Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Autocratic strategies for infinitely iterated multiplayer social dilemma games ⋆

E. Martirosyan * **A. Govaert** * **M. Cao** *

* *ENTEG, Faculty of Science and Engineering, University of Groningen, The Netherlands (e-mail: {e.n.martirosyan, a.govaert, m.cao}@rug.nl)*

Abstract: In this work, we present results on autocratic strategies in infinitely repeated multiplayer games. Extending the previously developed theory for two-player games, we formulate necessary conditions for the existence of autocratic strategies in a standard multiplayer social dilemma game, namely the public goods game. The infinitely repeated game is designed with a discount factor that reduces the values of the future payoffs. The contribution of this work is an adaptation of existing theory on autocratic strategies to multiplayer games with arbitrary action spaces. We first show the existence of an autocratic strategy that uses a finite set of points from a continuous action space. Then, using a strategy concentrated on two points of the continuous interval representing the autocrat's available actions, we show the necessary conditions for the existence of autocratic strategies in the context of the public goods game.

Keywords: Social dilemmas, autocratic strategies, repeated games, multiplayer games.

1. INTRODUCTION

Social dilemma games model interactions among players where each one decides whether to sacrifice her own interests in favor of the greater good. The emergence of such behaviors in social sciences (Webster and Sell (2014)) motivates researchers to explore dilemmas and search for mechanisms that enable players to follow a particular behavior. For example, mechanisms that encourage cooperation have been developed for social dilemma games (Nowak (2007), Sigmund (2010), Nowak (2013), Hauser et al. (2014)). One proposed mechanism to establish cooperation, using reciprocity of the cooperation behavior, was described in Henrich et al. (2001), Nowak and Sigmund (2005). However, for large populations, these mechanisms fail due to the difficulty of tracking one's opponents, or even to influence them (Hauert and Schuster (1997), Suzuki and Akiyama (2007)). Another example of a strategy used for sustaining cooperation in iterated games is the tit-for-tat strategy (Axelrod and Axelrod (1984)), where a player copies the action that the opponent adopted in the previous round.

The Iterated Prisoner's Dilemma (IPD) is a classic example of social dilemma games that has a long history of being exploited for describing a variety of phenomena appearing in different fields (Chong et al. (2007)). In 2012, a new class of strategy, called zero-determinant strategies (Press and Dyson (2012)), was presented in the context of IPD. That strategy is of an autocratic type since it allows a player to unilaterally enforce a linear relationship between her own payoff and the co-players' payoff. Loosely speaking, the zero-determinant strategy is a sort of ultimatum strategy that gives a player a unilateral control over the distribution of the payoffs. Of particular interest is a strategy that allows a focal player to set a generous share of payoff to the co-players. Another particular case is the equalizer strategy with which a focal player sets her own payoff within a feasible range. For other means of exerting control on games, we refer the reader to surveys such as Marden and Shamma (2015), Riehl et al. (2018).

Naturally, the discovery of zero-determinant strategies provoked much research on its properties and possible applications, mostly for infinitely repeated games with discount factors. Hilbe et al. (2013) studied zero-determinant strategies in the context of evolutionary game theory. Ichinose and Masuda (2017) investigated the existence of zero-determinant strategies in the two-player case. Hilbe et al. (2015) and Hilbe et al. (2014) made an essential step by extending existing results for multiplayer games. Govaert and Cao (2019) analyzed zero-determinant strategies in infinitely repeated multiplayer games in the context of social dilemma games. All previously mentioned works on zero-determinant strategies considered games with finite action spaces. Thus, the work of McAvoy and Hauert (2015) is particularly interesting because, inspired by the zero-determinant strategy, it introduced an autocratic strategy for iterated games with arbitrary action spaces.

In our work, we extend the result in McAvoy and Hauert (2015) for multiplayer games and apply it to a social dilemma game, namely public goods game. Using the general theorem on the existence of autocratic strategies (McAvoy and Hauert (2015)), we develop a corollary that provides conditions on the existence of autocratic strategies concentrated at n points of the continuous interval representing the autocrat's space of available actions. Exploiting the case for $n = 2$, i.e., a two-point autocratic strategy, we derive results that describe the

⋆ The work was supported in part by by the European Research Council (ERC-CoG-771687) and the Netherlands Organization for Scientific Research (NWO-vidi-14134).

existence and the limitations of special types of autocratic strategies in the context of the public goods game.

The public goods game, as a social dilemma, also belongs to another important type of games - aggregative games, where the payoff of a player is affected by some aggregate behavior of her opponents. Aggregative games are widely used for control of a behavior of self-interested but coupled agents (Chen et al. (2014), Barrera and Garcia (2014), Ye and Hu (2015)). Hence, the developed theory can be used in the context of control problems by using an autocratic strategist as a controller for achieving a desirable behavior.

