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Feedback stabilization: the algebraic view

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Abstract: Based on an abstract algebraic setting we provide an elementary derivation of the Youla-Kucera parametrization of stabilizing controllers and also an alternative, coordinate free, approach of the problem. For this latter case, in contrast to the Youla-Kucera approach, the parameter set is not universal but its elements can be generated by a universal algorithm. We also emphasise the natural continuity property of this parametrization compared to the Youla-Kucera case. Extending the framework to the LFT loops we show by using elementary tools that every controller which stabilizes the interior loop of the generalized plant also stabilizes the LFT loop.

Keywords: controller parametrisation; controller blending; stability guarantee

1. INTRODUCTION AND MOTIVATION

Although geometry is one of the richest areas for mathematical exploration it seems to have been applied to a limited extent by engineers and elementary geometrical treatment is often considered difficult to understand. To put geometry and geometrical thought in a position to become a reliable engineering tool, a certain mechanism is needed that translates geometrical facts into a more accessible form for everyday algorithms. Coordinates, in general, are the most essential tools for the applied disciplines that deal with geometry. Klein proposed group theory as a mean of formulating and understanding geometrical constructions.

In Szabó et al. [2014] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate various control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The main concern of our work is to highlight the deep relation that exists between the seemingly different fields of geometry, algebra and control.

In Szabó and Bokor [2015, 2016], Szabó et al. [2017] we have shown that in contrast to the classical Youla-Kucera approach, there is a parametrisation of the entire controller set which can be described entirely in a coordinate free way, i.e., just by using the knowledge of the plant G and of the given stabilizing controller K_0. The corresponding parameter set is given in geometric terms, i.e., by providing an associated algebraic (semigroup, group) structure. Moreover, it turns out that the geometry of stable controllers is surprisingly simple.

While the Kleinian view makes the link between geometry and group theory, through different representations and homomorphism the abstract group theoretical facts obtain an algebraic formulation that opens the way to engineering applications. This interplay between geometry, algebra and control theory is what we are interested in our investigations. This paper focuses on the algebraic aspects of the approach.

We would like to stress that it is a very fruitful strategy to try to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction; finally the solution of the original control problem can be formulated in an algorithmic way by transposing the geometric ideas into the proper algebraic terms. Accordingly, we suppose only that our objects (systems), plants and controllers, are elements of a suitable ring while stability is a property, which is inherited by addition and multiplication of the systems

The reader customised with system classes, like LTI, LPV (linear parameter varying), nonlinear, switching, etc. might find our presentation quite informal. We stress that geometry – and also algebra – does not deal with the existence and the actual nature of the objects that are the primitives of the given geometry but rather captures the "rules" they obeys to. It gives the abstract structures that can be, for a given application, associated with actual objects, i.e., responds to the question "what can be done with these objects" rather than "how to synthesise the object having a given property (e.g., stability)".

In the first part of this paper we illustrate this fact through the example of the Youla-Kucera parametrization, Kucera [1975], Youla, Jabr and Bongiorno [1976], Kevickzy and Bányász [2015]. For a modern algebraic based treatment see Quadrat [2005, 2006] On an abstract background and emphasising a loop-transformation view we provide here a completely elementary proof and discussion of the result, going well beyond its original content. It is revealed that once we leave the fixed controller (plant) axis, this approach does not have any particular advantages. Even simple questions, like relating a stably perturbed stable pair (G_0, K_0) to some stable neighbourhood of the origin, have nontrivial answers in the framework. We emphasize that even in the standard setting of rational transfer functions the related Double Youla-Kucera approach, see, e.g., Schrama et al. [1992], provides only a partial answer to the problem.

The second part of the paper revisits the coordinate free approach to the problem. We show that it is easy to provide bounds for the simultaneous stable perturbation of the stable pair (G_0, K_0) that guarantees stability. The coordinate free parametrization is also continuous in this sense, i.e., a neighbourhood of the origin in the parameter space is related to controllers from a neighbourhood of K_0. We also provide a completely new, algebraic development of the controller blending, complementing the already presented geometric based parametrization with additional results.

Finally we extend the approach to the LFT framework, showing that the already introduced blending operator still works in this context, too. The main result concerning this topic shows that every stabilizing controller of the interior loop is also a stabilizing controller for the entire LFT loop.

2. BASIC SETTINGS

2.1 Möbius transformations

Let us consider the operator matrices Σ and $\tilde{\Sigma}$ which satisfy the double Bézout identity

$$\tilde{\Sigma}\Sigma = \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (1)$$

By the matrix inversion lemma we have

$$\exists\, M^{-1} \quad \Leftrightarrow \quad \exists\, \tilde{V}^{-1} = V - NM^{-1}U \quad (2)$$

$$\exists\, V^{-1} \quad \Leftrightarrow \quad \exists\, \tilde{M}^{-1} = M - UV^{-1}N \quad (3)$$

$$\exists\, M^{-1}, V^{-1} \quad \Leftrightarrow \quad \exists\, \tilde{V}^{-1}, \tilde{M}^{-1} \quad (4)$$

Multiplying (1) by the right (left) translations, i.e., the upper (lower) block triangular matrices ρ_Q (λ_S), we get

$$\begin{pmatrix} \tilde{V} + Q\tilde{N} & -\tilde{U} - Q\tilde{M} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U + MQ \\ N & V + NQ \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -S\tilde{V} - \tilde{N} & S\tilde{U} + \tilde{M} \end{pmatrix} \begin{pmatrix} M + US & U \\ N + VS & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

i.e., according to (4), $\exists (V + NQ)^{-1} \Leftrightarrow \exists (\tilde{V} + Q\tilde{N})^{-1}$ and $\exists (M + US)^{-1} \Leftrightarrow \exists (S\tilde{U} + \tilde{M})^{-1}$, moreover

$$(U + MQ)(V + NQ)^{-1} = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M})$$

$$(N + VS)(M + US)^{-1} = (\tilde{M} + S\tilde{U})^{-1}(\tilde{N} + S\tilde{V}).$$

