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Input-output Decoupling and Linearization of Nonlinear Multi-input Multi-output Time-varying Delay Systems

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Abstract: In Nicolau et al. [2018], the authors proposed a solution (in the form of an algorithm that constructs a causal and bounded feedback) for the input-output decoupling and linearization problem for the particular class of two-input two-output time-varying delay systems. In this paper, we generalize that algorithm to the multi-input multi-output case. The main idea is to introduce, at each step when the input-output decoupling is not possible, an artificial delay for the inputs that appear "too early" in the system.

Keywords: Nonlinear time-varying delay systems, input-output decoupling and linearization.

1. INTRODUCTION

Input-output decoupling and linearization is an important tool in nonlinear control theory which aims to construct a feedback transformation for which the input-output map of the feedback modified systems is linear. This problem is well know for nonlinear control systems without delays and has been extended to nonlinear control systems with multiple delays Germani et al. [1998], Germani et al. [2003], Marquez-Martinez and Moog [2004], Oguchi and Watanabe [1998], Oguchi et al. [2002], see also Califano et al. [2013], Garcia-Ramirez et al. [2016]. These previous works consider constant time-delays. Motivated by applications, the class of time-varying delay systems has been studied mainly from a stability analysis point of view (see, e.g., Fridman [2014], Germani et al. [2000] and the references therein), but the problem of input-output decoupling and linearization for that class of systems is still largely open. Therefore, the goal of this paper is to study the input-output decoupling and linearization for nonlinear time-varying delay systems.

The main difficulty when dealing with such systems is that the linearizing feedback has to be constructed from a relation that involves recursively the input and its successive delays. Therefore, the constructed control may not be bounded nor causal. In Nicolau et al. [2018], the authors considered the particular case of two-input two-output time-varying delay systems and proposed, for that class of systems, a solution (in the form of an algorithm constructing a causal and bounded feedback) for the input-output decoupling and linearization problem, yielding a maximal loss of observability (see Isidori and Moog [1988] for that notion). Related results for the single-input single-output case can be found in Haidar et al. [2019]. In this paper, we generalize the algorithm of Nicolau et al. [2018] to the multi-input multi-output case and similarly to the aforementioned paper, we adopt the so-called standard solution implying the maximal loss of observability. This solution is classical for systems without delays. However, when applied to time-delay systems, it may provide a non causal or non bounded feedback transformation.

The algorithm is constructive and gives sufficient conditions for designing a causal and bounded linearizing feedback. As for the two-input two-output case, the main idea is to introduce an artificial delay for the inputs that appear "too early" in the system (see Descusse and Moog [1985], Nijmeijer and Respondek [1988] for a related method for nonlinear systems without delays where the inputs that appear "too early" are precompensated). The main difficulty is that, contrary to the two-input two-output case, where only one control could show up before the second one, for the general case, at each new iteration, several different controls may appear "too early". The endpoint criterion of the algorithm is more complicated than that proposed in Nicolau et al. [2018]. In fact, the main difficulty that still remains to be solved is that we are not able to give a upper bound for the number of iterations. The algorithm stops if we succeed to construct a new object with suitable properties allowing the construction of a bounded and causal feedback or after a maximal number of iteration that has to be defined by the user The paper is organized as follows. In Section 2, we give some notations and definitions. In Section 3, we discuss the input-output decoupling algorithm, which is the main result of the paper. Finally, we illustrate it by an example in Section 4.

2. DEFINITIONS AND PROBLEM STATEMENT

Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $\|\cdot\|$ and $\mathbb{R}_+$ the set of non-negative real numbers. For a real matrix $A = (a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, we define $\|A\|_{\sup} = \sup_{ij}(|a_{ij}|)$.

Definition 1. (δ^i-operator). *Let $\theta : \mathbb{R} \mapsto [0, \bar{\theta}]$ be a sufficiently smooth bounded time-varying delay function supposed to be known and satisfying $\frac{d\theta}{dt} < 1$. Denote by $\bar{\theta} > 0$ its supremum. Consider the recursive relation*

$$\tau_{i+1} = \tau_i - \theta \circ \tau_i, \quad \text{for } i \geq 0, \quad \text{where } \tau_0(t) = t.$$

We denote by δ^i the time delay operator that shifts the time from t to $\tau_i(t)$, that is

$$\delta^0\sigma(t) = \sigma(t) \quad \text{and} \quad \delta^i\sigma(t) = \sigma(\tau_i(t)), \quad \text{for } i \geq 1,$$

where σ is defined on an interval containing $[t - i\bar{\theta}, t]$. Applied on functions composition, resp., on functions product, the delay operator δ^i acts as

$$\delta^i\varphi(t, \sigma(t)) = \varphi(\tau_i, \delta^i\sigma(t)) = \varphi(\tau_i, \sigma(\tau_i)), \quad \text{for } i \geq 0, \quad (1)$$

resp., as

$$\delta^i\varphi(t) \cdot \sigma(t) = (\delta^i\varphi(t)) \cdot (\delta^i\sigma(t)), \quad \text{for } i \geq 0, \quad (2)$$

i.e., the delay spreads to the right. If parentheses are present, i.e., we have $(\delta^i\varphi(t)) \cdot \sigma(t)$, then the delay affects only the first function (here φ).