The paper is organized as follows. Section 2 sets the problem, describes the sequence of the game, and provides formal definitions of memory-one strategies and social dilemma games. In Section 3, we provide the extended theorem on autocratic strategies in repeated multiplayer games and the corollary that states conditions on n-point strategies. In Section 4, we provide results on the existence of autocratic strategies in the public goods game. Section 5 concludes the work, summarizing the results as well as describing future plans.

2. PROBLEM FORMULATION

2.1 Infinitely repeated N-player game

In this paper, we consider N-player non-cooperative games in which each player repeatedly chooses an action from an arbitrary (i.e., discrete or continuous) space of available actions, basing his choice on the outcomes of previous rounds. For each player $i \in \{1, \dots, N\}$, the space of available actions is a measurable space S_i (equipped with a σ-algebra, hence $(S_i, \mathcal{F}(S_i))$, in the following referred simply by S_i).

After T rounds, a history h^T is generated as a sequence of action profiles played by players in each round, i.e.

$$h^T := \left(h_0^T, \dots, h_{T-1}^T\right), \tag{1}$$

where $h^T \in \mathfrak{H}^T := \prod_{t=0}^{T-1} S_1 \times \cdots \times S_N$, with $\mathfrak{H}^T$ being the set of all possible histories that can be generated before round T, and the vector $h_t^T = \left(x_1^{(t)}, \dots, x_N^{(t)}\right) \in S_1 \times \cdots \times S_N$ describes the particular action profile generated by the N players at round t.

We consider a game with a discount factor $\lambda \in (0, 1)$. Thus, a single round payoff to player i at moment t is given by $\lambda^t u_i\left(x_i^{(t)}, x_{-i}^{(t)}\right)$, where $x_{-i}^{(t)} \in S_{-i} = \prod_{j \neq i} S_j$ denotes an action profile chosen by the opponents at round t. The normalized cumulative payoff to player i after $T+1$ interactions is given by

$$\pi_i^T = \frac{1-\lambda}{1-\lambda^{T+1}} \sum_{t=0}^{T} \lambda^t u_i\left(x_i^{(t)}, x_{-i}^{(t)}\right). \tag{2}$$

From this, we easily extrapolate the normalized cumulative payoff to player i in the infinitely repeated game as

$$\pi_i = (1-\lambda) \sum_{t=0}^{\infty} \lambda^t u_i\left(x_i^{(t)}, x_{-i}^{(t)}\right). \tag{3}$$

Following McAvoy and Hauert (2015), we define a strategy for every player i as a map

$$\sigma_i \colon \mathfrak{H} \to \Delta(S_i), \tag{4}$$

where $\mathfrak{H} := \bigsqcup_{T \geq 0} \mathfrak{H}^T$ (with $\mathfrak{H}^0 := \emptyset$) being the set of all possible histories and $\Delta(S_i)$ being the space of probability measures, i.e., the simplex, on S_i.
In this work, we focus on an important class of strategies, namely *memory-one strategy*, that takes into account only the last played action profile, hence

$$\sigma_i[h^T] = \sigma_i[h_{T-1}^T]. \tag{5}$$

The strategy σ_i is the conditional probability of picking an action by player i from her space of available actions S_i, given an action profile played in the previous round. Thus, for a given action profile (x_i, x_{-i}), $\sigma_i[x_i, x_{-i}](s)$, which denotes the probability that i uses s after (x_i, x_{-i}) is played, satisfies the following

$$\int_{s \in S_i} \sigma_i[x_i, x_{-i}](s)\, ds = 1. \tag{6}$$

The initial strategy of player i, used in the first round, is denoted as

$$\sigma_i[\emptyset](s) = \sigma_i^0(s). \tag{7}$$

Remark 1. In Press and Dyson (2012), it was shown that a longer memory does not give an advantage if a fixed game (same allowed moves and same payoffs at every iteration) is indefinitely repeated. Thus, all the results presented in this paper, can be extended for the case of players with longer memory strategies.

2.2 Social Dilemma

In this work, we investigate the existence of autocratic strategies in the public goods game that belongs to the class of social dilemma games. Below, we provide a few standing assumptions on N-player social dilemma games with continuous action space represented by the unit interval, analogous to those made in Hilbe et al. (2014) but extended for the case of continuous action spaces.

Assumption 2. For establishing the social dilemma, we assume

(1) irrespective of the own strategy, players prefer the other group members to cooperate, i.e., for $x_{-i} = (x_j)_{j \neq i}$, $x'_{-i} = (x'_j)_{j \neq i}$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} x'_j$, then the following is satisfied
$$u_i(x_i, x_{-i}) > u_i(x_i, x'_{-i}), \quad \forall i \in \{1, \dots, N\};$$
(2) within any mixed group, defectors obtain strictly higher payoffs than cooperators, i.e., for $x_i > x'_i$, the following must be true
$$u_i(x_i, x_{-i}) < u_i(x'_i, x_{-i}), \quad \forall i \in \{1, \dots, N\};$$
(3) mutual cooperation is favored over mutual defection, i.e., for $(x_i, x_{-i}) = (1, \dots, 1)$ and $(x'_i, x'_{-i}) = (0, \dots, 0)$, it holds that
$$u_i(x_i, x_{-i}) > u_i(x'_i, x'_{-i}), \quad \forall i \in \{1, \dots, N\}.$$

Next, we define the public goods game.

