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State estimation of a benchmark two-tank problem: A Carleman linearization approach

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Abstract: The two-tank problem is often considered as a challenging benchmark problem of process control, owing to its non-linear nature and non-minimum phase behavior. Non-linearity arises due to the dependence of non-linear outlet flow associated with the tank level. Hence, it is significant to investigate the dynamics of the system. This paper utilizes a novel idea of transforming the non-linear Stochastic Differential Equations (SDEs) to an appropriate form of the SDEs that preserves non-linear effects. In this paper, first, the Carleman linearization technique is explored to arrive at the bi-linearized two-tank SDEs. Then, we utilize the Fokker-Planck equation for the estimation of the bi-linearized two-tank problem. The theoretical results corroborated with numerical simulations highlight the effectiveness of the proposed Carleman linearization-based estimation method in contrast to the benchmark EKF-prediction method, i.e. without observation.

Keywords: State estimation, Carleman linearization, Kronecker product, Itô Stochastic differential rule, Fokker-Planck equation, two-tank system.

1. INTRODUCTION

Multi-connected tank systems possessing non-linear behaviour are ubiquitous in process industries, such as iron and steel, petrochemical, paper-making, food processing, or water purification industries. Moreover, because of the highly non-linear nature, the two-tank system is considered a challenging task and becomes a benchmark problem in the control system (Smith and Doyle, 1988). In this paper, the structure of two interacting tanks placed one above the other is considered. The schematic is shown in Fig.1. Here, water is pumped from the bottom water basin and fed to tank 1, having an outlet as a water input to the tank 2. Ordinary Differential Equations (ODEs) of the two-tank system (Gouta et al., 2019) are given in (1)

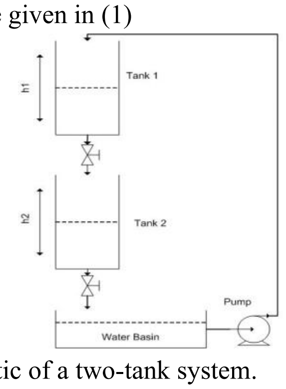

Fig. 1. Schematic of a two-tank system.

$$\dot{h}_1 = \frac{k_p}{A}u - \frac{c}{A}\sqrt{2gh_1}\,,\;\; \dot{h}_2 = \frac{c}{A}\sqrt{2gh_1} - \frac{c}{A}\sqrt{2gh_2}\,, \tag{1}$$

where h_1, h_2 denote the water-level in tank 1 and tank 2, respectively, see Fig. 1. The terms k_p, A, c, g and u are pump constant, area of each tank, cross-sectional area of orifice, the gravitational force, and the voltage applied to the pump, respectively. The cross-sectional area of the orifice of both the tanks and the area of both the tanks is the same.

Here, we describe some of the appealing results on two-tank system problems briefly. Hedi et al. (2018) presented a technique to estimate the fault in the pipeline of the two-tank system. Parameter estimation for the coupled tank system using Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) is explored in (Seung et al., 2017). Estimation of the fault detection problem in the two-tank system using a Kalman filter can be found in Khalid et al. (2011). More significant details regarding the exposure to estimation and filtering can be found in (Dochain, 2003; Nørgaard et al., 2000; Garcia et al., 2019; Zhang and Zhang, 2019; Huang and Dey, 2007). Despite the fecundity of various estimator design techniques for the non-linear system, results related to the stochastic version of the two-tank are scarcely available. More precisely, estimation results related to the bilinear SDEs framework unifying the Carleman linearization based on the Fokker-Planck equation are not available.

Moreover, fluctuation in the input signal to the pump of the two-tank system structure is a reality, which obeys the non-linear stochastic evolution equation. Hence, a question arises that is it possible to transform the non-linear SDEs into an alternative framework of SDEs that accounts for nonlinear effects and offer simplified analysis? The answer is yes. In order to answer this question, first, we apply the Carleman linearization (Carleman, 1932) to the benchmark two-tank problem. Then, the conditional moment equations are derived by utilizing the Fokker-Planck equation for the estimation of the Carleman linearized bilinear two-tank Itô SDEs.

The paper organization is as follows: *Section* 2 explains mathematical preliminaries utilized for the development of the theoretical results. *Section* 3 presents the formalization of Itô stochastic differential two-tank system and the development of the conditional moment evolution equations from the Fokker-Plank equation for the bilinear two-tank Itô SDEs. *Section* 4 illustrates the numerical simulation results of the proposed method in comparison with the benchmark EKF-prediction method.

