{1\delta}^{(2n_1)} \\ y_{2\delta}^{(2n_2)} \\ \vdots \\ y_{m\delta}^{(2n_m)} \end{bmatrix} = \mathcal{B} \begin{bmatrix} u_{1\delta} \\ u_{2\delta} \\ \vdots \\ u_{m\delta} \end{bmatrix} + \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_m(t) \end{bmatrix}, \quad (27)$$

with $\sum_{i=1}^m 2n_i = 2n$, $\mathcal{B} \in \mathbb{R}^{m \times m}$. It is assumed that the constant square, $m \times m$, matrix, \mathcal{B} , is invertible, although, not necessarily, diagonal.

3. THE MOVING CRANE WITH FLEXIBILITY

Consider the moving crane, shown in figure 1.

We begin the modeling process by establishing the positions of all centers of mass. The center of mass of the cart is, in the (x, y) plane, $(x, 0)$; that of the first link is (x_1, y_1) and the center of mass for the second link is just (x_2, y_2) . We have then,

$$x_M = x, \quad y_M = \text{constant} \\ x_1 = x + L_{c1} \sin \theta_1, \quad y_1 = L_{c1} \cos \theta_1 \\ x_2 = x + L_1 \sin \theta_1 + L_{c2} \sin \theta_2, \\ y_2 = L_1 \cos \theta_1 + L_{c2} \cos \theta_2$$

The corresponding velocities are:

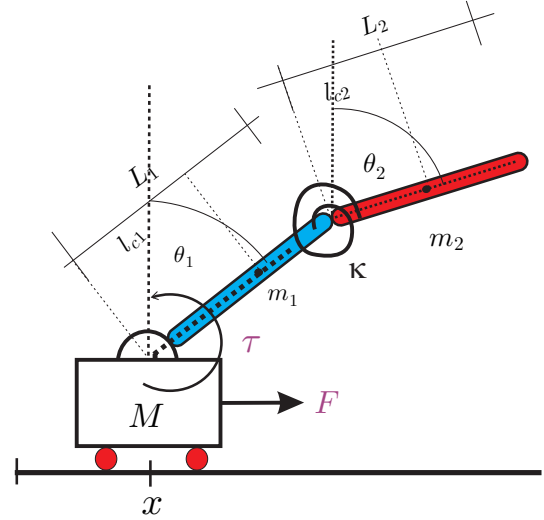


Fig. 1. The moving crane with flexibility

$$\dot{x}_M = \dot{x}, \\ \dot{x}_1 = \dot{x} + L_{c1} \dot{\theta}_1 \cos \theta_1, \quad \dot{y}_1 = -L_{c1} \dot{\theta}_1 \sin \theta_1, \\ \dot{x}_2 = \dot{x} + L_1 \dot{\theta}_1 \cos \theta_1 + L_{c2} \dot{\theta}_2 \cos \theta_2, \\ \dot{y}_2 = -L_1 \dot{\theta}_1 \sin \theta_1 - L_{c2} \dot{\theta}_2 \sin \theta_2.$$

The kinetic energy is given by,

$$K = \frac{1}{2} (M + m_1 + m_2) \dot{x}^2 + \frac{1}{2} (m_1 L_{c1}^2 + m_2 L_1^2 + I_1) \dot{\theta}_1^2 \\ + \frac{1}{2} (m_2 L_{c2}^2 + I_2) \dot{\theta}_2^2 + (m_1 L_{c1} + m_2 L_1) \dot{x} \dot{\theta}_1 \cos \theta_1 \\ + m_2 L_{c2} \dot{x} \dot{\theta}_2 \cos \theta_2 + m_2 L_1 L_{c2} \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2),$$

while the potential energy is found to be:

$$\mathcal{V} = m_1 g L_{c1} \cos \theta_1 + m_2 g (L_1 \cos \theta_1 + L_{c2} \cos \theta_2) \\ + \frac{1}{2} \kappa (\theta_2 - \theta_1)^2 - \tau \theta_1 - F x$$