Notation 1. Let q be the maximal order of the time delay operator acting on a map σ. Then the δ-operator denotes

$$\delta\sigma(t) = (\delta^0\sigma(t), \ldots, \delta^q\sigma(t)). \quad (3)$$

We study input-output decoupling of multi-input multi-output nonlinear time-varying delay systems of the form

$$\begin{cases} \dot{x}(t) = f(\delta x(t), t) + \sum_{j=1}^{m} g_j(\delta x(t), t)u_j(t), \\ y_i(t) = h_i(\delta x(t), t), \quad 1 \leq i \leq m, \end{cases} \quad (4)$$

where $t \geq 0$, with initial condition

$$x(\mathrm{t}) = \zeta(\mathrm{t}), \text{ for all } \mathrm{t} \in [-\mu, 0], \quad (5)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, all vector fields $f, g_j : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}^n$ and functions $h_i : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}$ are supposed sufficiently smooth. The integer q corresponds to the maximal delay order explicitly involved in f, g_j and h_i, for $1 \leq i \leq m$, (this does not mean that all of them depend explicitly on $\delta^q x(t)$). Recall that by (3), we have $\delta x(t) = (\delta^0 x(t), \ldots, \delta^q x(t))$. The initial condition ζ belongs to the Banach space of continuous functions from $[-\mu, 0]$ into $\mathbb{R}^n$, where μ is a sufficiently large integer, and the input $u(\cdot)$ is a Lebesgue measurable function. We also assume that system (4) is forward complete guaranteeing the existence of solutions on $[-\mu, +\infty)$ for each determined u. Moreover, we suppose that all outputs are independent (i.e., there is no nontrivial relation between the outputs and the delayed outputs).

Remark 1. Our results can be generalized to general time-varying delay systems of the form $\dot{x}(t) = f(\delta x(t), t) + \sum_{j=1}^{m}\sum_{\ell=1}^{s} g_j^\ell(\delta x(t), t)\delta^\ell u_j(t)$, with s the maximal order of delays involved in the controls, but considering that general form would considerably complicate the notations.

In the case of the input-output decoupling problem, the output is connected to the control only indirectly through the state. To achieve input-output decoupling and linearization, we must find a direct relation between the inputs and the outputs of the system. In general, this is done by successive differentiation of the outputs h_i until the inputs appear in the resulting derivative equations. An important tool when differentiating is the Lie derivative, see Haidar et al. [2019] for its definition in the context of time-varying delay, but we will not use that notion here (even if what follows is related). Similarly to Nicolau et al. [2018] (but without using the Lie derivative), for each output we define its relative degree and then construct the decoupling matrix of the system which will be seen as a polynomial of the δ^ℓ-operator with matrix coefficients.

The derivative of h_i, for $1 \leq i \leq m$, can be expressed as:

$$\frac{\mathrm{d}h_i(\delta x(t), t)}{\mathrm{d}t} = \frac{\partial h_i(\delta x(t), t)}{\partial t} + \sum_{\ell=0}^{q}\left[\frac{\partial h_i(\delta x(t), t)}{\partial \delta^\ell x} \cdot \dot{\tau}_\ell(t) \cdot \delta^\ell\left(f(\delta x(t), t) + \sum_{j=1}^{m} g_j(\delta x(t), t) \cdot u_j(t)\right)\right]$$

yielding

$$\dot{h}_i = \frac{\partial h_i}{\partial t} + \sum_{\ell=0}^{q}\frac{\partial h_i}{\partial \delta^\ell x}\cdot\dot{\tau}_\ell\cdot(\delta^\ell f) + \sum_{j=1}^{m}\sum_{\ell=0}^{q}\frac{\partial h_i}{\partial \delta^\ell x}\cdot\dot{\tau}_\ell\cdot(\delta^\ell g_j)\cdot(\delta^\ell u_j),$$

where $\dot{h}_i = \frac{\mathrm{d}h_i}{\mathrm{d}t}$ and, in order to simplify the notation, the arguments $\delta x(t), t$ have been omitted. Recall that q denotes the maximal delay order explicitly involved in system (4) and observe that when computing the time-derivative of h_i, we introduce, via the terms $\delta^\ell\left(f + \sum_{j=1}^{m} g_j u_j\right)$, new delays that affect both x and u variables. Therefore, in order to compute $\dot{h}_i$ for all $t \geq 0$, in addition to (5), we also need an initial condition on u:

$$u(\mathrm{t}) = \psi(\mathrm{t}), \text{ for all } \mathrm{t} \in [-\mu, 0]. \quad (6)$$

We suppose that ζ and ψ are known on a sufficiently large interval $[-\mu, 0]$ and that they satisfy the differential equations of (4) on that interval. The number of new delays introduced in the derivative $\dot{h}_i$ is actually related to the maximal delay order appearing in h_i only. Remark also that, even if h_i does not explicitly depend on t, i.e., $h_i = h_i(\delta x(t))$, the derivative $\dot{h}_i$ does depend on t (through the terms $\dot{\tau}_\ell(t)$), explaining why we considered from the beginning non autonomous systems.