It is convenient to introduce the following Möbius transformations: the plant transformation

$$P = \mathcal{P}_{\Sigma}(S) = (N + VS)(M + US)^{-1}, \quad (5)$$

$S \in \mathrm{dom}\mathcal{P}_{\Sigma} = \{\exists\, (M + US)^{-1}\}$ and its counterpart, the controller transformation

$$K = \mathcal{K}_{\Sigma}(Q) = (U + MQ)(V + NQ)^{-1}, \quad (6)$$

$Q \in \mathrm{dom}\mathcal{K}_{\Sigma} = \{\exists\, (V + NQ)^{-1}\}$, along with their dual versions

$$P = \mathcal{P}^{d}_{\tilde{\Sigma}}(S) = (\tilde{M} + S\tilde{U})^{-1}(\tilde{N} + S\tilde{V}), \quad (7)$$

and

$$K = \mathcal{K}^{d}_{\tilde{\Sigma}}(Q) = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}), \quad (8)$$

respectively. Note that we have the following fundamental duality properties:

$$\mathcal{P}^{d}_{\tilde{\Sigma}}(S) = \mathcal{P}_{\Sigma}(S), \quad \mathrm{dom}\mathcal{P}_{\Sigma} = \mathrm{dom}\mathcal{P}^{d}_{\tilde{\Sigma}}, \quad (9)$$

and

$$\mathcal{K}^{d}_{\tilde{\Sigma}}(Q) = \mathcal{K}_{\Sigma}(Q), \quad \mathrm{dom}\mathcal{K}_{\Sigma} = \mathrm{dom}\mathcal{K}^{d}_{\tilde{\Sigma}}. \quad (10)$$

respectively.

Recall that the lower and upper LFT is defined as

$$\mathcal{F}_l(P, K) = P_{zw} + P_{zu}K(I - GK)^{-1}P_{yw}$$

and

$$\mathcal{F}_u(P, \Delta) = G + P_{yw}\Delta(I - P_{zw}\Delta)^{-1}P_{zu},$$

respectively, see Figure 2(b). Then, the lower and upper LFT representation of the Möbius transformations (5) and (6) are

$$P = \mathcal{P}_{\Sigma}(S) = \mathcal{F}_u(\hat{\Sigma}_{P,K_0}, S), \quad (11)$$

$$K = \mathcal{K}_{\Sigma}(Q) = \mathcal{F}_l(\check{\Sigma}_{P,K_0}, Q), \quad (12)$$

respectively, where

$$\hat{\Sigma} = \begin{pmatrix} -M^{-1}U & M^{-1} \\ V - NM^{-1}U & NM^{-1} \end{pmatrix} = \begin{pmatrix} -\tilde{U}\tilde{M}^{-1} & \tilde{V} - \tilde{U}\tilde{M}^{-1}\tilde{N} \\ \tilde{M}^{-1} & \tilde{M}^{-1}\tilde{N} \end{pmatrix}, \quad (13)$$

$$\check{\Sigma} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix} = \begin{pmatrix} \tilde{V}^{-1}\tilde{U} & \tilde{V}^{-1} \\ \tilde{M} - \tilde{N}\tilde{V}^{-1}\tilde{U} & -\tilde{N}\tilde{V}^{-1} \end{pmatrix}. \quad (14)$$

2.2 Stable feedback connections

A central concept of control theory is that of the feedback and the stability of the feedback loop. For practical reasons our basic objects, the systems, i.e., plants and controllers, are causal. Stability is actually a continuity property of a certain map, more precisely a property of boundedness and causality of the corresponding map. Boundedness also involves some topology. In what follows we consider linear systems, i.e., the signals are elements of some normed linear spaces and an operator means a linear map that acts between signals. Thus, boundedness of the systems is regarded as boundedness in the induced operator norm.

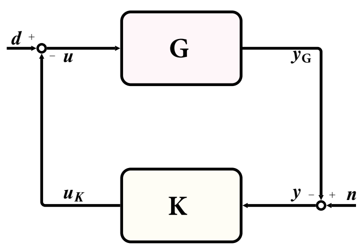


Fig. 1. Basic feedback loop

To fix the ideas let us consider the feedback-connection depicted on Figure 1. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, g = \begin{pmatrix} u \\ y_G \end{pmatrix}, k = \begin{pmatrix} u_K \\ y \end{pmatrix}, z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathfrak{H},$$

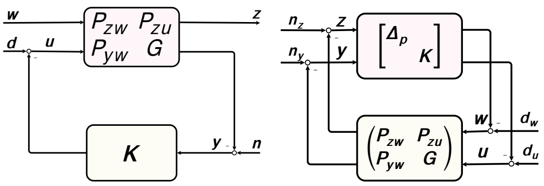
where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and we suppose that the signals are elements of the Hilbert space $\mathfrak{H}_1, \mathfrak{H}_2$ (e.g., $\mathfrak{H}_i = \mathfrak{L}^{n_i}[0, \infty)$) endowed by a resolution structure which determines the causality concept on these spaces. In this model the plant G and the controller K are linear causal maps. For more details on this general setting, see Feintuch [1998].