2. MATHEMATICAL PRELIMINARIES

2.1 Carleman Linearization

Carleman linearization is one of the important methods in system theory almost 80 years ago (Carleman, 1932). The technique was developed to transform sets of polynomial ordinary differential equations into infinite-dimensional bilinear system representation. Based on the order of the non-linearity, we choose a suitable finite order of Carleman linearization to arrive at the finite-dimensional bilinear system representation. Carleman linearization is applied to extend the state vectors of original state variables. An advantage of this method is the embedding dynamics of non-linear systems in corresponding bilinear representations (Kowalski and Steeb, 1991). For Carleman linearization the Kronecker product is the cornerstone, i.e., for a given $A \in R^{n\times m}$ and $B \in R^{s\times t}$, their Kronecker product (Brewer, 1978) can be written as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & \dots & \dots & a_{nm}B \end{pmatrix}. \quad (2)$$

Consider state equations represented by

$$\dot{\xi}_t = f(t,\xi_t) + g(t,\xi_t)u_t, \quad (3)$$

where ξ_t is the $n \times 1$ state vector with u_t as a scalar input signal. Replacing the right-side of (3) by power series representation and adopting the Kronecker-product notation $\otimes$ in (2) (Rugh, 1981)

$$f(t,\xi_t) = A_{0t} + A_{1t}\xi_t^{(1)} + A_{2t}\xi_t^{(2)} + \ldots + A_{2t}\xi_t^{(N)} + \ldots \quad (4a)$$

$$g(t,\xi_t) = G_{0t} + G_{1t}\xi_t^{(1)} + G_{2t}\xi_t^{(2)} + \ldots + G_{2t}\xi_t^{(N-1)} + \ldots \quad (4b)$$

where $\xi_t^{(2)} = (\xi_t \otimes \xi_t)$ and $\xi_t^{(N)} = \xi_t \otimes \xi_t \ldots \otimes \xi_t$.

Embedding (4a)-(4b) in (3),

$$\dot{\xi}_t = \sum_{k=0}^{N} A_{kt}\xi_t^{(k)} + \sum_{k=0}^{N-1} G_{kt}\xi_t^{(k)}u_t + \ldots \quad (5)$$

Here, $\xi_t^{(k)}$ in (5) has $\binom{n+k-1}{k}$ independent state variables, where n and k represents the dimension of the state vector ξ_t and the Kronecker power of the state, respectively. For example, if $k = 2$, then $\xi_t \in R^n$ and $\xi_t^{(2)} \in R^{\binom{n+1}{2}}$. The present concept of Carleman linearization for the specific case of the two-tank problem with its Itô SDEs, $n = 2$, and $k = 2$ is explained in *Section* 3 of the paper.

2.2 Fokker-Planck for bilinear SDEs

The realization of estimation procedures for non-linear SDEs is an interesting problem (Sharma, 2008). The Fokker-Planck equation is utilized for developing estimation theories for given initial states and also for the realization of prediction algorithms for continuous dynamics. Consider the Itô SDE set up,

$$d\xi_t = f(t,\xi_t)dt + G(t,\xi_t)dB_t, \quad (6)$$

Here, from the Fokker-Planck equation, the conditional mean and conditional variance evolutions for the Itô SDE described in (6) are as below

$$d\hat{\xi}_i = \hat{f}_i(t,\xi_t)dt, \quad (7a)$$

$$dP_{ij} = (\overbrace{\xi_i f_j}^{\wedge} - \hat{\xi}_i \hat{f}_j + \overbrace{f_i \xi_j}^{\wedge} - \hat{f}_i \hat{\xi}_j + \overbrace{(G\varphi_w G^T)_{ij}(\xi_t, t)}^{\wedge})dt, \quad (7b)$$

where $\hat{\xi}_i = E(\xi_i(t)|\xi_{t_0}, t_0)$. As Carleman linearization order contributes to the dimension of the augmented state vector and reduces the non-linear stochastic differential system to bilinear stochastic differential system. The bilinear SDE in Itô setting is recast as