Example 3. (*Public Goods Game*) Each of N players makes a decision - whether or not to contribute into a public pot. Every player $i = \{1, \dots, N\}$ contributes $x_i \in [0, 1]$. Her payoff function is given by

$$u_i(x_i, x_{-i}) := \frac{rc \sum_{k=1}^{N} x_k}{N} - cx_i. \tag{8}$$

In the public goods game, Assumption 2 leads to the following conditions on r and c

$$1 < r < N \quad \text{and} \quad c > 0. \tag{9}$$

In the following section, we explain what an autocratic strategy is.

3. AUTOCRATIC STRATEGIES

In the remainder of the paper, we denote an action profile as $\mathbf{x} = (x_i, x_{-i})$. Below, we introduce the N-player extension of autocratic strategies for discounted games with an arbitrary action space. The original theorem was developed in McAvoy and Hauert (2015) for the two players case. The undiscounted case can be recovered by setting $\lambda = 1$.

Theorem 4. Suppose that $\sigma_i[\mathbf{x}]$ is a memory-one strategy for player i and let σ_i^0 be player i's initial strategy. If, for some bounded function $\psi(\cdot)$ and fixed $\alpha, \gamma \in \mathbb{R}$, $\boldsymbol{\beta} = \{\beta_j \in \mathbb{R} | j \neq i\}$, the equation

$$\alpha u_i(\mathbf{x}) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}) + \gamma = \psi(x_i) \\ -\lambda \int_{s\in S_i} \psi(s)\, d\sigma_i[\mathbf{x}](s) - (1-\lambda) \int_{s\in S_i} \psi(s)\, d\sigma_i^0(s)\,, \tag{10}$$

holds for all $\mathbf{x} \in S_i \times S_{-i}$, then (σ_i, σ_i^0) enforce the linear payoff relationship

$$\alpha\pi_i + \sum_{j\neq i} \beta_j \pi_j + \gamma = 0 \tag{11}$$

for any strategies of i's $N-1$ opponents.

Here we give the sketch of the proof. Extending the game to N-player case, we prove theorem by following the steps of the proof for the 2-player case (see supplementary material for McAvoy and Hauert (2015)). The proof is a direct result of the generalization of Akin's Lemma (Akin (2015)). The lemma, extended to N-player case, relates strategies of N players, and the sequence of play to the initial action of the autocratic strategist.

In fact, the theorem is difficult to apply because the integral, $\int_{s\in S_i} \psi(s)\, d\sigma_i[\mathbf{x}](s)$, in general cannot be solved explicitly. As a result, one is unable to directly find all possible pairs (σ_i, σ_i^0). However, it is possible to show that under specific conditions, an autocratic strategist can use (σ_i, σ_i^0) concentrated on a finite set of points of S_i and unilaterally enforce (11). In other words, an autocratic strategist can enforce the linear payoff relationship (11) by using a finite number of actions from the space of available actions while all the opponents are employing continuous action spaces.

Corollary 5. Let $\alpha, \gamma \in \mathbb{R}$, $\boldsymbol{\beta} = \{\beta_j \in \mathbb{R} | j \neq i\}$, $\{s_k\}_{k=1}^n \in S_i$, $x_{-i} \in S_{-i}$, and there exists a bounded function $\psi(s_k)$ defined for all $k \in \{1, \ldots, n\}$. Suppose there exists a pair (σ_i, σ_i^0) such that

$$\sigma_i[\mathbf{x}_m](s_k) := p_k(\mathbf{x}_m) \geq 0\,, \quad \sum_{q=1}^{n} p_q(\mathbf{x}_m) = 1\,, \\ \sigma_i^0(s_k) := p_k^0 \geq 0\,, \quad \sum_{q=1}^{n} p_q^0 = 1\,, \tag{12}$$

where $\mathbf{x}_m = (s_m, x_{-i}) \in S_i \times S_{-i}$, satisfying for all $m \in \{1, \ldots, n\}$

$$\alpha u_i(\mathbf{x_m}) + \sum_{j\neq i} \beta_j u_j(\mathbf{x_m}) + \gamma + (\psi(s_n) - \psi(s_m)) \\ -(1-\lambda)\Bigg(\sum_{k\neq n} (\psi(s_n) - \psi(s_k)) p_k^0\Bigg) \\ = \lambda\Bigg(\sum_{k\neq n} (\psi(s_n) - \psi(s_k)) p_k(\mathbf{x}_m)\Bigg)\,, \tag{13}$$

Then, this pair (σ_i, σ_i^0) of the autocratic strategist i enforces

$$\alpha\pi_i + \sum_{j\neq i} \beta_j \pi_j + \gamma = 0\,, \tag{14}$$

for any strategies of i's $N-1$ opponents.