If $\dot{h}_i$ does not involve explicitly the control u or any delayed control $\delta^\ell u$, for $\ell \geq 1$, i.e., $\dot{h}_i$ is a function of $(\delta x(t), t)$ only, we repeat recursively the above differentiation until u or $\delta^\ell u$, for some $\ell \geq 1$, appear. The smallest integer ρ_i such that $h_i^{(\rho_i)}$ depends explicitly on u or $\delta^\ell u$, for some $\ell \geq 1$, is called the relative degree of the output h_i. More precisely:

$$\begin{cases} \dfrac{\partial h_i^{(s)}}{\partial \delta^\ell u_j} \equiv 0, \text{ for all } 1 \leq s \leq \rho_i - 1, 1 \leq j \leq m, \ell \geq 0, \\ \dfrac{\partial h_i^{(\rho_i)}}{\partial \delta^\ell u_j} \neq 0, \text{ for some } 1 \leq j \leq m \text{ and some } \ell \geq 0. \end{cases}$$

Assumption 1. We work locally around a given initial condition and suppose that the system evolves far from singularities. So when we say that a function does not vanish, we mean that it is nonzero for any $t \geq 0$ (or any t in the domain of validity of (4)) implying, in particular, that relative degrees and ranks are constant with respect to time.

Let d denote the maximal delay order of u involved in the derivatives $h_i^{(\rho_i)}$, for $1 \leq i \leq m$, (recall that when computing $h_i^{(s+1)}$ by differentiating $h_i^{(s)}$, we introduce as many new delays as in $h_i^{(s)}$). According to our notations, we have $d \leq q \times (\max_{1 \leq i \leq m} \rho_i)$. Denote

$$\alpha_i(\delta x(t), t) = \frac{\partial h_i^{(\rho_i - 1)}}{\partial t} + \sum_{\ell=0}^{d}\frac{\partial h_i^{(\rho_i-1)}}{\partial \delta^\ell x}\cdot\dot{\tau}_\ell\cdot\delta^\ell f, \; 1 \leq i \leq m,$$

and

$$a_{ij}^\ell(\delta x(t), t) = \frac{\partial h_i^{(\rho_i-1)}}{\partial \delta^\ell x}\cdot\dot{\tau}_\ell\cdot\delta^\ell g_j, \; 1 \leq i, j \leq m, \; 1 \leq \ell \leq d.$$

With that notations, for $1 \leq i \leq m$, we have

$$h_i^{(\rho_i)} = \alpha_i(\delta x(t),t) + \sum_{j=1}^{m}\sum_{\ell=0}^{d} a_{ij}^{\ell}(\delta x(t),t)\delta^{\ell}u_j(t),$$

where, according to the definition of the relative degree and of the integer d, for each $1 \leq i \leq m$, at least one function a_{ij}^{ℓ} does not vanish, and there exist integers $1 \leq i,j \leq m$ such that $a_{ij}^{d}(\delta x(t),t) \neq 0$. For $1 \leq i,j \leq m$, define the δ-polynomials

$$a_{ij}(\delta] = \sum_{\ell=0}^{d} a_{ij}^{\ell}(\delta x(t),t)\delta^{\ell}.$$

Each $a_{ij}(\delta]$ is associated to u_j in the expression of $h_i^{(\rho_i)}$:

$$h_i^{(\rho_i)} = \alpha_i(\delta x(t),t) + \sum_{j=1}^{m} a_{ij}(\delta]u_j(t), \tag{7}$$

where we use the fact that the delay operator spreads to the right, see (2). We can now define the $(m \times m)$-decoupling matrix

$$\mathcal{A}(\delta] = (a_{ij}(\delta])_{1\leq i,j\leq m},$$

which is seen as a (matrix) δ-polynomial (whose coefficients are $(m \times m)$-real matrices) and can be developed with respect to the delay operator as follows:

$$\mathcal{A}(\delta] = \mathcal{A}^0(\delta x(t),t)\delta^0 + \cdots + \mathcal{A}^d(\delta x(t),t)\delta^d, \tag{8}$$

where the $(m \times m)$ matrices $\mathcal{A}^{\ell}$, $0 \leq \ell \leq d$, are given by

$$\mathcal{A}^{\ell}(\delta x(t),t) = \left(a_{ij}^{\ell}(\delta]\right)_{1\leq i,j\leq m}. \tag{9}$$

Finally, by $\alpha(\delta x(t),t)$ we will mean the vector of m smooth functions whose i-entry is $\alpha_i(\delta x(t),t)$ of (7).

Definition 2. Consider a δ-polynomial of the form $\mathcal{B}(\delta] = \sum_{\ell=0}^{d}\mathcal{B}^{\ell}(\delta x(t),t)\delta^{\ell}$, whose coefficients are $(m \times m)$-real matrices and assume $\mathcal{B}^d(\delta x(t),t) \neq 0$. We call minimal degree of $\mathcal{B}$ the order of its first coefficient non identically zero, i.e., the integer $0 \leq k \leq d$ such that

$$\mathcal{B}^k(\delta x(t),t) \neq 0 \quad \text{and} \quad \mathcal{B}^{\ell}(\delta x(t),t) \equiv 0,\ \forall\ 0 \leq \ell < k,$$

where 0 is the $(m \times m)$ zero-matrix. The integer d is called the maximal degree of $\mathcal{B}$.

Notation 2. Denote by p the minimal degree of the δ-polynomial $\mathcal{A}(\delta]$, given by (8)-(9), and associated to (4).

Definition 3. The problem of input-output decoupling is solvable for system (4) if each output h_i admits a finite relative degree ρ_i and if there exist a bounded and causal feedback $u(t)$ and an integer k such that, for $t \geq \tau_p^{-1}(0)$,

$$\mathcal{A}(\delta]u(t) = -\alpha(\delta x(t),t) + \delta^k v(t) \quad \text{and} \quad k \geq p, \tag{10}$$

where p is the minimal degree of $\mathcal{A}$. For such $u(t)$ and k, the feedback modified system satisfies

$$h_i^{(\rho_i)} = \delta^k v_i(t),\ \text{for } 1 \leq i \leq m,$$

where v is the new control (assigned with respect to the properties that we want to achieve). Moreover, the system is said input-output decoupled and linearizable with delay if $k > 0$ (resp., without delay if $k = 0$).