$$d\xi_t = (A_{0t} + A_t\xi_t)dt + D_t\xi_t\, dB_t + G_t dB_t, \quad (8)$$

The conditional mean and conditional variance evolutions for the Itô bilinear stochastic differential equation (8) are obtained by utilizing (7a)-(7b), i.e.,

$$d\hat{\xi}_t = (A_{0t} + A_t\hat{\xi}_t)dt, \quad (9a)$$

$$\begin{aligned} dP_t = (&P_t A_t^T + A_t P_t + G_t G_t^T + G_t \hat{\xi}_t^T D_t^T \\ &+ D_t \hat{\xi}_t G_t^T + D_t \overbrace{\xi_t \xi_t^T}^{\wedge} D_t^T)dt, \end{aligned} \quad (9b)$$

where $\overbrace{\xi_t\xi_t^T}^{\wedge} = E(\xi_t\xi_t^T|t_0, \xi_{t_0}) = P_t + \hat{\xi}_t\hat{\xi}_t^T$. The detailed proof of conditional mean and conditional variance for bilinear SDE can be found in (Bhatt and Sharma, 2019).

3. APPLICATION TO TWO-TANK PROBLEM

3.1 *Itô stochastic differential equation for two-tank system*

The stochasticity is attributed to the fluctuating voltage signal u of the pump. Now, recalling the two-tank ODEs (1) the stochastic version, of the two-tank system, is formalized under the influence of noise in the input voltage signal, which is depicted below

$$d\xi_1 = (\frac{k_p}{A}u - \frac{c}{A}\sqrt{2g_1\xi_1})dt + \frac{k_p\alpha}{A}dB_t, \tag{10a}$$

$$d\xi_2 = (\frac{c}{A}\sqrt{2g\xi_1} - \frac{c}{A}\sqrt{2g\xi_2})dt, \tag{10b}$$

where α is the diffusion noise intensity. The term $w_t = \frac{dB_t}{dt}$, where dB_t is the Brownian process. , $(h_1\ h_2)^T$ are replaced with standard notations $(\xi_1\ \xi_2)^T$. The Carleman linearization approach to non-linear SDEs leads to the bilinear SDEs. For this characteristic, it is also referred to as a bi-linearization method. Since SDEs of the two-tank system accounts for non-linearity, embedding the Carleman linearization will lead to bilinear two-tank SDEs. Considering the Carleman linearization order two, the augmented state vector becomes $(\xi_t\ \ \xi_t^{(2)})^T$, where

$$\xi_t = (\xi_1\ \xi_2)^T, \tag{11}$$

$$\xi_t^{(2)} = \xi_t \otimes \xi_t = (\xi_1\ \xi_2)^T \otimes (\xi_1\ \xi_2)^T = (\xi_1^2\ \xi_1\xi_2\ \xi_2^2)^T. \tag{12}$$

The dimension of the state vector and the Kronecker power associated with the state $\xi_t^{(2)}$ is $\begin{pmatrix} n+k-1 \\ k \end{pmatrix}$, where n denotes the number of states and k denotes the Carleman linearized order. For the two-tank SDE system, n is 2 and k is 2, so the dimension of the Carleman linearized state vector $\xi_t^{(2)}$ becomes three. After accounting the independent state variables of Carleman linearized state vector order two, the dimension of the augmented state vector for two-tank SDE system is five, i.e.

$$d\begin{pmatrix} \xi_t \\ \xi_t^{(2)} \end{pmatrix} = (d\xi_1\ d\xi_2\ \ d\xi_1^2\ d\xi_1\xi_2\ \ d\xi_2^2\ )^T. \tag{13}$$

Thus, utilizing the Carleman linearization method mentioned in *Section* 2.1 and applying the stochastic differential rule (Karatzas and Shreve, 1988; p. 154) for (10a)-(10b), the following Carleman linearized evolution equations for augmented states in (13) are obtained:

$$\begin{aligned} d\xi_1 = (&\frac{k_p}{A}u - \frac{3c}{8A}\sqrt{2g}\sqrt{\xi_{1s}} - \frac{3c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{1s}}}\xi_1 \\ &+ \frac{c}{8A}\frac{\sqrt{2g}}{\xi_{1s}^{\ 1.5}}\xi_1^2)dt + \frac{k_p\alpha}{A}dB_t, \end{aligned} \tag{14a}$$