The feedback connection is called well-posed if for every $w \in \mathfrak{H}$ there is a unique g and k such that $w = g + k$ (causal invertibility) and the pair (G, K) is called stable if the map $w \to z$ is a bounded causal map, i.e., the pair (G, K) is called well-posed if the inverse

$$\mathcal{H}(G,K) = \begin{pmatrix} I & K \\ G & I \end{pmatrix}^{-1} = \begin{pmatrix} S_u^K & S_c^K \\ S_g^K & S_y^K \end{pmatrix} = \\ = \begin{pmatrix} (I-KG)^{-1} & -K(I-GK)^{-1} \\ -G(I-KG)^{-1} & (I-GK)^{-1} \end{pmatrix} \tag{15}$$

exists (causal invertibility), and it is called stable if all the block elements are stable.

Stability of the LFT loop means that the causal map $\mathcal{L}(P, K)$ that relates the signals (z, u, y) to (w, d, n) is invertible and the inverse map is stable, see Figure 2(a). It turns out that this is equivalent to the stability of the extended feedback loop for $\Delta_p = 0_p$, see Figure 2(b).



(a) Performance (b) Stability

Fig. 2. Stability of LFTs

As a consequence, stability questions of LFT loops can be reduced to the investigation of the configuration determined by $(P, \mathrm{diag}(0, K))$. It is obvious that the LFT loop is well–defined if and only if (G, K) is well defined. However, it is less obvious whether this claim remains true for stability.

3. YOULA-KUCERA PARAMETRIZATION

A fundamental result concerning the stabilization problem related to the basic feedback connection depicted on Figure 1 is description of the set of stabilizing controllers. For a fixed plant G_0 a standard assumption is that among the stable factorizations there exists a special one, called double coprime factorization, i.e., $G_0 = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, and there are causal bounded systems $U, V, \tilde{U}$ and $\tilde{V}$, with invertible V and $\tilde{V}$, such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \tilde{\Sigma}_{G_0}\Sigma_{G_0} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{16}$$

see, e.g., Vidyasagar [1985], Feintuch [1998]. The existence of a double coprime factorization implies feedback stabilizability, actually $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ is a stabilizing controller.

Note, that besides the trivial non-uniqueness of the coprime factors, the entire approach has a considerable freedom in the choice of the given elements, like Σ_{G_0}, which makes possible to embed a given system in different frameworks. While the standard setting considers rational LTI systems, the entire approach can be applied for linear parameter varying (LPV) systems or even switched systems. Even if G_0 is set to be an LTI one, nothing prevents us to chose the blocks of Σ_{G_0} to be LPV systems, see, e.g., Szabó and Bokor [2018].

In what follows let $\mathbb{S}$ and $\mathbb{Q}$ be the set of stable plants compatible with the dimension of G_0 and K_0, respectively. Let us denote by (G_0, K_0) an initial stable pair and assume that it corresponds to the double coprime factorization $(\tilde{\Sigma}, \Sigma)$, i.e., (16) holds with $G_0 = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$. After this preparatory introduction we can state the Youla-Kucera type result in its most general form as:

Theorem 3.1. Based on our standing assumptions and notations introduced above:

Youla: every stable pair (G_0, K) is given by a suitable controller transformation, i.e.,

$$K = K_Q = \mathcal{K}^d_{\tilde{\Sigma}}(Q) = \mathcal{K}_\Sigma(Q), \tag{17}$$

where

$$Q \in \mathrm{dom}\mathcal{K}_\Sigma \cap \mathbb{Q} = \mathrm{dom}\mathcal{K}^d_{\tilde{\Sigma}} \cap \mathbb{Q} = \mathbb{Q}^0_\Sigma. \tag{18}$$

Dual Youla: every stable pair (G, K_0) is given by a suitable plant transformation, i.e.,

$$G = G_S = \mathcal{P}^d_{\tilde{\Sigma}}(S) = \mathcal{P}_\Sigma(S), \tag{19}$$

where

$$S \in \mathrm{dom}\mathcal{P}_\Sigma \cap \mathbb{S} = \mathrm{dom}\mathcal{P}^d_{\tilde{\Sigma}} \cap \mathbb{S} = \mathbb{S}^0_\Sigma. \tag{20}$$

Proof: By symmetry it enough to prove the Youla-Kucera part. Sufficiency is easy to be checked and it is left out for brevity. For necessity consider the identity

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} I & K \\ G_0 & I \end{pmatrix} = \begin{pmatrix} \tilde{V} - \tilde{U}G_0 & \tilde{V}K - \tilde{U} \\ -\tilde{N} + \tilde{M}G_0 & -\tilde{N}K + \tilde{M} \end{pmatrix} = \\ = \begin{pmatrix} M^{-1} & \tilde{V}K - \tilde{U} \\ 0 & -\tilde{N}K + \tilde{M} \end{pmatrix} \tag{21}$$

Thus, with $Q = (\tilde{V}K - \tilde{U})(-\tilde{N}K + \tilde{M})^{-1}$ we have

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix}^{-1} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} M & -MQ \\ 0 & (-\tilde{N}K + \tilde{M})^{-1} \end{pmatrix}, \tag{22}$$

i.e., MQ and $(-\tilde{N}K + \tilde{M})^{-1}$ is stable. It remains to show that Q is stable. Observe, see (8), that

$$K = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) = (U + MQ)(V + NQ)^{-1}$$

and thus

$$-\tilde{N}K + \tilde{M} = (-\tilde{N}U - \tilde{N}MQ + \tilde{M}V + \tilde{M}NQ)(V + NQ)^{-1} \\ = (V + NQ)^{-1}$$

Putting everything together, it follows that $U + MQ$ and $V + NQ$ are stable, hence

$$Q = (I \ 0) \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} U + MQ \\ V + NQ \end{pmatrix}$$

is also stable, as desired.