Remark 2. Equation (10) leads to a solution for the problem of input-output decoupling yielding a maximal loss of observability, Isidori and Moog [1988]. An interesting question is how to obtain a general causal and bounded feedback (with or without maximal loss of observability). For the constant-delay case, we send the reader to Marquez-Martinez and Moog [2004], Baibeche and Moog [2015], where the right-hand side of (10) is replaced by a polynomial of $\delta^i v$; the generalization of their solution to the time-varying delay case is not straightforward (and its adaptation will be studied elsewhere).

Remark 3. In Definition 3, when we say that a bounded feedback exists, we implicitly suppose that the zero-dynamics or, equivalently, the unobserved (with the help of $(h_1,\ldots,h_m)$) part of (4), if there is one, is stable. In general, this is not enough to guarantee the boundedness of u constructed from an equation of form (10) and sufficient conditions are given by Lemma 1, in Section 3.

Remark 4. In (10), we consider $\delta^k v$ instead of v (that would lead to a linear input-output map for which the delay is completely compensated) because we do not request to know v in the future. Indeed, if we replace $\delta^k v$ by v in (10), in the particular case where a feedback of the form $v = \Psi(\delta x(t),t)$ is applied, advances appear in (10) if the lowest delay order in Ψ is smaller than p. Another question is why we do not simply take $\delta^p v$ (instead of $\delta^k v$, with $k \geq p$). This point will become clear when presenting our algorithm where transformations involving some artificial delays will be applied. Observe that, even if we take $\delta^k v$ instead of v, advances may derive from the term α. That will be excluded by causality conditions.

3. INPUT-OUTPUT DECOUPLING ALGORITHM

Two problems arise when constructing a feedback u from an equation of form (10). Since u is described by a recursive equation [1], the solution may not be bounded or causal. Lemma 1 gives sufficient conditions guaranteeing that the feedback u constructed from an equation of the form

$$\mathcal{B}(\delta]u(t) = \beta(\delta x(t),t) + \delta^k v, \tag{11}$$

where k is the minimal degree of $\mathcal{B}(\delta]$, stays bounded when the right-hand side of (11) is bounded.

Lemma 1. Consider equation (11) where the δ-polynomial $\mathcal{B}(\delta]=\sum_{\ell=k}^{d}\mathcal{B}^{\ell}(\delta x(t),t)\delta^{\ell}$ is such that its $(m\times m)$-matrix $\mathcal{B}^k$ is invertible for all $t \geq \tau_k^{-1}(0)$. Assume that v is such that $\beta(\delta x(t),t)+\delta^k v(t)$ is bounded over $[\tau_k^{-1}(0),+\infty)$, and that the initial condition $u(\mathfrak{t}) = \psi(\mathfrak{t})$ is bounded over $[-d\bar{\theta},0]$. If $k = d$, i.e., $\mathcal{B}(\delta] = \mathcal{B}^d(\delta x(t),t)\delta^d$ with $\mathcal{B}^d$ invertible, then the solution $u(t)$ of (11) is always bounded. If $k < d$ and there exists a constant $\gamma > 1$ such that

$$\sup_{t\geq\tau_k^{-1}(0)} \left\| \left(\mathcal{B}^k\right)^{-1}\mathcal{B}^{\ell} \right\|_{\sup} \leq \frac{1}{2\gamma(d-k)}, \quad \text{for all } \ell > k, \tag{12}$$

then, for any $t \geq 0$, the solution $u(t)$ of (11) verifies

$$\|u(t)\| \leq \frac{\gamma}{\gamma-1}\sup_{s\geq\tau_k^{-1}(0)} \left\| \left(\mathcal{B}^k\right)^{-1}\left(\beta(\delta x(s),s)+\delta^k v(s)\right) \right\| + \varepsilon(t),$$

where $\varepsilon(t)$ tends to 0 when t tends to $+\infty$.

Proof. The proof is similar to that of [Nicolau et al. 2018, Lemma 1] for the two-input two-output case. □

One of the most important conditions of Lemma 1 is the invertibility of $\mathcal{B}^k$, the first nonzero matrix-coefficient of the δ-polynomial $\mathcal{B}(\delta]$. It is $\mathcal{B}^k$ that have to be inverted in order to compute u in function of v from (11). Its inverse is also involved in (12), which is actually the condition assuring that u stays bounded when $\mathcal{B}(\delta]$ contains at least two delays of different order.

[1] By "recursive" we point out that the construction of u requires a recursive prediction of the values of v over the intervals $[0,\tau_p^{-1}(0)]$ and $[\tau_i^{-1}(0),\tau_{i+1}^{-1}(0)]$, for $i \geq p$, the integer p being the minimal degree of $\mathcal{A}(\delta]$, see Definition 2 and Notation 2.