$$\begin{aligned} d\xi_2 = (&\frac{3c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{1s}}}\xi_1 - \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{1s}^{\ 1.5}}\xi_1^{\ 2} - \frac{3c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{2s}}}\xi_2 \\ &+ \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{2s}^{\ 1.5}}\xi_2^2)dt, \end{aligned} \tag{14b}$$

$$\begin{aligned} d\xi_1^{\ 2} = ((&\frac{2k_p}{A}u(t) - \frac{2c\sqrt{2g}\sqrt{\xi_{1s}}}{A} + \frac{5c\sqrt{2g}\sqrt{\xi_{1s}}}{4A})\xi_1 \\ &- \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\xi_1^{\ 2})dt + \frac{k_p^2\alpha^2}{A^2} + \frac{2k_p\alpha}{A}\xi_1 dB_t), \end{aligned} \tag{14c}$$

$$\begin{aligned} d\xi_1\xi_2 = (&-(\frac{c\sqrt{2g}\sqrt{\xi_{1s}}}{8A} + \frac{3c\sqrt{2g}\sqrt{\xi_{2s}}}{8A})\xi_1 + (\frac{k_p u}{A} - \frac{3c\sqrt{2g}\sqrt{\xi_{1s}}}{8A})\xi_2 \\ &+ \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}\xi_1^{\ 2} - (\frac{3c\sqrt{2g}}{4A\sqrt{\xi_{2s}}} + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}})\xi_1\xi_2)dt + \frac{k_p\alpha}{A}\xi_2 dB_t, \end{aligned} \tag{14d}$$

$$\begin{aligned} d\xi_2^{\ 2} = ((&\frac{3c\sqrt{2g}\sqrt{\xi_{1s}}}{4A} - \frac{3c\sqrt{2g}\sqrt{\xi_{2s}}}{4A})\xi_2 \\ &+ \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\xi_1\xi_2 - \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\xi_2^{\ 2})dt. \end{aligned} \tag{14e}$$

The above Carleman linearized states (14a)-(14e) of the two-tank problem can be rearranged in the bilinear SDE format of (8), which is given in *Appendix* A.

3.2 Fokker-Planck for Carleman linearized two-tank system

This section deals with the development of the conditional moment evolutions of the Carleman linearized two-tank SDEs (14a)-(14e). More specifically, the element-wise conditional mean evolution equations and the condition variance evolution equations of the bilinear two-tank SDEs, which is a consequence of (9a)-(9b), are obtained as

$$\begin{aligned} d\hat{\xi}_1 = (&\frac{k_p}{A}u - \frac{3c}{8A}\sqrt{2g}\sqrt{\xi_{1s}} - \frac{3c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{1s}}}\hat{\xi}_1 \\ &+ \frac{c}{8A}\frac{\sqrt{2g}}{\xi_{1s}^{\ 1.5}}P_{\xi_1} + \frac{c}{8A}\frac{\sqrt{2g}}{\xi_{1s}^{\ 1.5}}\hat{\xi}_1^{\ 2})dt, \end{aligned} \tag{15a}$$

$$\begin{aligned} d\hat{\xi}_2 = (&\frac{3c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{1s}}}\hat{\xi}_1 - \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{1s}^{\ 1.5}}P_{\xi_1} - \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{1s}^{\ 1.5}}\hat{\xi}_1^{\ 2} \\ &- \frac{c}{4A}\frac{\sqrt{2g}}{\sqrt{\xi_{1s}}}\hat{\xi}_2 + \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{2s}^{\ 1.5}}P_{\xi_2} + \frac{c}{A}\frac{\sqrt{2g}}{8\xi_{2s}^{\ 1.5}}\hat{\xi}_2^{\ 2})dt, \end{aligned} \tag{15b}$$

$$\begin{aligned} d\hat{\xi}_1^2 = ((&\frac{2k_p}{A}u - \frac{3c\sqrt{2g}\sqrt{\xi_{1s}}}{4A})\hat{\xi}_1 - \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}P_{\xi_1} \\ &- \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\hat{\xi}_1^{\ 2} + \frac{k_p^2\alpha^2}{A^2})dt, \end{aligned} \tag{15c}$$