Concerning the simultaneous perturbation of the plant and controller we have the following results that extend slightly the Double Youla-Kucera approach:

Theorem 3.2. If $G \in \mathrm{dom}\,\mathcal{P}_{\tilde{\Sigma}}$ and $K \in \mathrm{dom}\,\mathcal{K}_{\tilde{\Sigma}}$ then denote by $\gamma = \mathcal{P}_{\tilde{\Sigma}}(G)$ and $\kappa = \mathcal{K}_{\tilde{\Sigma}}(K)$, respectively. Then

a) (G, K) is stable if and only if (γ, κ) is stable.
b) If both γ and κ are stable, then (G, K) is stable if $\|\gamma\|\|\kappa\| < 1$. (**Double Youla**)

c) if $\gamma \in \mathbb{S}^0_\Sigma$, then the pair (G, K) is stable if and only $K = \mathcal{K}_\Sigma(q_\gamma)$ with $q_\gamma = q(I + \gamma q)^{-1} \in \operatorname{dom}\mathcal{K}_\Sigma$ if for some $q \in \mathbb{Q}$ and the pair (γ, q_γ) is stable.
d) If $\kappa \in \mathbb{Q}^0_\Sigma$, then the pair (G, K) is stable if and only if $G = \mathcal{P}_\Sigma(s_\kappa)$ with $s_\kappa = s(I + \kappa s)^{-1} \in \operatorname{dom}\mathcal{P}_\Sigma$ for some $s \in \mathbb{S}$ and the pair (s_κ, κ) is stable.

Proof: If $G \in \operatorname{dom}\mathcal{P}_{\tilde{\Sigma}}$ and $K \in \operatorname{dom}\mathcal{K}_{\tilde{\Sigma}}$ then

$$\tilde{\Sigma}\begin{pmatrix} I & K \\ G & I \end{pmatrix} = \begin{pmatrix} \tilde{V} - \tilde{U}G & \tilde{V}K - \tilde{U} \\ -\tilde{N} + \tilde{M}G & -\tilde{N}K + \tilde{M} \end{pmatrix} = = \begin{pmatrix} I & \kappa \\ \gamma & I \end{pmatrix}\begin{pmatrix} \tilde{V} - \tilde{U}G & 0 \\ 0 & -\tilde{N}K + \tilde{M} \end{pmatrix},$$

i.e., we have the following loop transformation formula

$$\mathcal{H}(\gamma, \kappa) = \begin{pmatrix} \tilde{V} - \tilde{U}G & 0 \\ 0 & -\tilde{N}K + \tilde{M} \end{pmatrix}\mathcal{H}(G, K)\Sigma =$$

$$\left[\begin{pmatrix} \tilde{V} & 0 \\ 0 & \tilde{M} \end{pmatrix} + \begin{pmatrix} 0 & -\tilde{U} \\ -\tilde{N} & 0 \end{pmatrix}\left\{\begin{pmatrix} I & K \\ G & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}\right]\mathcal{H}(G, K)\Sigma$$

$$= \begin{pmatrix} 0 & -\tilde{U} \\ -\tilde{N} & 0 \end{pmatrix}\Sigma + \begin{pmatrix} \tilde{V} & \tilde{U} \\ \tilde{N} & \tilde{M} \end{pmatrix}\mathcal{H}(G, K)\Sigma. \quad (23)$$

It follows that $\mathcal{H}(G, K)$ is stable if and only if (γ, κ) is stable. Then Double Youla-Kucera follows from the Small Gain Theorem.

By symmetry it enough to prove *c*). Observe that Youla-Kucera is a special case of *c*) for $\gamma = 0$. In particular, if $\gamma \in \mathbb{S}^0_\Sigma$, let us denote by

$$\lambda_\gamma = \begin{pmatrix} I & 0 \\ \gamma & 0 \end{pmatrix},$$

and recall that $G \in \operatorname{dom}\mathcal{P}_{\tilde{\Sigma}}$, i.e., $\gamma \in \operatorname{dom}\mathcal{P}_\Sigma$. It follows, that $(\lambda_{-\gamma}\tilde{\Sigma}, \Sigma\lambda_\gamma)$ is a double coprime factorization for G. Then from Youla-Kucera it follows that $K = \mathcal{K}_{\Sigma\lambda_\gamma}(q)$ for some stable q. Since $K \in \operatorname{dom}\mathcal{K}_{\tilde{\Sigma}}$ it follows that $I + \gamma q$ exists. Moreover, $\kappa = \mathcal{K}_{\tilde{\Sigma}}(K) = q(I + \gamma q)^{-1}$, as stated.

Note, also from Youla, that the pair (γ, κ) is stable if and only if $\kappa = q_\gamma = \bar{q}(I + \gamma\bar{q})^{-1}$ for some $\bar{q} \in \mathbb{Q}$. Observe, that $q_\gamma \in \operatorname{dom}\mathcal{K}_\Sigma$ implies that $\bar{q} \in \operatorname{dom}\mathcal{K}_{\Sigma\lambda_\gamma}$, i.e., with $K = \mathcal{K}_{\Sigma\lambda_\gamma}(\bar{q}) = \mathcal{K}_\Sigma(\kappa)$ the pair (G, K) is stable.