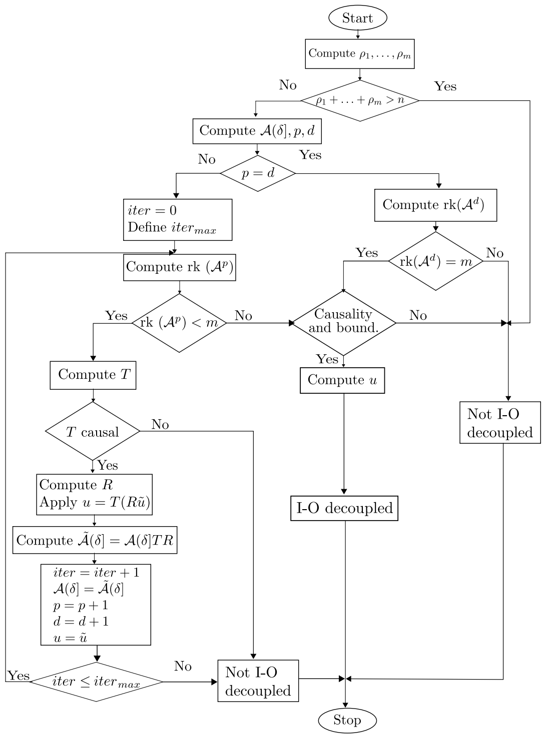

Fig. 1. Input-output decoupling algorithm

We present next the main result of the paper: a constructive input-output decoupling and linearization algorithm. Recall that $\mathcal{A}(\delta]$, given by (8)-(9), is the δ-polynomial associated to (4), and that p and d denote its minimal and maximal degree, resp. The main idea on which the algorithm is based is that if $\mathcal{A}^p$ is invertible and satisfies Lemma 1, then a bounded feedback can be constructed from (10), with $\mathcal{B}(\delta]$ being replaced by $\mathcal{A}(\delta]$, the integer k by p, and the m-valued β by α (defined with the help of (7) in Section 2). But, in general, $\mathcal{A}^p$ does not need to be invertible. In that case, the idea is to transform it in such a way that its (new) first nonzero matrix is invertible and satisfies Lemma 1. To that end, a two-step feedback transformation (T, R) is proposed at each iteration of the algorithm, which stops if we manage to construct a new δ-polynomial with the desired properties or after a maximum number of iterations defined by the user.

3.1 Algorithm

Fig. 1 summarizes the input-output decoupling algorithm whose steps are presented and commented below.

1) Calculate the relative degrees $\rho_1, \ldots, \rho_m$ associated to the outputs $h_1, \ldots, h_m$ of system (4).
   a) If $\rho_1 + \cdots + \rho_m > n$ (where $\dim x = n$), then the system cannot be decoupled and linearized and the algorithm stops.
   b) If $\rho_1 + \cdots + \rho_m \leq n$, then calculate the δ-polynomial $\mathcal{A}(\delta]$, its minimal (resp., maximal) degree p (resp., d).
2) If $p = d$, i.e., $\mathcal{A}(\delta] = \mathcal{A}^d(\delta x(t), t)\delta^d$, then compute $\mathrm{rk}\,(\mathcal{A}^d)$, the rank of the $(m \times m)$-real matrix $\mathcal{A}^d$.
   a) If $\mathrm{rk}\,(\mathcal{A}^d) = m$, i.e., $\mathcal{A}^d$ is invertible, then we can always construct from (10), with $k = d$, a bounded feedback u. If in addition
   $$\frac{\partial \alpha}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial \mathcal{A}^d}{\partial \delta^i x} \equiv 0, \qquad (13)$$
   for $0 \leq i \leq d-1$, then the feedback u is also causal, the system is input-output decoupled and linearized with its help and the algorithm stops.
   b) If $\mathrm{rk}\,(\mathcal{A}^d) < m$, then the system cannot be input-output decoupled and linearized and the algorithm stops.
3) If $p < d$, then set $iter = 0$ and define the maximum number of iterations $iter_{max} \geq d - p + 1$.
4) Identify the matrix $\mathcal{A}^p$ and compute its rank.
   a) If $\mathrm{rk}\,(\mathcal{A}^p) = m$, i.e., $\mathcal{A}^p$ is invertible, then check if Lemma 1 (and, in particular, condition (12)) is verified for $\mathcal{B}(\delta] = \mathcal{A}(\delta]$ and $\beta = -\alpha$.
   If (12) is satisfied, then we can construct from (11), with $\mathcal{B}(\delta] = \mathcal{A}(\delta]$ and $\beta = -\alpha$, a bounded feedback u. If in addition,
   $$\frac{\partial \alpha}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial \mathcal{A}^\ell}{\partial \delta^i x} \equiv 0, \qquad (14)$$
   for $0 \leq i \leq p-1$, $p \leq \ell \leq d$, the feedback u is also causal and the system is input-output decoupled and linearized and the algorithm stops.
   If (12) is not satisfied then the system cannot be input-output decoupled and linearized by our algorithm which stops.
   b) If $\mathrm{rk}\,(\mathcal{A}^p) = r < m$, then compute the m-valued vectors $\nu^1(\delta x(t), t), \ldots, \nu^{m-r}(\delta x(t), t)$, that always exist, such that
   $$\delta^p \nu^j \in \ker(\mathcal{A}^p), \quad 1 \leq j \leq m - r.$$
   If all ν^j can be chosen such that none of them contains any advances, then define the invertible $(m \times m)$-matrix
   $$T = \begin{pmatrix} D & \nu^1(\delta x(t), t) & \ldots & \nu^{m-r}(\delta x(t), t) \end{pmatrix}, \quad (15)$$
   where $D = (D_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq r$, and its construction is explained below. Since $\mathrm{rk}\,\mathcal{A}^p = r$, the matrix $\mathcal{A}^p$ contains r independent columns indexed $C^p_{j_s}$, $1 \leq s \leq r$. We put $D_{ij} = 1$, if $j = j_s$ and $i = s$, for $1 \leq s \leq r$, and $D_{ij} = 0$, in all other cases. If for all possible choices of vectors ν^j, $1 \leq j \leq m-r$, at least one of them always contains advances, then the transformation is not causal and the system cannot be input-output decoupled and linearized by our algorithm which stops.
5) Define the block-matrix
   $$R = \begin{pmatrix} \delta^1 \cdot I_r & 0 \\ 0 & I_{m-r} \end{pmatrix},$$
   where I_n denotes the $(n \times n)$-identity matrix (with $n = r$ and $n = m - r$, resp.). Introduce the following feedback transformation $u = T(R\tilde{u})$, that transforms the polynomial $\mathcal{A}(\delta]$ into:
   $$\tilde{\mathcal{A}}(\delta] = \mathcal{A}(\delta]TR = \mathcal{A}^p(\delta^p T)R\delta^p + \cdots + \mathcal{A}^d(\delta^d T)R\delta^d. \qquad (16)$$
   Rewrite (16) as
   $$\tilde{\mathcal{A}}(\delta] = \tilde{\mathcal{A}}^{p+1}\delta^{p+1} + \cdots + \tilde{\mathcal{A}}^{d+1}\delta^{d+1}, \text{ with}$$
   $$\tilde{\mathcal{A}}^{\ell+1} = \left(\mathcal{A}^\ell(\delta^\ell T)\right)\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} + \left(\mathcal{A}^{\ell+1}(\delta^{\ell+1} T)\right)\begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix},$$
   for $p \leq \ell \leq d$, and where $\mathcal{A}^{d+1}$ is the zero matrix.
6) Increase the number of iterations: $iter = iter + 1$. Set $\mathcal{A}(\delta] = \tilde{\mathcal{A}}(\delta]$, $p = p + 1$, $d = d + 1$, and $u = \tilde{u}$.
   a) If $iter > iter_{max}$, then the system cannot be decoupled and linearized with those choices of transformations T and our algorithm stops.
   b) If $iter \leq iter_{max}$, then return to step 4).