The proof can be found in Appendix A.

Note that we design player i strategy such that for all $\bar{x}_i \in S_i \setminus \{s_k\}_{k=1}^n$, the probability of being played is zero, i.e.,

$$\sigma_i[\mathbf{x}_m](\bar{x}_i) = 0\,, \quad \sigma_i^0(\bar{x}_i) = 0\,. \tag{15}$$

where $\mathbf{x}_m = (s_m, x_{-i})$ with $m \in \{1, \ldots, n\}$.

Now, we define an autocratic strategy concentrated only in two points of the space of available actions, i.e. we want to define a two-point autocratic strategy specifying the result of Corollary 5 for the $n = 2$ case.

Let $\alpha, \gamma \in \mathbb{R}$, $\boldsymbol{\beta} = \{\beta_j \in \mathbb{R} | j \neq i\}$, $s_1, s_2 \in S_i$. We set $\psi(s_1), \psi(s_2)$ as

$$\psi(s_1) = \psi_1 \in \mathbb{R}\,, \\ \psi(s_2) = \psi_1 + \frac{1}{\phi}\,, \text{ where } \phi \in \mathbb{R}_{>0}\,. \tag{16}$$

Then, according to (12), we set the memory-one strategy of player i as

$$\sigma_i[\mathbf{x}_1](s_1) := p(\mathbf{x}_1)\,, \\ \sigma_i[\mathbf{x}_1](s_2) := 1 - p(\mathbf{x}_1)\,, \\ \sigma_i[\mathbf{x}_2](s_1) := p(\mathbf{x}_2)\,, \\ \sigma_i[\mathbf{x}_2](s_2) := 1 - p(\mathbf{x}_2)\,. \tag{17}$$

and the initial strategy as

$$\sigma_i^0(s_1) := p_0\,, \\ \sigma_i^0(s_2) := 1 - p_0\,, \tag{18}$$

Then, (13) gives

$$p(\mathbf{x}_1) = \frac{1}{\lambda}(\phi(\alpha u_i(\mathbf{x}_1) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_1) + \gamma) \\ -(1-\lambda)p_0 + 1)\,, \\ p(\mathbf{x}_2) = \frac{1}{\lambda}(\phi(\alpha u_i(\mathbf{x}_2) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_2) + \gamma) \\ -(1-\lambda)p_0)\,, \tag{19}$$

and the constraints in (12) imply

$$(1-\lambda)p_0 - 1 \leq \phi(\alpha u_i(\mathbf{x}_1) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_1) + \gamma) \\ \leq (1-\lambda)(p_0 - 1)\,, \\ (1-\lambda)p_0 \leq \phi(\alpha u_i(\mathbf{x}_2) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_2) + \gamma) \leq \lambda \\ \leq \lambda + (1-\lambda)p_0\,. \tag{20}$$

Hence, (17) and (18) describe (σ_i, σ_i^0) pair of the autocratic strategist i that enforces

$$\alpha\pi_i + \sum_{j\neq i} \beta_j \pi_j + \gamma = 0\,, \tag{21}$$

for any strategies of i's $N-1$ opponents.

The two-point strategy (17), (18) provides a tool for studying the existence of autocratic strategies in multi-player games. We use it for investigating the public goods game described in Example 3. Further, considering that $x_i \in [0,1]$, we set $s_1 = 1$ and $s_2 = 0$ for an autocratic strategist.

4. AUTOCRATIC STRATEGIES IN THE PUBLIC GOODS GAME

4.1 Existence of the autocratic strategies

Firstly, we re-parameterize the current $(\alpha, \boldsymbol{\beta}, \gamma)$ setting as done in Hilbe et al. (2014), obtaining

$$\begin{aligned} \rho &= -\sum_{k\neq i}\beta_k\,, \quad \chi = \frac{\alpha}{\rho}\,, \\ \omega_{j\neq i} &= -\frac{\beta_j}{\rho}\,, \quad l = -\frac{\gamma}{\alpha-\rho}\,. \end{aligned} \tag{22}$$

Then, the enforceable payoff relationship takes the following form

$$\pi_{-i} = \chi\pi_i + (1-\chi)l\,, \tag{23}$$

where $\pi_{-i} = \sum_{j\neq i}\omega_j\pi_j$, χ, $l \in \mathbb{R}$, $\boldsymbol{\omega} = \{\omega_j \in \mathbb{R} | j \neq i\}$. While the probabilities (19) in the autocratic memory-one strategy (17) become

$$\begin{aligned} p(\mathbf{x}_1) &= \frac{1}{\lambda}(\phi\rho(\chi u_i(\mathbf{x}_1) - \sum_{j\neq i}\omega_j u_j(\mathbf{x}_1) + (1-\chi)l) \\ &\qquad -(1-\lambda)p_0 + 1)\,, \\ p(\mathbf{x}_2) &= \frac{1}{\lambda}(\phi\rho(\chi u_i(\mathbf{x}_2) - \sum_{j\neq i}\omega_j u_j(\mathbf{x}_2) + (1-\chi)l) \\ &\qquad -(1-\lambda)p_0)\,. \end{aligned} \tag{24}$$