$$\begin{aligned} d\widehat{\xi_1\xi_2} = (&-(\frac{c\sqrt{2g}\sqrt{\xi_{1s}}}{8A} + \frac{3c\sqrt{2g}\sqrt{\xi_{2s}}}{8A})\hat{\xi}_1 \\ &+ (\frac{k_p u}{A} - \frac{3c\sqrt{2g}\sqrt{\xi_{1s}}}{8A})\hat{\xi}_2 + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}P_{\xi_1} + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}\hat{\xi}_1^{\ 2} \\ &- (\frac{3c\sqrt{2g}}{4A\sqrt{\xi_{2s}}} + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}})P_{\xi_1\xi_2} - (\frac{3c\sqrt{2g}}{4A\sqrt{\xi_{2s}}} + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}})\hat{\xi}_1\hat{\xi}_2)dt, \end{aligned} \tag{15d}$$

$$d\hat{\xi}_2^2 = \left(\left(\frac{3c\sqrt{2g}\sqrt{\xi_{1s}}}{4A} - \frac{3c\sqrt{2g}\sqrt{\xi_{2s}}}{4A}\right)\hat{\xi}_2 + \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}P_{\xi_1\xi_2} + \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\hat{\xi}_1\hat{\xi}_2 - \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}P_{\xi_2} - \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}\hat{\xi}_2^{\,2}\right)dt. \tag{15e}$$

The notation $\hat{\xi}$ denotes the action of the conditional expectation operator on the term ξ.

Conditional variance evolution equations

$$dP_{\xi_1} = d(\hat{\xi}_1^2 - \hat{\xi}_1^{\,2}) = d\hat{\xi}_1^2 - d\hat{\xi}_1^{\,2} = \left(\frac{k_p^2\alpha^2}{A^2} - \frac{3c\sqrt{2g}}{2A}P_{\xi_1} - \frac{c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}\hat{\xi}_1 P_{\xi_1} - \frac{c\sqrt{2g}}{4A\xi_{1s}^{\;1.5}}\hat{\xi}_1^3\right)dt, \tag{16a}$$

$$dP_{\xi_2} = d(\hat{\xi}_2^2 - \hat{\xi}_2^{\,2}) = d\hat{\xi}_2^2 - d\hat{\xi}_2^{\,2} = \left(\frac{3c\sqrt{2g}}{2A\sqrt{\xi_{1s}}}P_{\xi_1\xi_2} - \frac{3c\sqrt{2g}}{2A\sqrt{\xi_{2s}}}P_{\xi_2} + \frac{c\sqrt{2g}}{4A\xi_{1s}^{\;1.5}}\hat{\xi}_1^{\,2}\hat{\xi}_2 + \frac{c\sqrt{2g}}{4A\xi_{1s}^{\;1.5}}\hat{\xi}_2 P_{\xi_1} - \frac{c\sqrt{2g}}{4A\xi_{2s}^{\;1.5}}\hat{\xi}_2 P_{\xi 2}\right)dt, \tag{16b}$$

$$dP_{\xi_1\xi_2} = d(\widehat{\xi_1\xi_3}) - \hat{\xi}_1 d\hat{\xi}_2 - \hat{\xi}_2 d\hat{\xi}_1 - d\hat{\xi}_1 d\hat{\xi}_2 = \left(-\frac{c\sqrt{2g}\sqrt{\xi_{1s}}}{8A}\hat{\xi}_1 - \frac{3c\sqrt{2g}\sqrt{\xi_{2s}}}{8A}\hat{\xi}_1 + \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}P_{\xi_1} - \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{2s}}}P_{\xi_1\xi_2} - \frac{3c\sqrt{2g}}{4A\sqrt{\xi_{1s}}}P_{\xi_1\xi_2} + \frac{c\sqrt{2g}}{8A\xi_{1s}^{\;1.5}}\hat{\xi}_1^{\,3} + \frac{c\sqrt{2g}}{8A\xi_{1s}^{\;1.5}}\hat{\xi}_1 P_{\xi_1} - \frac{c\sqrt{2g}}{8A\xi_{2s}^{\;1.5}}\hat{\xi}_1 P_{\xi_2} - \frac{c\sqrt{2g}}{8A\xi_{2s}^{\;1.5}}\hat{\xi}_1\hat{\xi}_1^{\,2} - \frac{c\sqrt{2g}}{8A\xi_{1s}^{\;1.5}}\hat{\xi}_2 P_{\xi_1} - \frac{c\sqrt{2g}}{8A\xi_{1s}^{\;1.5}}\hat{\xi}_2\hat{\xi}_1^{\,2}\right)dt. \tag{16c}$$