Remark 3.1. Observe that due to the domain condition not the entire plane defined by the stable (G, K) pairs is mapped through $(\mathcal{P}_{\tilde{\Sigma}}(G), \mathcal{K}_{\tilde{\Sigma}}(K))$. Given (G_0, K_0) it would be interesting to infer that some neighbourhood, e.g., a perturbation $(G_0 + \Delta_g, K_0 + \Delta_k)$ with sufficiently small (in norm) stable Δ_g and Δ_k is also a stable pair. Since we cannot apply the small gain theorem, in general, Youla-Kucera parametrization is not of too much help here.

We encounter the same obstruction on the parameter side: from Theorem 3.2, *c*) we can infer that $\|\gamma\|\|q\| < 1$ ensures q_γ is stable, moreover (γ, q_γ) is stable. The problem is that there is no obvious bound that would guarantee well-posedness (the domain condition) for every (γ, q_γ) in the neighbourhood of the origin.

The problem is present even for Youla/dual Youla. As an example, the identity

$$\begin{pmatrix} M + \Delta & U \\ N & V \end{pmatrix} = \begin{pmatrix} M & U \\ N & V \end{pmatrix}\begin{pmatrix} I & 0 \\ S_\Delta & I \end{pmatrix}\begin{pmatrix} I + \tilde{V}\Delta & 0 \\ 0 & I \end{pmatrix},$$

with $S_\Delta = \tilde{N}\Delta(I + \tilde{V}\Delta)^{-1}$ reveals that $\exists(M + \Delta)^{-1}$ if and only if $S_\Delta \in \operatorname{dom}\mathcal{P}_\Sigma$, provided that $I + \tilde{V}\Delta$ is unimodular. The latter condition holds for $\|\Delta\| < 1/\|\tilde{V}\|$. However, concerning the former condition neither part can be inferred by applying elementary tools, even for sufficiently small Δs.

Remark 3.2. Another deficiency of the parametrization is revealed by attempting to define the indirect (Youla) blending as

$$K = \mathcal{K}_\Sigma((\mathcal{K}_{\tilde{\Sigma}}(K_1) + \mathcal{K}_{\tilde{\Sigma}}(K_2))), \quad (24)$$

where an obstruction appears if the sum of the Youla parameters is not in the domain of $\mathcal{K}_\Sigma$.

3.1 (G, K) vs. $(P, \bar{K})$ stability

Stability questions of LFT loops can be reduced to the investigation of the configuration determined by $(P, \bar{K})$, where $\bar{K} = \operatorname{diag}(0, K)$. It is obvious that the LFT loop is well–defined if and only if (G, K) is well defined. However, it is less obvious whether this claim remains true for stability. It is a fundamental application of the Youla-Kucera parametrization, that if there exists a double coprime factorization of G, then the stabilizing controller set of an LFT loop coincides with the set of all stabilizing controllers K of G, and the closed–loop for a stabilizing controller is given by

$$\mathcal{F}_l(P, K) = q_1 + q_2 Q q_3, \quad (25)$$

where Q is the Youla-Kucera parameter of K relative to the given double coprime factorization of G and q_1, q_2, q_3 are stable systems. We will show that this result is valid in general, without the assumption on the existence of a double coprime factorization.

4. A COORDINATE FREE CONTROLLER PARAMETRIZATION

It turns out that stability of the feedback loop implies a series of algebraic properties for the plant, controller and different sensitivities. In what follows we are to highlight the basic relations of this algebraic structure: we will focus, in particular, on a controller parametrization which is tight to this coordinate (factorization) free framework.

Note that by a direct computation, i.e., without any reference to some particular factorization of the plant or of the controller, we have

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ G_0 & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix}(K - K_0)\,(0 \;\; I).$$

Applying two times the matrix inversion lemma we get

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & K_0 \\ G_0 & I \end{pmatrix}^{-1} - \begin{pmatrix} S_u \\ S_g \end{pmatrix} R\,(S_g \;\; S_y), \quad (26)$$

i.e.,

$$\mathcal{H}(G_0, K) = \mathcal{H}(G_0, K_0)\begin{pmatrix} I - RS_g & -RS_y \\ 0 & I \end{pmatrix} = \quad (27)$$

$$= \begin{pmatrix} I & -S_u R \\ 0 & I - S_g R \end{pmatrix}\mathcal{H}(G_0, K_0), \quad (28)$$

with

$$R = (K - K_0)(I + S_g(K - K_0))^{-1} \quad (29)$$

and then

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ G_0 & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} R(I - S_g R)^{-1} \,(0\ I)\,. \quad (30)$$

Analogous to the Youla-Kucera parametrization one has

$$K = \mathcal{K}_{\Gamma_{G_0,K_0}}(R) = \mathcal{F}_l(\Psi_{G_0,K_0}, R), \quad (31)$$
$$R = \mathcal{K}_{\Gamma^{-1}_{G,K_0}}(K) = \mathcal{F}_l(\Phi_{G_0,K_0}, K), \quad (32)$$

with

$$\Gamma_{G_0,K_0} = \begin{pmatrix} S_u & K_0 \\ -S_g & I \end{pmatrix}, \quad \Psi_{G_0,K_0} = \begin{pmatrix} K_0 & I \\ I & S_g \end{pmatrix}, \quad (33)$$

$$\Gamma^{-1}_{G_0,K_0} = \begin{pmatrix} I & -K_0 \\ S_g & S_y \end{pmatrix}, \Phi_{G_0,K_0} = \begin{pmatrix} -K_0 S_y^{-1} & S_u^{-1} \\ S_y^{-1} & G_0 \end{pmatrix}, \quad (34)$$

with the parameter space

$$\mathbb{R}_{(G_0,K_0)} = \{\mathcal{F}_l(\Phi_{G_0,K_0}, K) \,|\, (G_0, K) \text{ stable}\}. \quad (35)$$

Observe that $\{0, K_0\} \subset \mathbb{R}_{(G_0,K_0)}$, moreover the representation independent set

$$\mathbb{Q}_{(G_0,K_0)} = \{Q \,|\, Q \text{ stable}, (I - S_g Q)^{-1} \text{ exists}\} \subset \mathbb{R}_{(G_0,K_0)},$$

i.e., we know a'priori a significant part of $\mathbb{R}_{(G_0,K_0)}$. We have revealed the geometry of this set by showing the intimate relation of the stability preserving controller blending and the corresponding operation on this parameter space.