In the case of public goods game (Example 3), (24) are

$$\begin{aligned} p(\mathbf{x}_1) &= \frac{1}{\lambda}\left(\phi\rho\left(\chi\frac{rc\left(\sum_{j\neq i}x_j+1\right)}{N} - c\right) - \sum_{j\neq i}\omega_j\cdot\right. \\ &\left.\left(\frac{rc\left(\sum_{k\neq i}x_j+1\right)}{N} - cx_j\right) + (1-\chi)l\right) - (1-\lambda)p_0 + 1\Bigg), \\ p(\mathbf{x}_2) &= \frac{1}{\lambda}\left(\phi\rho\left(\chi\frac{rc\sum_{j\neq i}x_j}{N} - \sum_{j\neq i}\omega_j\cdot\right.\right. \\ &\left.\left(\frac{rc\sum_{k\neq i}x_j}{N} - cx_j\right) + (1-\chi)l\right) - (1-\lambda)p_0 + 1\Bigg)\,. \end{aligned} \tag{25}$$

Below, we provide a criterion for considering the payoff relation (23) to be enforceable. It is based on the fact that mapping $\sigma_i[\mathbf{x}]$ should belong to $\Delta(S_i)$, i.e. the probabilities $p(s_1, x_{-i})$, $p(s_2, x_{-i})$ are both in the unit interval.

Definition 6. The linear payoff relationship (23) with $(\chi, \boldsymbol{\omega}, l)$ for the infinitely repeated N-players public goods game with the discount factor $\lambda \in (0,1)$ is said to be *enforceable* if there exists $\sigma_i^0 \to \Delta(S_i)$ such that $\sigma_i(\mathbf{x})$ is in $\Delta(S_i)$.

Applying this definition to the public goods game, we get the following proposition that states necessary conditions on $\rho, \chi, \boldsymbol{\omega}, l$.

Proposition 7. The existence of the enforceable payoff relation $(\chi, \boldsymbol{\omega}, l)$ for the infinitely repeated N-player public goods game, satisfying Assumption 2, with continuous action space and discount factor $\lambda \in (0,1)$, requires the following necessary conditions

$$\begin{aligned} -\frac{1}{N-1} \leq -\min\boldsymbol{\omega} < \chi < 1\,, \\ \rho > 0\,, \\ 0 \leq l \leq rc - c\,. \end{aligned} \tag{26}$$

The proof can be found in Appendix B.

Note that the lower and upper bounds on l are the payoffs to a single player if $x_k = 0$ and $x_k = 1$ for all $k \in \{1, \ldots, N\}$, respectively.

We study conditions on the existence of the four most studied autocratic strategies that are formulated in Table 1. The proofs of the propositions presented below result from conditions (12) and are avoided in the paper.

Table 1. The mostly studied autocratic strategies

Autocratic strategy	Parameters value
Fair	$\chi = 1$
Generous	$l = rc - c$ and $0 < \chi < 1$
Extortionate	$l = 0$ and $0 < \chi < 1$
Equalizer	$\chi = 0$

Proposition 8. In the N-player repeated public goods game, satisfying Assumption 2, with the continuous action space and discount factor $0 < \lambda < 1$, the existence of

(1) the fair strategy is not possible;
(2) the generous strategy requires necessary condition $p_0 = 1$;
(3) the extortionate strategy requires necessary condition $p_0 = 0$.

For the equalizer strategy, we have $\chi = 0$ and, as a result, the payoff relation (23) becomes

$$\pi_{-i} = l\,. \tag{27}$$

Literally, the autocratic strategist i sets the payoff to $N-1$ opponents. The proposition below provides bounds on the value of l that player i can unilaterally enforce. Before we introduce the bounds, we define the indicator function as

$$\mathbb{1}(y) = \begin{cases} 1\,, & y \geq 0\,, \\ 0\,, & y < 0\,. \end{cases} \tag{28}$$

Proposition 9. In the N-player repeated public goods game, satisfying Assumption 2, with continuous action space and discount factor $0 < \lambda < 1$, the equalizer strategy that enforces $(0, \boldsymbol{\omega}, l)$ does not exist for l not satisfying the following condition

$$\begin{aligned} &\max\left(0, c\sum_{j\neq i}\left(\omega_j - \frac{r}{N}\right)\mathbb{1}\left(\omega_j - \frac{r}{N}\right)\right) \leq l \\ &\leq \min\left(rc - c, c\sum_{j\neq i}\left(\frac{r}{N} - \omega_j\right)\mathbb{1}\left(\omega_j - \frac{r}{N}\right) + \frac{rc}{N}\right)\,. \end{aligned} \tag{29}$$

Remark 10. In the case of the equalizer strategy ($\chi = 0$), Proposition 7 sets the following constraint

$$0 < \min\boldsymbol{\omega} \leq \frac{1}{N-1}\,. \tag{30}$$

That means the following - in the case of equalizer strategy, the autocrat cannot exclude any opponent from the payoff relation (23).