4. NUMERICAL SIMULATIONS

The efficacy of the proposed Carleman linearization-based estimation methods is adjudged by performing a numerical simulation study. The estimation results of the proposed method are also compared to the results generated from the Extended Kalman Filter-based prediction (EKF-predicted) method. The proposed estimation method of this paper is implemented in MATLAB© on Intel(R) Core(TM) i5-5200U laptop CPU clocked at 2.20 GHz with 8.00GB RAM. The operating point values and nominal parameter values are depicted in Table 1 (Gouta et al., 2019). The initial value of the input signal u in Table 1 corresponds to the steady-state value 16 cm for the water-level of both the tanks.

Table 1. Operating points and nominal values

Parameters	Values	Units
K_p	18.5183	cm^3/Vs
A	144	cm^2
c	0.502	cm^2
g	981	cm/s^2
u	4.803	V
ξ_{1s}	16	cm
ξ_{2s}	16	cm

The closeness of the Carleman linearized bilinear SDEs (14a)-(14e) to the exact two-tank non-linear SDEs (10a)-(10b) is displayed in Fig. 2. Figs. 2(a) and 2(b) show the true trajectories vs. Carleman linearized trajectories for the initial water-level of both the tanks set to $\xi_1(0) = \xi_2(0) = 2\ cm$, with $\alpha = 0.7$ and initial variance $P_{\xi_1}(0) = P_{\xi 2}(0) = 1$, $P_{\xi_1\xi_2}(0) = 0$. Similarly, Figs. 2(c) and 2(d) show the true trajectories vs. Carleman linearized trajectories for the initial water-level of both the tanks set to 28 cm.

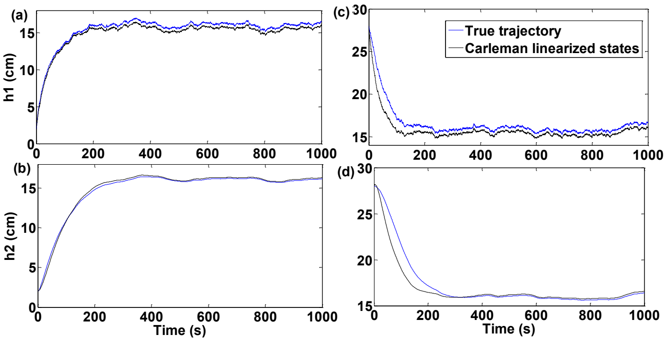

Fig. 2. Comparison of the true state trajectories vs. Carleman linearized state trajectories.

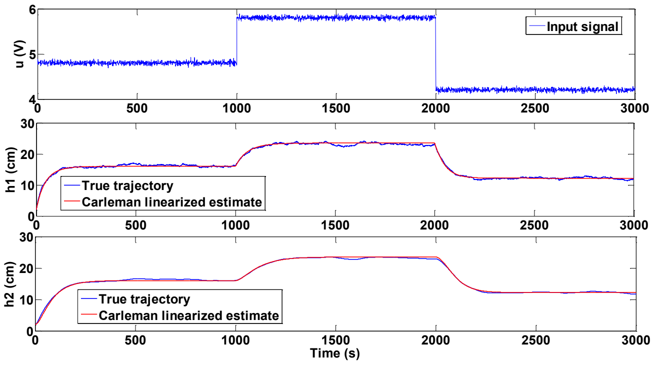

Fig. 3. Conditional mean trajectories under a series of step variations in the input signal.

Fig. 3 shows the performance of the proposed estimation method for a series of step variations in the input signal. The input u is increased from 4.803 V to 5.8 V and then decreased to 4.2 V. The responses of the true trajectories, i.e., non-linear SDEs and the Carleman linearized estimated conditional mean trajectories of both the tanks are shown in Fig. 3.

In order to reveal the true potential of the proposed estimation method, it is compared with the EKF-predicted estimates. Fig. 4 shows the comparison of the Carleman linearized estimated state trajectories, the EKF-predicted state trajectories, and the true state trajectories. Note that the true state trajectories and The Carleman linearized estimated state trajectories are a consequence (13a)-(13b)and (18a)-(18e). Note that the EKF-predicted trajectories are the non-linear predicted estimate without observations (Jazwinski, 1970; p. 278).