The Lagrangian of the system is just

$$\mathcal{L} = \frac{1}{2} [\dot{x}, \dot{\theta}_1, \dot{\theta}_2] \mathcal{M}(q) \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ - m_1 g L_{c1} \cos \theta_1 - m_2 g (L_1 \cos \theta_1 + L_{c2} \cos \theta_2) \\ - \frac{1}{2} \kappa (\theta_2 - \theta_1)^2 + \tau \theta_1 + F x \quad (28)$$

with

$$\mathcal{M}(q) = \begin{bmatrix} M + m_1 + m_2 & (m_1 L_{c1} + m_2 L_1) \cos \theta_1 \\ (m_1 L_{c1} + m_2 L_1) \cos \theta_1 & m_1 L_{c1}^2 + m_2 L_1^2 + I_1 \\ m_2 L_{c2} \cos \theta_2 & m_2 L_1 L_{c2} \cos(\theta_1 - \theta_2) \\ m_2 L_{c2} \cos \theta_2 & m_2 L_1 L_{c2} \cos(\theta_1 - \theta_2) \\ m_2 L_{c2}^2 + I_2 \end{bmatrix} \quad (29)$$

The crane model is under-actuated, since the three degrees of freedom, needed to specify the positions of the centers of mass of the three mass elements, are actuated only by two external control inputs.

3.1 Equilibria and end effector position inverse kinematics.

A natural equilibrium point is: $\bar{\theta}_1 = \bar{\theta}_2 = 0$. For this particular configuration, the torque delivered by the rotational spring is zero.

For an arbitrary equilibrium position, parameterized by the first link equilibrium angle, $\bar{\theta}_1$, one must numerically solve for the corresponding equilibrium value $\bar{\theta}_2$, of, θ_2 , from:

$$\sin \theta_2 = -\frac{\kappa}{m_2 g L_{c2}}(\bar{\theta}_1 - \theta_2) \quad (30)$$

From the above relation (30), it is clear that: $\bar{\theta}_2 = \varphi(\bar{\theta}_1)$.

The equilibrium input torque, $\bar{\tau}$, is obtained from:

$$\bar{\tau} = -(m_1 g L_{c1} + m_2 g L_1) \sin \bar{\theta}_1 - \kappa(\varphi(\bar{\theta}_1) - \bar{\theta}_1), \quad (31)$$

while the equilibrium for the input force is just $\bar{F} = 0$.

The end effector position coordinates, (X, Y) , are related to the configuration coordinates, (x, θ_1, θ_2) , by:

$$\begin{aligned} X &= x + L_1 \sin \theta_1 + L_2 \sin \theta_2, \\ Y &= L_1 \cos \theta_1 + L_2 \cos \theta_2 \end{aligned} \quad (32)$$

For a given equilibrium position of the end effector (\bar{X}, \bar{Y}) , one has to solve for $\bar{\theta}_1$ and \bar{x} from the set of transcendental equations:

$$\begin{aligned} \bar{X} &= \bar{x} + L_1 \sin(\bar{\theta}_1) + L_2 \sin(\varphi(\bar{\theta}_1)), \\ \bar{Y} &= L_1 \cos(\bar{\theta}_1) + L_2 \cos(\varphi(\bar{\theta}_1)) \end{aligned} \quad (33)$$

This may be eased by building a table of pairs $(\bar{\theta}_1, \varphi(\bar{\theta}_1))$.