5. CONCLUSION

Extending existing results, we have developed a theory on autocratic strategies in multiplayer games. Firstly, we have given the necessary conditions for the existence of the infinite-dimensional autocratic strategy in multiplayer games. Next, we have shown that the autocrat is able to enforce the linear relationship between her payoff and the co-players' payoffs using a strategy concentrated at a finite number of points of the continuous interval. Exploiting the case of the two-point autocratic strategy, we have focused on investigating the existence of the autocratic strategy in general as well as of its special types in the public goods game, that belongs to the class of social dilemma games. We have shown that for the existence of the extortionate strategy, the autocrats should contribute nothing in the first round, while for the existence of the generous one, it is the opposite, i.e., the autocrat contributes the highest possible value. Also, our results have shown that it is impossible to enforce the fair relationship between players' payoffs. Finally, we have derived bounds on the value that the autocrat can assign to its opponents in the equalizer strategy case. For the future work, we firstly want to derive sufficient conditions for the existence of the autocratic strategy in social dilemma games with continuous action spaces as well as applying the developed theory to games with non-linear payoff functions. Another direction is to study the effect of uncertainty on the existence of autocratic strategies, e.g., the uncertainty, described by a distribution function, of the discount factor.

ACKNOWLEDGEMENTS

The authors are grateful to Carlo Cenedese for reading an earlier draft of this manuscript and for giving valuable suggestions on its content.

REFERENCES

Akin, E. (2015). What you gotta know to play good in the iterated prisoner's dilemma. *Games*, 6, 175–190.

Axelrod, R. and Axelrod, R. (1984). *The Evolution of Cooperation*. Basic Books.

Barrera, J. and Garcia, A. (2014). Dynamic incentives for congestion control. *IEEE Transactions on Automatic Control*, 60, 1–1.

Chen, J.Q., Li, Y., Louie, R., and Vucetic, B. (2014). Autonomous demand side management based on energy consumption scheduling and instantaneous load billing: An aggregative game approach. *IEEE Transactions on Smart Grid*, 5, 1744–1754.

Chong, S.Y., Humble, J., Kendall, G., Li, J., and Yao, X. (2007). *The Iterated Prisoner's Dilemma: 20 Years On*, chapter 1, 1–21. World Scientific Publishing Co. Pte. Ltd.

Govaert, A. and Cao, M. (2019). Zero-determinant strategies in finitely repeated n-player games. *IFAC-PapersOnLine*, 52, 150–155.

Hauert, C. and Schuster, H. (1997). Effects of increasing the number of players and memory size in the iterated prisoner's dilemma: A numerical approach. *Proceedings of the Royal Society B: Biological Sciences*, 264, 513–519.

Hauser, O., Peysakhovich, A., and Nowak, M. (2014). Cooperating with the future. *Nature*, 511, 220–3.

Henrich, J., Boyd, R., Bowles, S., Camerer, C., Fehr, E., Gintis, H., and McElreath, R. (2001). Cooperation, reciprocity and punishment in fifteen small-scale societies. *Santa Fe Institute, Working Papers*, 91.

Hilbe, C., Nowak, M.A., and Sigmund, K. (2013). Evolution of extortion in iterated prisoner's dilemma games. *Proceedings of the National Academy of Sciences*, 110(17), 6913–6918.

Hilbe, C., Wu, B., Traulsen, A., and Nowak, M. (2014). Cooperation and control in multiplayer social dilemmas. *Proceedings of the National Academy of Sciences of the United States of America*, 111.

Hilbe, C., Wu, B., Traulsen, A., and Nowak, M. (2015). Evolutionary performance of zero-determinant strategies in multiplayer games. *Journal of theoretical biology*, 341.

Ichinose, G. and Masuda, N. (2017). Zero-determinant strategies in finitely repeated games. *Journal of Theoretical Biology*, 438.

Marden, J. and Shamma, J. (2015). Chapter 16. game theory and distributed control. *Handbook of Game Theory with Economic Applications*, 4, 861–899.

McAvoy, A. and Hauert, C. (2015). Autocratic strategies for iterated games with arbitrary action spaces. *Proceedings of the National Academy of Sciences*, 113.

Nowak, M. (2007). Review five rules for the evolution of cooperation. *Science (New York, N.Y.)*, 314, 1560–3.

Nowak, M. (2013). Human cooperation. *Trends in cognitive sciences*, 17.