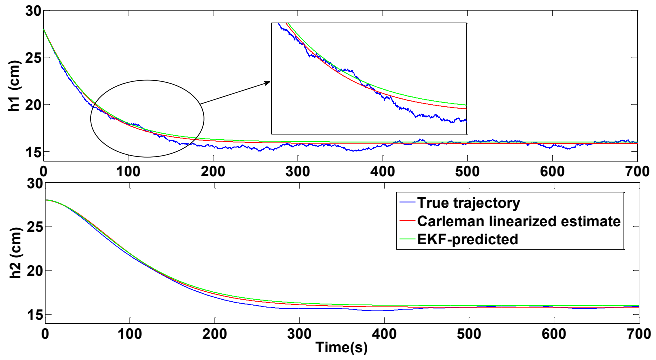

Fig. 4. Comparative conditional mean trajectories for both the tanks.

Fig. 4 reveals that Carleman linearized estimates are closer to the true trajectories in comparison to the EKF-predicted trajectories for both the tanks. Fig. 5 shows a comparison between two Absolute Prediction Error (APE) evolutions for both the states. The maximum APE obtained with the Carleman linearized estimate of tank 1 is 0.9, which is less than the maximum APE 1.1 of EKF-predicted.

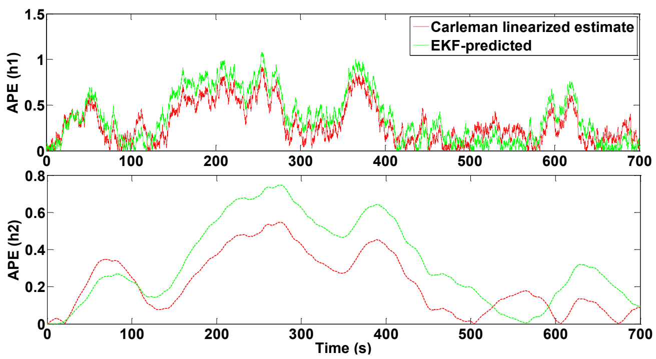

Fig. 5 Absolute Prediction Error (APE) trajectories for both the tanks.

Similarly, for tank 2 the maximum APE of Carleman linearized estimate is 0.54, and that of the EKF-predicted is 0.75. The less value of the absolute prediction error, associated with the proposed method, confirms greater closeness of the estimated trajectories via the Carleman linearization with true state trajectories.

The conditional variance trajectory in Fig. 6 gives the random fluctuations in the mean trajectory. The less variance suggests the better estimate, which is indicative of less random fluctuations in the most probable trajectory. The conditional variance trajectories associated with the Carleman linearized estimate are generated using the conditional variance evolution equations (19a)-(19c). Hence, the Carleman linearization-based estimation gives a better estimate than the EKF-predicted.

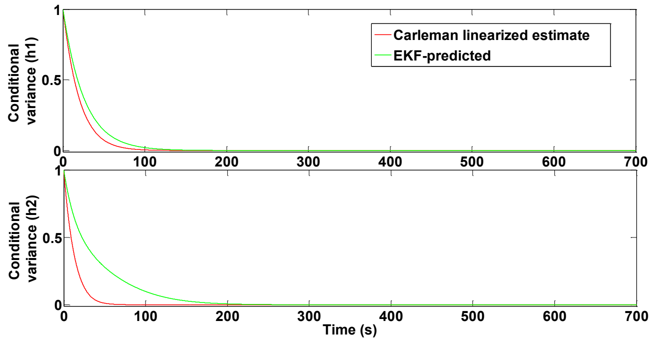

Fig. 6. Comparative conditional variance trajectories for both the tanks.

5. CONCLUSION

In this paper, we have demonstrated the universality of the Carleman linearization technique for a non-linear stochastic system via the potential two-tank problem. The Carleman linearized two-tank states account for the non-linear effects, stochasticity and allow straightforward bi-linear structure. The estimation theory of this paper has utilized the Itô framework and combined the Carleman linearization with the Fokker-Planck equation to develop the conditional moment equations for the two-tank problem. Numerical simulations of the paper revealed the superiority and overall better performance of the proposed Carleman linearization-based estimation theory. The proposed method is recommended for the filtering, and stochastic control procedures of the Carleman linearized two-tank setup. That is under investigation.