3.2 The tangent linearization model

The second order expansion of the Lagrangian around the equilibrium point: $(\bar{x}, \bar{\theta}_1, \bar{\theta}_2) = (0, 0, 0)$, with $\bar{\tau} = 0$, $\bar{F} = 0$, yields:

$$\begin{aligned} \mathcal{L}_\delta &= \frac{1}{2} \begin{bmatrix} \dot{x}_\delta & \dot{\theta}_{1\delta} & \dot{\theta}_{2\delta} \end{bmatrix} \mathcal{M}(\bar{q}) \begin{bmatrix} \dot{x}_\delta \\ \dot{\theta}_{1\delta} \\ \dot{\theta}_{2\delta} \end{bmatrix} \\ &- \frac{1}{2} \begin{bmatrix} x_\delta & \theta_{1\delta} & \theta_{2\delta} \end{bmatrix} \mathcal{K} \begin{bmatrix} x_\delta \\ \theta_{1\delta} \\ \theta_{2\delta} \end{bmatrix} + \begin{bmatrix} x_\delta & \theta_{1\delta} & \theta_{2\delta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_\delta \\ \tau_\delta \end{bmatrix} \end{aligned} \quad (34)$$

where,

$$\begin{aligned} \mathcal{M}(\bar{q}) &=: \mathcal{M} =: \{\mathcal{M}_{ij}\} \\ &\begin{bmatrix} M + m_1 + m_2 & (m_1 L_{c1} + m_2 L_1) & m_2 L_{c2} \\ (m_1 L_{c1} + m_2 L_1) & m_1 L_{c1}^2 + m_2 L_1^2 + I_1 & m_2 L_1 L_{c2} \\ m_2 L_{c2} & m_2 L_1 L_{c2} & m_2 L_{c2}^2 + I_2 \end{bmatrix} \end{aligned} \quad (35)$$

and,

$$\mathcal{K} = \{\mathcal{K}_{ij}\} := - \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 g L_{c1} + m_2 g L_1 - \kappa & \kappa \\ 0 & \kappa & m_2 g L_{c2} - \kappa \end{bmatrix} \quad (36)$$

The tangent linearization model, around the equilibrium point, $(\bar{\theta}_1, \bar{\theta}_2) = (0, 0)$, is given by direct application of the Euler-Lagrange formalism, (8), on the incremental

Lagrangian expression (34), using, as generalized coordinates, the incremental configuration variables:

$$\mathcal{M} \begin{bmatrix} \ddot{x}_\delta \\ \ddot{\theta}_{1\delta} \\ \ddot{\theta}_{2\delta} \end{bmatrix} = -\mathcal{K} \begin{bmatrix} x_\delta \\ \theta_{1\delta} \\ \theta_{2\delta} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_\delta \\ \tau_\delta \end{bmatrix} \quad (37)$$

Tedious, but straightforward, manipulations, involving (23), show that the linearized lagrangian system is controllable and, hence, flat. The independent flat outputs are given by,

$$\begin{aligned} y_{1\delta} &= (M + m_1 + m_2)x_\delta + (m_1 L_{c1} + m_2 L_1)\theta_{1\delta} \\ &\quad + m_2 L_{c2}\theta_{2\delta} =: \mathcal{M}_{11}x_\delta + \mathcal{M}_{12}\theta_{1\delta} + \mathcal{M}_{13}\theta_{2\delta} \\ y_{2\delta} &= m_2 L_{c2}x_\delta + m_2 L_{c2}L_1\theta_{1\delta} + (m_2 L_{c2}^2 + I_2)\theta_{2\delta} \\ &=: \mathcal{M}_{31}x_\delta + \mathcal{M}_{32}\theta_{1\delta} + \mathcal{M}_{33}\theta_{2\delta} \end{aligned} \quad (38)$$

The differential parametrization of the generalized incremental position variables, in terms of the incremental flat outputs (and its time derivatives), is obtained from the following set of linear equations:

$$\begin{bmatrix} y_{1\delta} \\ y_{2\delta} \\ \dot{y}_{2\delta} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \\ 0 & \mathcal{K}_{32} & \mathcal{K}_{33} \end{bmatrix} \begin{bmatrix} x_\delta \\ \theta_{1\delta} \\ \theta_{2\delta} \end{bmatrix} \quad (39)$$

On the other hand, from (35)-(38), it follows that the incremental control input variables F_δ and τ_δ can be expressed as

$$F_\delta = \ddot{y}_{1\delta}, \quad \tau_\delta = \frac{y_{2\delta}^{(4)}}{\beta_{22}} - \left(\frac{\beta_{21}}{\beta_{22}} \right) \dot{y}_{1\delta} + \dots \quad (40)$$

with β_{21} , β_{22} being constant gains.