If we keep the controller K_0 fixed, we can define an analogous parametrization for the stabilizing plants, which is skipped du to space limitations. Note, that in contrast to Youla-Kucera parametrization, controllers and plants for fixed plants (controllers) are related by different maps apart from the remarkable case, when on of the components G_0, or K_0, respectively, is stable.

Note that if K_0 is stable, then $\mathbb{R}_{(G_0,K_0)} = \mathbb{Q}_{(G_0,K_0)}$ and

$$\begin{pmatrix} I & -K_0 \\ S_g & S_y \end{pmatrix} \begin{pmatrix} S_u & K_0 \\ -S_g & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad (36)$$

provides a double coprime factorization given in terms of the original data. Moreover, we have the following result:

Theorem 4.1. $\mathbb{R}_{(G_0,K_0)} = \mathbb{Q}_{(G_0,K_0)}$ if and only if K_0 is stable.

For the reverse direction observe that $K_0 \in \mathbb{R}_{(G_0,K_0)}$, which is the parameter that corresponds to the stabilizing controller $K = 2K_0 - K_0 G_0 K_0$, see (48).

At this point it is immediate to observe the advantages of this parametrization compared to the Youla-Kucera parametrization concerning the issue raised in Remark 3.1. Considering a stable perturbation $\delta_k = K - K_0$, a small gain argument for (29) shows that there is a stable perturbation ball Δ, contained in the ball with radius $\frac{1}{\|S_g\|}$, such that the pair $(G_0, K_0 + \delta_k)$ is stable for all $\delta_k \in \Delta$. It is interesting to note that the corresponding parameters will be contained in a stable ball. In particular, if the controller K_0 is strongly stabilizing, then all the controllers from $K_0+\Delta$ are strongly stabilizing. Due to the symmetry, analogous role is played by S_c for G_0. Actually, the identity

$$\mathcal{H}(G_0 + \delta_g, K_0 + \delta_k) = (I + \mathcal{H}(G_0, K_0) \begin{pmatrix} 0 & \delta_k \\ \delta_g & 0 \end{pmatrix})^{-1} \mathcal{H}(G_0, K_0) \quad (37)$$

shows that for a sufficiently small stable perturbation $(G_0 + \delta_g, K_0 + \delta_k)$ will also be stable.

Starting from the identities

$$K - K_0 = R(I - S_g R)^{-1}, \quad (38)$$
$$I + (K - K_0)S_g = (I - RS_g)^{-1}, \quad (39)$$
$$I + S_g(K - K_0) = (I - S_g R)^{-1}, \quad (40)$$

we can infer that

$$\begin{aligned} I - RS_g &= (I + (K - K_0)S_g)^{-1} = \\ &= (I - K_0 G_0)(I - K G_0)^{-1} = S_u^{-1} S_u^K, \quad (41) \\ I - S_g R &= (I + S_g(K - K_0))^{-1} = \\ &= (I - G_0 K)^{-1}(I - G_0 K_0) = S_y^K S_y^{-1}. \quad (42) \end{aligned}$$

From (26) we have

$$(I\ K_0)\,\mathcal{H}(G_0, K) \begin{pmatrix} K_0 \\ I \end{pmatrix} = K_0 - R, \quad \text{i.e.,}$$
$$R = (K - K_0) S_y^K S_y^{-1} = S_u^{-1} S_u^K (K - K_0). \quad (43)$$

Let us denote by $\Delta_a^b = K_a - K_b$. Then, from (41)

$$\begin{aligned} &(I + (K_a - K_0)S_g)(I + (K_b - K_0)S_g)^{-1} = \\ &= (I - K_a G_0)(I - K_b G_0)^{-1}, \end{aligned}$$

i.e.,

$$(I + \Delta_a^0 S_g^0)(I + \Delta_b^0 S_g^0)^{-1} = (I + \Delta_b^a S_g^a)^{-1} \quad (44)$$
$$(I + S_g^0 \Delta_a^0)^{-1}(I + S_g^0 \Delta_b^0) = (I + S_g^b \Delta_a^b)^{-1}. \quad (45)$$

These relations are very similar to some operator valued cross ratios, and might be the starting point of some distance definition on the stable neigbourhood of (G_0, K_0). More precisely, observe that this ratio is stable for any $\Delta_a^0, \Delta_b^0 \in B_{1/\|S_g\|}$. Moreover, if we associate to Δ_a^0 a length defined by $\|I + \Delta_a^0 S_g^0\|$ (or $\|(I + \Delta_a^0 S_g^0)^{-1}\|$) then we can use the same formula for every hyperbolic ball with center in, e.g., K_a. Due to space limitation this topic will be developed elsewhere.

Having (37) one would be tempted to relate δ_k and δ_g to some parameters. Unfortunately, in general, this cannot be done. Neither we can provide a universal blending rule for stable pairs, i.e., an operation which renders a stable pair for two given stable pairs. However, somehow surprisingly, in the fixed plant (controller) case this is possible. The rest of this section is dedicated to provide some details.