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APPENDIX A. BILINEAR SDEs OF TWO-TANK SYSTEM

Recall the standard bilinear SDE form, i.e.,

$$d\xi_t = (A_{0t} + A_t\xi_t)dt + D_t\xi_t dB_t + G_t dB_t, \qquad \text{(A.1)}$$

where the state vector $\xi_t = (\xi_i^{(k)})$, and k is the Kronecker power, $1 \le k \le N$ $1 \le i \le \sum_{1\le k\le N} \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$. Note that the augmented state vector $\xi_t \in R^{\sum_{1\le k\le N}\binom{n+k-1}{k}}$. In the partitioned matrix-vector format, the two-tank Carleman linearized SDE becomes

$$d\begin{pmatrix} \xi_t \\ \xi_t^{(2)} \end{pmatrix} = \left(\begin{pmatrix} A_{01}(t) \\ A_{02}(t) \end{pmatrix} + \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} \xi_t \\ \xi_t^{(2)} \end{pmatrix} \right) dt + \left(\begin{pmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{pmatrix} \begin{pmatrix} \xi_t \\ \xi_t^{(2)} \end{pmatrix} + \begin{pmatrix} G_{1t} \\ G_{2t} \end{pmatrix} \right) dB_t. \qquad \text{(A.2)}$$

For notational convenience, the time 't' is stated as an input argument of submatrices in (A.2) and a subscript of matrices in (A.1). For the specific case of two-tank problem,

$$A_{01}(t) = \begin{pmatrix} \dfrac{k_p u}{A} - \dfrac{3\beta b_1}{8} \\ 0 \end{pmatrix}, \quad A_{02}(t) = (\frac{k_p^2\alpha^2}{A^2} \quad 0 \quad 0)^T,$$

$$A_{11}(t) = \begin{pmatrix} -\dfrac{3\beta}{4b_1} & 0 \\ \dfrac{3\beta}{4b_1} & \dfrac{3\beta}{4b_2} \end{pmatrix}, \quad A_{12}(t) = \begin{pmatrix} \dfrac{\beta}{8\xi_{1s}{}^{1.5}} & 0 & 0 \\ -\dfrac{\beta}{8\xi_{1s}{}^{1.5}} & 0 & \dfrac{\beta}{8\xi_{2s}{}^{1.5}} \end{pmatrix},$$

$$A_{21}(t) = \begin{pmatrix} \dfrac{2k_p u}{A} - \dfrac{3\beta b_1}{4} & 0 \\ -\dfrac{\beta b_1}{8} - \dfrac{3\beta b_2}{8} & \dfrac{k_p u}{A} - \dfrac{3\beta b_2}{4} \\ 0 & \dfrac{3\beta b_1}{4} - \dfrac{3\beta b_2}{4} \end{pmatrix},$$

$$A_{22}(t) = \begin{pmatrix} -\dfrac{3\beta b_1}{2b_1} & 0 & 0 \\ \dfrac{3\beta b_1}{4b_1} & -\dfrac{3\beta}{4b_2} - \dfrac{3\beta}{4b_1} & 0 \\ 0 & \dfrac{3\beta}{2b_1} & -\dfrac{3\beta}{2b_1} \end{pmatrix},$$

$$D_{21}(t) = \begin{pmatrix} \dfrac{2k_p\alpha}{A} & 0 & 0; 0 & \dfrac{k_p\alpha}{A} & 0 \end{pmatrix}^T, \quad G_{1t} = (\frac{k_p\alpha}{A} \quad 0)^T,$$

where $\beta = c\sqrt{2g}/A$, $b_1 = \sqrt{\xi_{1s}}$ and $b_2 = \sqrt{\xi_{2s}}$.

For zero matrices, i.e., $D_{11}(t) = (D_{11}^{ij})$, where $1 \le i \le 2$, $1 \le j \le 2$ and $D_{11}^{ij}(t) = 0$, $D_{12}(t) = (D_{12}^{ij})$, where $1 \le i \le 2$, $1 \le j \le 3$ and $D_{12}^{ij}(t) = 0$, $D_{22}(t) = (D_{22}^{ij})$, where $1 \le i \le 3$, $1 \le j \le 3$ and $D_{22}^{ij}(t) = 0$. $G_{2t} = (G_{2t}^{ij})$, where $1 \le i \le 3$, and $G_{2t}^{ij} = 0$.