$$\mathcal{A}^p = \begin{pmatrix} a_{11}^p & \dots & a_{1m}^p \\ \vdots & & \vdots \\ a_{m1}^p & \dots & a_{mm}^p \end{pmatrix}$$
$$\downarrow T$$
$$\mathcal{A}^p(\delta^p T) = \left(\begin{array}{ccc|ccc} a_{1i_1}^p & \dots & a_{1i_r}^p & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1}^p & \dots & a_{mm}^p & 0 & \dots & 0 \end{array}\right)$$
$$\downarrow R$$
$$\tilde{\mathcal{A}}^{p+1} = \left(\begin{array}{ccc|ccc} a_{1i_1}^p & \dots & a_{1i_r}^p & \tilde{a}_{1r+1}^{p+1} & \dots & \tilde{a}_{1m}^{p+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1}^p & \dots & a_{mm}^p & \tilde{a}_{mr+1}^{p+1} & \dots & \tilde{a}_{mm}^{p+1} \end{array}\right)$$

Fig. 2. Passage from $\mathcal{A}^p$ to $\tilde{\mathcal{A}}^{p+1}$.

3.2 Discussion

Remark 5. (*Role of T*). If $\mathrm{rk}\,(\mathcal{A}^p) = r < m$, the transformation T changes the polynomial $\mathcal{A}(\delta]$ such that, after its application, the first nonzero matrix $\mathcal{A}^p$ has its $m-r$ last columns identically zero. Notice that it is not T that acts on $\mathcal{A}^p$, but $\delta^p T$ (hence the construction of T with its $m-r$ last columns spanning, after the application of a delay of order p, the kernel of $\mathcal{A}^p$). Since $\mathcal{A}^p$ is not identically zero and $\mathrm{rk}\,(\mathcal{A}^p) = r < m$, the dimension of the kernel is $m-r$ and the choice of its generators (the vectors $\delta^p \nu^j$'s, $1 \le j \le m-r$) is not unique (and, obviously, T is not unique either). An important question is how to choose them. The vectors ν^j have to be chosen such that T is causal, that is, none of them may contain any advances (they depend on δx only). Such a (causal) choice may not be possible, and, in that case the system cannot be input-output decoupled and linearized by our algorithm.

Furthermore, D is also far from being unique. In fact, any transformation of form (15) with the components D_{ij}, $1 \le i \le m$, $1 \le j \le r$, not necessarily constant, as assumed in step 4.b), but being any functions of $(\delta(x), t)$ such that T is invertible, would give a new matrix $\mathcal{A}^p$ with its $m-r$ last columns being identically zero. Here, we take the simplest possible choice: D is actually the identity $(r \times r)$-matrix to which we inserted $m-r$ identically zero lines such that, after applying T, the r independent columns of $\mathcal{A}^p$ are preserved (see Fig. 2).