4.1 Geometric description of the parameters

The basic observation

$$\begin{pmatrix} I & K \\ G & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - GK \end{pmatrix} \quad (46)$$

can be iterated as

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ G_0 & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - G_0 K_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - G_0 K_2 \end{pmatrix}. \quad (47)$$

which leads to the operation

$$K = K_1(I - G_0 K_2) + K_2 = K_1 \boxdot_G K_2, \quad (48)$$

under which well-posed controllers form a group $(\mathbb{W}_G, \boxdot_G)$. The unit of this group is the zero controller $K = 0_K$ and the corresponding inverse elements are given by

$$K^{\boxminus_G} = -K(I - G_0 K)^{-1}. \quad (49)$$

Note that

$$I - GK^{\boxminus_G} = (I - G_0 K)^{-1}. \quad (50)$$

While not all elements of $\mathbb{W}_G$ are stabilizing, e.g., 0_K is not stabilizing for an unstable plant, the set of stabilizing controllers endowed with this operaton, $(\mathbb{G}_G, \boxdot_G)$ is a semigroup. Note, that

$$(I - GK)^{-1} = (I - GK_2)^{-1}(I - GK_1)^{-1}. \quad (51)$$

It is a routine calculation to show that the blending of the inverses is related to the original blending as:

$$K = K_1 \boxdot_G K_2 \quad \text{iff} \quad K^{\boxminus_G} = K_2^{\boxminus_G} \boxdot_G K_1^{\boxminus_G}. \quad (52)$$

Let us write (47) as

$$\begin{pmatrix} I & K \\ G_0 & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ G_0 & I \end{pmatrix} \begin{pmatrix} I & S_c \\ 0 & S_y \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - G_0K_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - G_0K_2 \end{pmatrix}.$$

to obtain, according to (28),

$$\begin{pmatrix} I & -S_uR \\ 0 & I - S_gR \end{pmatrix} \mathcal{H}(G_0, K_0) = = \begin{pmatrix} I & S_c^{K_2} \\ 0 & S_y^{K_2} \end{pmatrix} \begin{pmatrix} I & S_c^{K_1} \\ 0 & S_y^{K_1} \end{pmatrix} \begin{pmatrix} I & K_0 \\ 0 & I - G_0K_0 \end{pmatrix} \mathcal{H}(G_0, K_0).$$

We have, see (41), (42) and (43), that

$$\begin{pmatrix} I & S_c^{K_1} \\ 0 & S_y^{K_1} \end{pmatrix} \begin{pmatrix} I & K_0 \\ 0 & I - G_0K_0 \end{pmatrix} = \begin{pmatrix} I & K_0 - K_1(I - S_gR_1) \\ 0 & I - S_gR_1 \end{pmatrix} = = \begin{pmatrix} I & -R_1 + K_0S_gR_1 \\ 0 & I - S_gR_1 \end{pmatrix} = \begin{pmatrix} I & -S_uR_1 \\ 0 & I - S_gR_1 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} I & -S_uR \\ 0 & I - S_gR \end{pmatrix} = \begin{pmatrix} I & -S_uR_2 \\ 0 & I - S_gR_2 \end{pmatrix} \begin{pmatrix} I & S_c \\ 0 & S_y \end{pmatrix} \begin{pmatrix} I & -S_uR_1 \\ 0 & I - S_gR_1 \end{pmatrix}$$

Analogous to the controller case, this factorization reveals that on $\mathbb{R}_{G_0,K_0}$ we have the (compatible) blending rule

$$R_2 \odot_{G_0,K_0} R_1 = K_0 + S_uR_1 + R_2S_y - R_2S_yS_gR_1. \quad (53)$$

The semigroup $(\mathbb{G}_G, \boxdot_G)$ does not have a unit, in general. However, if there is a stable stabilizing controller K_0, then $(\mathbb{G}_G, \boxtimes_G)$ with

$$K_1 \boxtimes_G K_2 = K_1 \boxdot_G K_0^{\boxminus_G} \boxdot_G K_2$$

is a semigroup with a unit (K_0). This may happen only if the plant is strongly stabilizable. If we denote by $\mathbb{S}_G$ the set of strongly stabilising controllers, then if this set is not empty, then $(\mathbb{S}_G, \boxtimes_G)$ with the operation (blending) defined as

$$K = K_1 \boxtimes_G K_2 = K_1 \boxdot_G K_0^{\boxminus_G} \boxdot_G K_2 = = K_2 + (K_1 - K_0)(I - GK_0)^{-1}(I - GK_2) \quad (54)$$

is the group of strongly stable controllers, where $K_0 \in \mathbb{S}_G$ is arbitrary. The corresponding inverse is given by

$$K^{\boxtimes_G^{-1}} = K_0 - (K - K_0)(I - GK)^{-1}(I - GK_0). \quad (55)$$

For the stable controllers the parameter blending is more simple:

$$R_2 \otimes_{G_0,K_0} R_1 = R_2 + R_1 - R_2S_gR_1, \quad (56)$$

$$R^{\otimes_{G_0,K_0}^{-1}} = -R(I - S_gR)^{-1}. \quad (57)$$

It is interesting to compare this with (53) in view of the fact that in this case Youla-Kucera corresponding to (36) met the coordinate free context, see Theorem 4.1.