Nowak, M. and Sigmund, K. (2005). Evolution of indirect reciprocity. *Nature*, 437, 1291–8.

Press, W. and Dyson, F. (2012). Iterated prisoners dilemma contains strategies that dominate any evolutionary opponent. *Proceedings of the National Academy of Sciences of the United States of America*, 109, 10409–13.

Riehl, J., Ramazi, P., and Cao, M. (2018). A survey on the analysis and control of evolutionary matrix games. *Annual Reviews in Control*, 45.

Sigmund, K. (2010). The calculus of selfishness. *The Calculus of Selfishness*.

Suzuki, S. and Akiyama, E. (2007). Evolution of indirect reciprocity in groups of various sizes and comparison with direct reciprocity. *Journal of theoretical biology*, 245, 539–52.

Webster, M. and Sell, J. (2014). *Laboratory Experiments in the Social Sciences: Second Edition*. Academic Press.

Ye, M. and Hu, G. (2015). Game design and analysis for price-based demand response: An aggregate game approach. *IEEE Transactions on Cybernetics*, 47.

Appendix A. PROOF OF COROLLARY 5

From Theorem 4, for fixed $\alpha, \gamma \in \mathbb{R}$ and $\boldsymbol{\beta} = \{\beta_j \in \mathbb{R} | j \neq i\}$, (10) needs to be satisfied for some bounded function $\psi\colon \{s_1, \ldots, s_n\} \to \mathbb{R}$ and $\mathbf{x}_m \in S_i \times S_{-i}$ for all $m \in \{1, \ldots, n\}$. Thus,

$$\alpha u_i(\mathbf{x}_m) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_m) + \gamma = \psi(s_m) \\ -\lambda \int_{s\in S_i} \psi(s)\, d\sigma_i[\mathbf{x}_m](s) - (1-\lambda)\int_{s\in S_i} \psi(s)\, d\sigma_i^0(s)\,. \quad \text{(A.1)}$$

should hold. Formally, the memory-one strategy, described in (12) and (15), can be defined for all $x_i \in S_i$ and $\mathbf{x} \in S_i \times S_{-i}$ as

$$\sigma_i[\mathbf{x}](x_i) := \sum_{k=1}^{n} p_k(\mathbf{x})\delta_{s_k}(x_i)\,, \\ \sigma_i^0(x_i) := \sum_{k=1}^{n} p_k^0 \delta_{s_k}(x_i)\,, \quad \text{(A.2)}$$

where δ_{s_k} is the Dirac measure on S_i centered at s_k. Because the design of the memory-one strategy (A.2) restricts i's actions to n points and Theorem 4 makes no assumptions on the action space, we may assume that $S_i = \{s_1, \dots, s_n\}$. Substituting (A.2) in (A.1), we get

$$\psi(s_m) - \lambda\Big(\sum_{k\neq n} \psi(s_k)p_k(\mathbf{x}_m) + \psi(s_n)\Big(1 - \sum_{k\neq n} p_k(\mathbf{x}_m)\Big)\Big) \\ -\Big(1-\lambda\Big)\Big(\sum_{k\neq n} \psi(s_k)p_k^0 + \psi(s_n)\Big(1-\sum_{k\neq n} p_k^0\Big)\Big) \\ = \alpha u_i(\mathbf{x}_m) + \sum_{j\neq i} \beta_j u_j(\mathbf{x}_m) + \gamma\,. \quad \text{(A.3)}$$

Rearranging the terms, we get (13). Also, due to the constraints in (12), we ensure that the memory one strategy belongs to the simplex, as defined in (4). This completes the proof. ■

Appendix B. PROOF OF PROPOSITION 7

Firstly, we consider the following two cases

(1) $x_i = s_1$ and $x_{-i} = (1, \dots, 1)$, i.e., all players contribute 1;
(2) $x_i = s_2$ and $x_{-i} = (0, \dots, 0)$, i.e., all players contribute 0.

According to (25), the cases give the following

$$p(\mathbf{x}_1) \\ = \frac{1}{\lambda}(\phi\rho((\chi-1)(rc-c) + (1-\chi)l) - (1-\lambda)p_0 + 1)\,, \quad \text{(B.1)}$$

$$p(\mathbf{x}_2) = \frac{1}{\lambda}(\phi\rho(1-\chi)l - (1-\lambda)p_0)\,. \quad \text{(B.2)}$$

Further, we check under which conditions both (B.1) and (B.2) stay between $[0, 1]$. Thus, we study the following inequalities

$$0 \leq \phi\rho((\chi-1)(rc-c) + (1-\chi)l) - (1-\lambda)p_0 + 1 \leq \lambda, \quad \text{(B.3)}$$

$$0 \leq \phi\rho(1-\chi)l - (1-\lambda)p_0 \leq \lambda\,. \quad \text{(B.4)}$$

After rearranging terms and multiplying (B.3) by -1, we get

$$(1-p_0)(1-\lambda) \leq \phi\rho(1-\chi)(rc-c-l) \leq 1-(1-\lambda)p_0, \quad \text{(B.5)}$$

$$(1-\lambda)p_0 \leq \phi\rho(1-\chi)l \leq (1-\lambda)p_0 + \lambda\,. \quad \text{(B.6)}$$