The incremental flat output vector, $(y_{1\delta}, y_{2\delta})$, exhibits, then, a vector relative degree: $(2, 4)$. Note that the sum of the component values of this vector, matches the order of the system. Indeed, one easily verifies that,

$$\begin{aligned} \ddot{y}_{1\delta} &= F_\delta, \\ y_{2\delta}^{(4)} &= \beta_{21}F_\delta + \beta_{22}\tau_\delta + \xi_2(t) \end{aligned} \quad (41)$$

3.3 Problem formulation

It is desired to move the end effector, located at the tip of the under-actuated arm, from an initial equilibrium point: $(\bar{X}_{init}, \bar{Y}_{init})$, towards a final desired equilibrium point, $(\bar{X}_{final}, \bar{Y}_{final})$, in a certain, given, time interval: $[t_{init}, t_{final}]$, provided both equilibria are located within the working area of the moving crane.

With the knowledge of the initial and final values of the end effector position vector in the plane (X, Y) , the initial and final values of the generalized position coordinates equilibria may be computed for the nonlinear plant model. These set of values, in turn, allow for the prescription of the two incremental flat output reference trajectories $(y_{1\delta}^*(t), y_{2\delta}^*(t))$, via two Bèzier polynomials, smoothly interpolating between their corresponding initial and final equilibrium values of the rest-to-rest maneuver.

It has been shown (See (5)) that, classical, transfer function output feedback control schemes, are equivalent to ROESO based ADRC controllers, specified for each flat output dynamics.

Let,

$$\begin{bmatrix} 1 & 0 \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} F_\delta \\ \tau_\delta \end{bmatrix} = \begin{bmatrix} [y_{1\delta}^*]^{(2)} - \left[\frac{k_2 s^2 + k_1 s + k_0}{s(s+k_3)} \right] (y_{1\delta} - y_{1\delta}^*) \\ [y_{2\delta}^*]^{(4)} - \left[\frac{\gamma_4 s^4 + \gamma_3 s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0}{s(s^3 + \gamma_7 s^2 + \gamma_6 s + \gamma_5)} \right] (y_{2\delta} - y_{2\delta}^*) \end{bmatrix} \quad (42)$$

A ROESO based ADRC realization of the first incremental flat output controller is achieved by setting the classical compensator gains, $\{k_3, k_2, k_1, k_0\}$, to match those of a fourth order characteristic polynomial combining a high-gain ROESO with a moderate gain linear feedback state controller as:

$$s^4 + k_3 s^3 + \dots + k_0 \equiv (s^2 + \lambda_1^F s + \lambda_0^F)(s^2 + g_1^F s + g_0^F)$$

while the second controller synthesis is obtained via,

$$s^8 + \gamma_7 s^7 + \dots + \gamma_0 \equiv (s^4 + \lambda_3^\tau s^3 + \dots + \lambda_0^\tau)(s^4 + g_3^\tau + \dots + g_0^\tau)$$

3.4 Trajectory Planning

Since, $\bar{\theta}_{1,init} = \bar{\theta}_{2,init} = 0$ and $\bar{x}_{init} = 0$, then:

$$\begin{aligned} \bar{x}_{init} = \bar{X}_{init} = 0, \quad \bar{Y}_{init} = L_1 + L_2, \quad \bar{\tau} = 0, \quad \bar{F} = 0 \\ \bar{y}_{1\delta}(t_{init}) = 0, \quad \bar{y}_{2\delta}(t_{init}) = 0 \end{aligned}$$