Remark 6. (*Role of R*). The transformation R simply introduces an artificial delay in the r first controls (after the application of T, they correspond to the inputs that appear "too early" into the system), while the remaining inputs are unchanged. The first nonzero coefficient of the new δ-polynomial $\tilde{\mathcal{A}}(\delta]$ is no longer the coefficient of order p, but that of order $p+1$ (the minimal degree of the new polynomial increases from p to $p+1$) and the new corresponding matrix $\tilde{\mathcal{A}}^{p+1}$, inherits its r first (independent) columns from $\mathcal{A}^p$ and its $m-r$ last columns from $\mathcal{A}^{p+1}(\delta^{p+1}T)$, see Fig. 2 (where $\tilde{a}_{ij}^{p+1}$ denotes the (ij)-component of the transformed matrix $\mathcal{A}^{p+1}(\delta^{p+1}T)$). By introducing artificial delays, also the maximal degree of the δ-polynomial increases from d to $d+1$, the new $\tilde{\mathcal{A}}^{d+1}$ being given by:

$$\tilde{\mathcal{A}}^{d+1} = \left(\mathcal{A}^d(\delta^d T)\right)\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc|ccc} a_{1i_1}^d & \dots & a_{1i_r}^d & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1}^d & \dots & a_{mm}^d & 0 & \dots & 0 \end{array}\right). \tag{17}$$

Remark 7. (*From an iteration to the next one*). In order to simplify the understanding, suppose that we have just completed the first iteration $iter = 1$. At the second one, step 4) is applied on the new polynomial $\tilde{\mathcal{A}}$ whose new first non zero matrix is given by $\tilde{\mathcal{A}}^{p+1}$, see Fig. 2. By construction, the r first columns of $\tilde{\mathcal{A}}^{p+1}$ are independent, so the rank $\tilde{r}$ of $\tilde{\mathcal{A}}^{p+1}$ (we use the tilde symbol for the objects related to $\tilde{\mathcal{A}}$ and for the transformations introduced at the second iteration) is greater than or equal to r (which is the rank of the original $\mathcal{A}^p$ and was computed at the first iteration). It follows that the new transformation $\widetilde{T}$ is of the following form $\widetilde{T} = (\tilde{D}\ \tilde{\nu}^1 \dots \tilde{\nu}^{m-\tilde{r}})$, with $\tilde{D}$ an $(m\times\tilde{r})$-matrix whose first $(r\times r)$-block is simply the identity matrix (that preserves the r first columns of $\tilde{\mathcal{A}}^{p+1}$) and whose remaining components are as in step 4.b) and preserve the $\tilde{r}-r$ independent columns among the last $m-r$ last columns of $\tilde{\mathcal{A}}^{p+1}$. An important observation is that, after the application of the new transformation $\widetilde{T}$, the zero columns of the matrix $\tilde{\mathcal{A}}^{d+1}$, see (17), are not preserved: each zero column is replaced by a linear combination of the r-first columns of (17), with coefficients equal to the components of the vectors $\tilde{\nu}^j$'s. So, after the application of the transformation

$$\tilde{R} = \begin{pmatrix} \delta^1 \cdot I_{\tilde{r}} & 0 \\ 0 & I_{m-\tilde{r}} \end{pmatrix},$$

the second to last matrix of the new δ-polynomial will not inherit $m-\tilde{r}$ identically zero-columns, but $m-\tilde{r}$ linear combination of the r-first columns of (17) (and its rank may increase by this last transformation). This is one of the reasons for which we are not able to give a upper bound for the maximum number of iterations, see Remark 8 below. Finally, notice that the last matrix of the new δ-polynomial (which is now associated to the delay of order $d+2$) has exactly form (17).

Remark 8. (*Maximum number of iterations*). At first sight, it seems that the maximum number of iterations cannot exceed $d-p+1$. Indeed, from Fig. 2 and relation (17), we have the feeling that, after $d-p+1$ iterations, the first nonzero matrix inherits only identically zero columns and thus its rank cannot grow anymore, but, as explained in Remark 7, this is not what really happens. A natural question arises: how to define the endpoint criterion? In the proposed algorithm, the maximum number of iterations is fixed by the user and, since it is closely related to the maximal delay order involved in the system, it should be chosen accordingly. It is however important to have an upper bound for that number which is one of the main difficulties that remains to be solved (for now, we are only able to say that the upper bound should be greater than $d-p+1$, where p and d are, resp., the minimal and maximal degree of the original $\mathcal{A}(\delta]$, given by (8)-(9)).

Remark 9. (*Choice of T*). The choice of T is important, because the parameters involved in T play an essential role for obtaining a bounded and causal control and they have to be chosen such that condition (12) is satisfied for the (new) δ-polynomial whose first nonzero matrix is invertible (that new δ-polynomial, if it exists, is obtained at step 5) after a certain number of iterations). Indeed, at each iteration, step 4), and in particular checking Lemma 1, is applied on the new matrix $\tilde{\mathcal{A}}^{p+1}$ that depends on the previous one $\mathcal{A}^p$, but also on the transformation T applied before (in fact, $\tilde{\mathcal{A}}^{p+1}$ depends on all transformations T of all previous steps). It is thus clear that the desired properties of the control depends on the choices of T (or equivalently, on the choices of D and vectors ν^j). Here, we simply take the simplest form for D and any vectors ν^j. It will be interesting to investigate which are the best choices (that is, those for which (12) is satisfied) and if they can be identified. Unfortunately, there is no an algorithmic way to do it because the family of transformations is parameterized by functional parameters.

Remark 10. (*Sufficiency*). We propose sufficient conditions depending on certain choices made at each iteration, and if we are not able to decouple and linearize the system (via a bounded and causal feedback) with those choices, this does not necessarily mean that the system cannot be decoupled and linearized.