Combining both inequalities, we get

$$(1-\lambda) \leq \phi\rho(1-\chi)(rc-c) \leq 1+\lambda\,. \quad \text{(B.7)}$$

Considering that $0 < \lambda < 1$, $\phi > 0$ and $rc - c > 0$ (from (9)), we can conclude that

$$0 < \rho(1-\chi)\,. \quad \text{(B.8)}$$

Next, we consider another situation - there is only player among N players that contributes 0, i.e.,

(1) $x_i = s_1$ and $x_{-i} = (1, \dots, 1, 0, 1, \dots, 1)$, i.e., that player is among i's opponents (we denote him by j');
(2) $x_i = s_2$ and $x_{-i} = (1, \dots, 1)$, i.e., that player is player i.

These two cases give us two inequalities, which guarantee that both $p(s_1, x_{-i})$ and $p(s_2, x_{-i})$ are in $[0, 1]$,

$$0 \leq \phi\rho\Big(\chi\Big(\frac{rc(N-1)}{N} - c\Big) - \sum_{j\neq i, j\neq j'} \omega_j\Big(\frac{rc(N-1)}{N} - c\Big) \\ -\omega_{j'}\frac{rc(N-1)}{N}\Big) + (1-\chi)l\Big) - (1-\lambda)p_0 + 1 \leq \lambda\,, \\ 0 \leq \phi\rho\Big(\chi\frac{rc(N-1)}{N} - \sum_{j\neq 0}\omega_j\Big(\frac{rc(N-1)}{N} - c\Big) + (1-\chi)l\Big) \\ -(1-\lambda)p_0 \leq \lambda\,. \quad \text{(B.9)}$$

Rearranging terms, using the fact that $\omega_{j'} = 1 - \sum_{j\neq i, j\neq j'} \omega_j$ (due to $\sum_{j\neq i} \omega_j = 1$) and multiplying the first inequality by -1, we get

$$(1-\lambda)(1-p_0) \leq \phi\rho\Big(-\chi\Big(\frac{rc(N-1)}{N} - c\Big) + (1-\omega_{j'}) \\ \Big(\frac{rc(N-1)}{N} - c\Big) + \omega_{j'}\frac{rc(N-1)}{N} - (1-\chi)l\Big) \\ \leq 1 - (1-\lambda)p_0\,, \\ (1-\lambda)p_0 \leq \phi\rho\Big(\chi\frac{rc(N-1)}{N} - \Big(\frac{rc(N-1)}{N} - c\Big) \\ +(1-\chi)l\Big) \leq \lambda + (1-\lambda)p_0\,. \quad \text{(B.10)}$$

Combining two inequalities in (B.10), we get

$$1-\lambda \leq \phi\rho(\chi c + \omega_{j'}c) \leq 1+\lambda\,. \quad \text{(B.11)}$$

Considering that $0 < \lambda < 1$, $\phi > 0$ and $c > 0$ (from 3), we can conclude that

$$0 < \rho(\chi + \omega_{j'})\,. \quad \text{(B.12)}$$

Because we do not specify which player j' is defecting, (B.12) should hold for all $\omega_{j'} = \omega_j \in \boldsymbol{\omega}$. Next, we sum (B.8) and (B.12) to get

$$0 < \rho(1+\omega_j)\,, \quad \forall \omega_j \in \boldsymbol{\omega}\,. \quad \text{(B.13)}$$

Because at least one $\omega_j > 0$, one can derive necessary conditions $\rho > 0$ and $\chi < 1$ (from (B.8)). From (B.12), we get for all $\omega_j \in \boldsymbol{\omega}$

$$\chi + \omega_j > 0\,, \\ \min \boldsymbol{\omega} > -\chi\,. \quad \text{(B.14)}$$

Considering that $\chi < 1$ and $\sum_{j\neq i} \omega_j = 1$, one can conclude

$$-\frac{1}{N-1} \leq -\min\boldsymbol{\omega} < \chi < 1\,. \quad \text{(B.15)}$$

Finally, considering that $p_0 \in [0, 1]$ and $\lambda \in (0, 1)$, left hand-sides of (B.5) and (B.6) become

$$0 \leq \phi\rho(1-\chi)(rc-c-l)\,, \quad \text{(B.16)}$$

$$0 \leq \phi\rho(1-\chi)l\,. \quad \text{(B.17)}$$

Using that $\phi, \rho > 0$ and $\chi < 1$, we conclude that

$$0 \leq l \leq rc - c\,. \quad \text{(B.18)}$$

Combining $\rho > 0$, (B.15) and (B.18) results, we complete the proof. ■