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Fault Detection and Identification for Nonlinear MIMO Systems Using Derivative Estimation

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Abstract: In this paper a method for fault detection and identification of affine input nonlinear systems is presented, which is based on derivative estimation with orthonormal Jacobi polynomials. A systematic approach is presented to derive a residual and a differential algebraic expression of the fault from the system description, which solely depends on measurable input and output signals as well as on their time derivatives. For this, a systematic algorithm is provided, which can be directly implemented in computer algebra packages. Furthermore, arbitrary disturbances are taken into account, by making use of a disturbance decoupling. Fault detection and identification is then achieved by polynomial approximation of the determined fault or residual expression. The results are illustrated for a faulty point-mass satellite model.

Keywords: Fault detection and identification, nonlinear systems, disturbance decoupling, algebraic derivative estimation.

1. INTRODUCTION

Safety in technical processes is becoming increasingly important for plants and systems, especially in industrial applications. As the latter become more and more complex the vulnerability to faults is increasing. In order to guarantee the safety of processes, fault detection and identification methods gain more importance. The literature provides many approaches for linear and nonlinear systems. A broad overview of these methods can be found in Chen and Patton (1999) or especially for linear systems in Ding (2008). The solutions for fault detection and diagnosis are are mainly divided into model-free (see e.g. Frank and Köppen-Seliger (1997), Heo and Lee (2018)) and model-based approaches (see e.g. Ding (2008)). Model-free approaches can be especially problematic if the available data is not sufficient to detect occurring faults and thus a potential safety risk can arise, which is why the focus here is on model-based approaches. Model-based approaches are mainly implemented with the use of observers. However, this can cause considerable problems for the implementation, especially with nonlinear systems. In addition, fault detection and identification is made even more difficult in the case of additional disturbances affecting the system, which is why the approaches in this case can become very complex (see, e.g., De Persis and Isidori (2001)). A possible solution to avoid the challenging design of nonlinear fault detection observers is to determine a fault expression in terms of the known signals and their time derivatives. This expression can be obtained from observability tests for the faulty nonlinear system. However, this is a challenging problem, since in addition disturbances must be taken into account and the individual derivative estimators have to be parameterized accordingly to evaluate the residual.

In this paper, an algebraic method for model-based fault detection and identification is presented, which can be used for affine input nonlinear MIMO systems in the presence of disturbances. For this, the residual is determined directly by using the observability map and the observability normal form. This allows to express the residual and the fault in terms of the known inputs and measurements as well as the time derivatives thereof. In order to take disturbances into account, the expressions are decoupled from them by annihilating the disturbance input matrix. It is demonstrated that the explicit expression for the residual and the fault allows a systematic implementation in standard computer algebra packages. A particular benefit of the presented approach is that it can be easily implemented for real world applications, since the proposed algebraic derivative estimation can be realized in real-time by evaluation of weighted sums (see Lomakin and Deutscher (2019)).

This paper is organized as follows: In the next section a formulation of the considered fault detection problem is set up. Afterwards the requirements, which the system must fulfill are defined and a method of general fault detection and identification is presented. In Section 4 the polynomial approximation, which was presented in Lomakin and Deutscher (2019), is briefly reviewed and the application on the obtained residual and fault estimate signal is provided. The proposed method is finally demonstrated by a simulation of a faulty point-mass satellite.

2. PROBLEM FORMULATION

In this paper the nonlinear affine input system

$$\dot{x} = a(x) + B(x)u + G(x)d + E(x)f \tag{1a}$$
$$y = c(x) \tag{1b}$$

is considered, in which the state x is defined on an open set $\mathcal{X} \subseteq \mathbb{R}^n$ with the initial state $x(0) = x_0 \in \mathcal{X}$. In (1a) $u \in \mathbb{R}^{n_u}$ is the input, $f \in \mathbb{R}^{n_f}$ and $d \in \mathbb{R}^{n_d}$ are the unknown fault and disturbance. The output $y \in \mathbb{R}^{n_y}$ of (1) is assumed to be available for measurement. All elements of the matrices $B(x), E(x), G(x)$ are assumed to be known and sufficiently smooth, i.e. $C^k(\mathcal{X})$, and the vector fields $a(x) \in \mathbb{R}^n$ and $c(x) \in \mathbb{R}^{n_y}$ are assumed to be a sufficiently smooth, i.e. are in $(C^k(\mathcal{X}))^n$ and $(C^k(\mathcal{X}))^{n_y}$. It is furthermore assumed that $G(x)$ and $E(x)$ have full column rank, i.e. $\operatorname{rank} G(x) = n_d, \forall x \in \mathcal{X}$ and $\operatorname{rank} E(x) = n_f, \forall x \in \mathcal{X}$ with $n_d, n_f \leq n_y$.