4. EXAMPLE

Consider the following example

$$\begin{aligned} \dot{x}_1 &= \cos(\delta^1 x_1)u_1 - \sin(\delta^1 x_1)u_2 \\ \dot{x}_2 &= \sin(\delta^1 x_1)u_1 + \cos(\delta^1 x_1)u_2 \\ \dot{x}_3 &= \delta^1 x_2 \\ \dot{x}_4 &= a(\delta x(t), t) + u_3 \end{aligned} \tag{18}$$

defined on $X =]-\frac{\pi}{2}, \frac{\pi}{2}[\times\mathbb{R}^3$, with $h_i = x_i$, $1 \le i \le 3$, initial conditions $x(\mathsf{t}) = \zeta(\mathsf{t})$ and $u(\mathsf{t}) = \psi(\mathsf{t})$, for $\mathsf{t} \in [-\bar{\theta}, 0]$, and $a(\delta x(t), t)$ a nonlinear smooth function such that $\frac{\partial a}{\partial x} \equiv 0$. The relative degrees are $(\rho_1, \rho_2, \rho_3) = (1, 1, 2)$. Step 1)-b) leads to the δ-polynomial $\mathcal{A}[\delta] = \mathcal{A}^0\delta^0 + \mathcal{A}^1\delta^1$, with

$$\mathcal{A}^0 = \begin{pmatrix} \cos(\delta^1 x_1) & -\sin(\delta^1 x_1) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{A}^1 = \dot{\tau}_1 \begin{pmatrix} 0 & 0 & 0 \\ \sin(\delta^2 x_1) & \cos(\delta^2 x_1) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{rk}\,\mathcal{A}^0 = 2 < 3$ (and is constant and equal to 2 for all $t \ge \tau^{-1}(0)$) the first and the third columns of $\mathcal{A}^0$ are independent and $\nu^1 = (\sin(\delta^1 x_1)\ \cos(\delta^1 x_1)\ 0)^T \in \ker \mathcal{A}^0$, so following step 4)-b), see (15), T is given by the first matrix of (19) below and is invertible on X. We now apply step 5) and introduce the transformation $u = T(R\tilde{u})$, with

$$T = \begin{pmatrix} 1 & 0 & \sin(\delta^1 x_1) \\ 0 & 0 & \cos(\delta^1 x_1) \\ 0 & 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \delta^1 & 0 & 0 \\ 0 & \delta^1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

The δ-polynomial $\mathcal{A}[\delta]$ becomes

$$\tilde{\mathcal{A}}[\delta] = \tilde{\mathcal{A}}^1\delta^1 + \tilde{\mathcal{A}}^2\delta^2, \quad \text{with} \tag{20}$$

$$\tilde{\mathcal{A}}^1 = \begin{pmatrix} \cos(\delta^1 x_1) & 0 & 0 \\ 0 & 0 & \dot{\tau}_1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{\mathcal{A}}^2 = \begin{pmatrix} 0 & 0 & 0 \\ \dot{\tau}_1 \sin(\delta^2 x_1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the new $\tilde{\mathcal{A}}(\delta]$, the minimal degree is $d = 1$, the associated matrix $\tilde{\mathcal{A}}^p = \tilde{\mathcal{A}}^1$ is everywhere invertible on X, and it is thus possible to find, on X, a feedback $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$, which satisfies the following equation:

$$\tilde{\mathcal{A}}^1\delta^1\tilde{u} = -\tilde{\mathcal{A}}^2\delta^2\tilde{u} - \alpha(\delta x(t), t) + \delta^1 v, \quad t \ge \tau_1^{-1}(0), \tag{21}$$

where $\alpha(\delta x(t), t) = (0\ 0\ a(\delta x(t), t)^T$. Condition (14) is satisfied (since $\frac{\partial a}{\partial x} \equiv 0$) and it follows that $\tilde{u}$ is causal. In order to conclude that the system is, indeed, input-output decoupled (via a bounded and causal transformation, see Definition 3), $\tilde{\mathcal{A}}(\delta]$ should also verify Lemma 1. Condition (12) translates into $|\sin(\delta^2 x_1)| < \frac{1}{2}$, thus a bounded feedback can be constructed for initial conditions $x_1 = \zeta_1$ belonging to $]-\frac{\pi}{6}, \frac{\pi}{6}[$ and sufficiently small delay functions.

As explained, the transformation T is not unique and some choices (maybe more complicated than those proposed by our algorithm) may lead to a simpler equation for $\tilde{u}$. For instance, consider T given by the first matrix of (22) below, which is invertible on X. By applying $u = T(R\tilde{u})$, where

$$T = \begin{pmatrix} \cos(\delta^1 x_1) & \sin(\delta^1 x_1) & 0 \\ -\sin(\delta^1 x_1) & \cos(\delta^1 x_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ R = \begin{pmatrix} \delta^1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^1 \end{pmatrix},$$

we obtain the following δ-polynomial: (22)

$$\tilde{\mathcal{A}}[\delta] = \tilde{\mathcal{A}}^1\delta^1, \quad \text{with } \tilde{\mathcal{A}}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dot{\tau}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

which is clearly simpler than (20) (which was given by the algorithm). Now, the control $\tilde{u}$ has to be constructed from:

$$\tilde{\mathcal{A}}^1\delta^1\tilde{u} = -\alpha(\delta x(t), t) + \delta^1 v, \quad t \ge \tau_1^{-1}(0),$$

and, in this case, $\tilde{u}$ is always bounded and causal (and, contrary to choice (19) of T, proposed by the algorithm, we do not need to verify Lemma 1).

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