The final values of the flat outputs may be obtained from the table 1,

3.5 Simulations Results

We consider a crane, with:

$$\begin{aligned} M &= 400[Kg], \quad m_1 = 100[Kg], \quad m_2 = 50[Kg], \\ L_{c1} &= 0.75[m], \quad L_1 = 2[m], \quad L_{c2} = 0.75[m], \quad L_2 = 1.5[m] \\ \kappa &= 800.0[N - m/rad] \end{aligned}$$

The numerical values above, establish numerical relations addressing the inverse kinematics for the final equilibrium conditions, with $\bar{x} = 0.0$, and $\bar{x} = 2.0$, we have:

Table 1. Inverse kinematics for some equilibria

\bar{x}	$\bar{\theta}_1$	$\bar{\theta}_2$	X_{eff}	Y_{eff}	$\bar{y}_{1\delta}$	$\bar{y}_{2\delta}$
0.0	$\pi/6$	0.876	2.153	2.691	116.9	64.19
0.0	$\pi/4$	1.216	2.821	1.935	172.0	93.47
0.0	$\pi/3$	1.505	3.229	1.097	226.0	121.34
2.0	$\pi/6$	0.876	4.152	2.691	1216.5	139.19
2.0	$\pi/4$	1.216	4.820	1.935	1272.6	168.47
2.0	$\pi/3$	1.506	5.229	1.098	1326.1	196.33

The graphs in figure 2, depict a smooth reference trajectory tracking task for the end effector position, starting from initial the equilibrium position $(\bar{X}_{init}, \bar{Y}_{init}) = (0, L_1 + L_2 = 3.5)[m]$, towards a desired final position specified by: $\bar{x} = 2.0[m]$, $(X_{final}, Y_{final}) = (4.153, 2.691)[m]$. This is indirectly achieved in terms of the desired initial and final incremental flat output values: $(y_{1\delta}(t_{init}), y_{2\delta}(t_{init})) = (0, 0)$, $(y_{1\delta}(t_{final}), y_{2\delta}(t_{final})) = (1216.5, 139.19)$ with $t_{init} = 0[s]$ and $t_{final} = 6 [s]$, using suitable Bèzier polynomials.

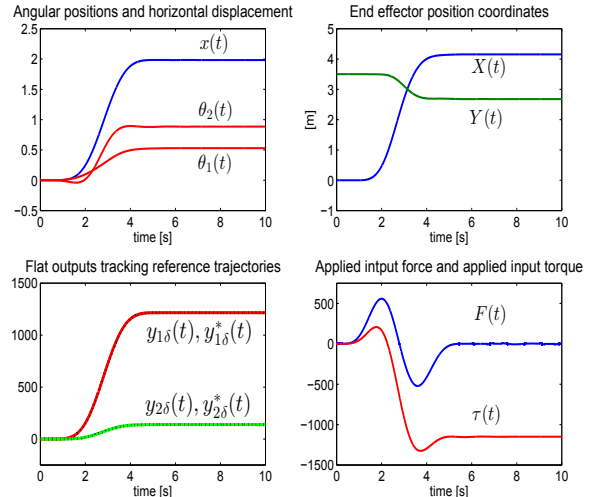


Fig. 2. ADRC controlled responses of moving crane.

4. CONCLUSIONS

We have presented a MIMO ADRC design procedure for lagrangian systems with a controllable tangent linearization, around a given equilibrium. The system needs not be differentially flat. A direct procedure is proposed for tangent linearization in MIMO controlled lagrangian systems, based on a second order expansion of the lagrangian function around an equilibrium. ADRC, designed on the basis of the controllable tangent model forces the nonlinear plant variables to behave as the controlled incremental variables. Thanks to a suitable equivalence between the ADRC method and classical compensation networks (See (5)), the controllability of the linearized system is shown to imply that the underlying nonlinear system can be controlled using only measurements of the incremental flat output variables. In turn, these special outputs require only generalized position measurements, with no need for generalized velocity observers. A multi-variable, under-actuated, mobile crane example, with joint flexibility, is used as an illustrative example, for trajectory tracking, with encouraging simulation results.

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