5. GEOMETRY OF THE LFT LOOP

We conclude this paper showing that the well-known and fundamental consequence of the Youla-Kucera parametrization, i.e., that every controller which stabilizes the interior plant of an LFT also stabilizes the entire LFT loop, can be deduced in a completely coordinate free way. The result reveals that this fact is an intrinsic, fundamental property of every stable feedback loop regardless to the existence of some particular factorization.

Recall that

$$\mathcal{L}_{P,K} = \begin{pmatrix} \mathcal{H}(G,K) & \mathcal{H}(G,K)\begin{pmatrix} 0 \\ -P_{yw} \end{pmatrix} \\ (-P_{zu}\ 0)\,\mathcal{H}(G,K) & \mathcal{F}_l(P,K) \end{pmatrix} \quad (58)$$

$$\mathcal{F}_l(P,K) = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw} = = P_{zw} - (P_{zu}\ 0)\,\mathcal{H}(G,K)\begin{pmatrix} 0 \\ P_{yw} \end{pmatrix} \quad (59)$$

Applying (27) and (28) we have

$$\mathcal{L}_{P,K} = \mathcal{L}_{P,K_0} - \begin{pmatrix} S_u \\ S_g \\ -P_{zu}S_u \end{pmatrix} R\,(S_g\ \ S_y\ \ -S_yP_{yw}). \quad (60)$$

Observe that the affine dependence of type (25) of $\mathcal{F}_l(P,K)$ on the parameter is trivially satisfied. Note, however, that R should not be stable, in general.

We already know that if $(P, \bar{K}_1)$ and $(P, \bar{K}_2)$ is stable, then $(P, \bar{K})$ is also stable with

$$\bar{K} = \bar{K}_1 \boxdot_P \bar{K}_2. \quad (61)$$

It is immediate to verify, that

$$\bar{K} = \bar{K}_1 \boxdot_P \bar{K}_2 = \bar{K}_1 + \bar{K}_2 - \bar{K}_1P\bar{K}_2 = = \text{diag}(0, K_1 + K_2 - K_1GK_2) = \text{diag}(0, K_1 \boxdot_G K_2),$$

i.e., $K = K_1 \boxdot_G K_2$. Thus, well-definedness and stability of LFT loops is also a geometric property.

Thus, if $(P, \bar{K}_0)$ is stable, by using the blending rule (53) and observing that $\text{diag}(0, \mathbb{Q}_\Sigma) \subset \mathbb{R}_{\bar{K}_0}$ we know by start a significant part of the stabilizing controllers for the LFT loop. Moreover, if we denote by $\mathbb{K}_{K_0}$ the set of controllers generated by using (53) and $\mathbb{Q}_\Sigma$, then these controllers will stabilize the LFT loop, too. In what follows we will show that the LFT loop is stabilized by exactly those controllers that stabilize G. Let us suppose that

$$\mathcal{L}_{P,K_0} = \begin{pmatrix} \mathcal{H}(G,K_0) & \begin{pmatrix} -S_cP_{yw} \\ -S_yP_{yw} \end{pmatrix} \\ (-P_{zu}S_u\ \ -P_{zu}S_c) & \mathcal{F}_l(P,K_0) \end{pmatrix}$$

and $\mathcal{H}(G,K)$ is stable. We would like to show that $\mathcal{L}_{P,K}$ is stable.

From (43) by direct verification we have that

$$P_{zu}S_uRS_g = P_{zu}\left[S_uKS_y^K - S_uK_0S_y^K\right]S_y^{-1}S_g = = -P_{zu}S_uS_u^KKG + P_{zu}S_cS_c^K \quad (62)$$

and

$$-P_{zu}S_uRS_y = -P_{zu}\left[S_uS_c^K - S_cS_y^K\right]S_y^{-1}S_y = = -P_{zu}S_uS_c^K + P_{zu}S_cS_y^K, \quad (63)$$

is stable. Analogously, we have that

$$-S_uRS_yP_{yw} = -S_u(K - K_0)S_y^KS_y^{-1}S_yP_{yw} = -[S_uS_c^K + S_cS_y^K]S_yP_{yw} - [S_cS_u^KKG - S_cS_g^K]S_cP_{yw} \quad (64)$$

and

$$\begin{aligned}&- S_gRS_yP_{yw} = GS_u(K - K_0)S_y^KS_y^{-1}S_yP_{yw} =\\&[S_gS_c^K + GKS_yS_y^K]S_yP_{yw}+\\&+ [GKS_yS_u^KKG - GKS_yS_g^K]S_cP_{yw} \qquad (65)\end{aligned}$$

is stable. Finally we can verify that

$$\begin{aligned}&P_{zu}S_uRS_yP_{yw} = P_{zu}S_u(K - K_0)S_y^KS_y^{-1}S_yP_{yw} =\\&= -P_{zu}S_u[S_c^KS_yP_{yw} - S_u^KKGS_cP_{yw}]+\\&+ P_{zu}S_c[S_y^KS_yP_{yw} - S_g^KS_cP_{yw}] \qquad (66)\end{aligned}$$

is stable.

Now, we are in a position to summarize the stabilizability condition of (lower) LFT loops:

Theorem 5.1. The stabilizing controller set of an LFT loop coincides with the set of all stabilizing controllers K of G_0.

6. CONCLUSIONS

Based on an abstract algebraic setting we provide an elementary characterization of the set of stabilizing controllers both for the well-known Youla-Kucera parametrization and also in a completely coordinate free way, without any reference to a coprime factorization. While in this latter case the parameter set is not universal, its elements can be generated by a universal algorithm based on the direct blending operation of the stabilizing controllers.

Extending the framework to the LFT loops we show by using elementary tools that every controller which stabilizes the interior loop of the generalized plant also stabilizes the LFT loop.

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