The fault detection and identification problems addressed in this paper are characterized as follows:

For a given system (1), find a *residual signal* r

$$r = \Phi(y, \dot{y}, ..., y^{(k_1)}, u, \dot{u}, ..., u^{(k_2)}) \in \mathbb{R}^{n_r}, \tag{2}$$

with $k_1, k_2 \in \mathbb{N}$ such that the conditions

(I) $\lim_{t\to\infty} \|r\| = 0, \qquad \forall\, f = 0$
(II) $r \neq 0, \qquad \forall\, f \neq 0$

are satisfied for any input u, any disturbance d and any initial state x_0. Then, the fault f can be detected by the residual signal r. If additionally to (I) and (II) for any two particular faults f_i and f_j, with $f_i, f_j \in \mathbb{R}^{n_f}$ and $f_j \neq f_i$, the corresponding residual signals r_i and r_j can be distinguished within any finite time interval $\mathcal{I}_t$, the fault f can also be isolated by the residual signal r. If furthermore an estimation for f_i and f_j can be derived from r for any finite time interval $\mathcal{I}_t$, the fault is assumed to be identifiable. Henceforth, the addressed problem will be regarded as the residual generation problem for fault detection (RGP-FD) and fault identification (RGP-FDI), respectively.

Remark 1. It should be noted that if further restrictions are made or additional information is available, unambiguous detection and/or identification is also possible if $n_d, n_f \geq n_y$. □

3. RESIDUAL GENERATION FOR FAULTDETECTION

In order to solve the presented fault detection and identification problem, it is necessary to determine a residual signal r, which can be calculated from the measurable quantities u and y, as well as their time derivatives only. For this purpose, the state x is successively replaced by these signals with aid of a local diffeomorphism and dynamic extension. Then, a differential algebraic expression is obtained to determine the fault f or the residual r according to (2). Subsequently, it is shown how the time derivatives of y and u in r are reconstructed by an algebraic derivative estimation and how fault detection and identification can be performed solely on the basis of y and u in the presence of unknown disturbances.

The aforementioned problem, of course, is linked to observability properties of the system (1). For this, the observability of the *drift system*

$$\dot{x} = a(x), \quad x(0) = x_0 \in \mathcal{X} \tag{3a}$$
$$y = c(x), \tag{3b}$$

resulting from (1) by omitting the input signals is recalled.

Definition 2. (Drift observability, see, e.g., Mora et al. (2000)) If there exist n_y constants $\bar{s}_1, ..., \bar{s}_{n_y}$ such that $\sum_{i=1}^{n_y} \bar{s}_i = n$ and the *observability map* $\Phi_o : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_1^{(\bar{s}_1-1)} \\ \vdots \\ y_{n_y} \\ \vdots \\ y_{n_y}^{(\bar{s}_{n_y}-1)} \end{bmatrix} = \Phi_o(x) = \begin{bmatrix} c_1(x) \\ \vdots \\ \mathrm{L}_a^{\bar{s}_1-1} c_1(x) \\ \vdots \\ c_{n_y}(x) \\ \vdots \\ \mathrm{L}_a^{\bar{s}_{n_y}-1} c_{n_y}(x) \end{bmatrix} \tag{4}$$

is a local diffeomorphism on a neighborhood $U(x^0)$ of x^0, then the system (1) is *drift observable* on $U(x^0)$. ◁

According to the implicit function theorem, (4) is locally diffeomorphic in a neighborhood of x^0, if and only if the corresponding Jacobian $\mathrm{d}\Phi_o$ satisfies

$$\det \mathrm{d}\Phi_o(x^0) \neq 0. \tag{5}$$

Remark 3. The selection of the parameters $\bar{s}_i$ may provide a degree of freedom, which can be used to reduce the derivative orders $\bar{s}_i$ of the outputs to avoid noisy measured values for fault diagnosis. □

In addition to the drift observability, the system (1) has to remain observable in the presence of the known inputs u for $f = 0$ and $d = 0$.

To this end, the coordinates ξ are defined according to

$$\xi = \begin{bmatrix} \xi_1^1 \\ \vdots \\ \xi_{\bar{s}_1}^1 \\ \vdots \\ \xi_1^{n_y} \\ \vdots \\ \xi_{\bar{s}_{n_y}}^{n_y} \end{bmatrix} = \begin{bmatrix} c_1(x) \\ \vdots \\ \mathrm{L}_a^{\bar{s}_1-1} c_1(x) \\ \vdots \\ c_{n_y}(x) \\ \vdots \\ \mathrm{L}_a^{\bar{s}_{n_y}-1} c_{n_y}(x) \end{bmatrix}. \tag{6}$$

Definition 4. (Complete uniform observability, see Gauthier and Bornard (1981)) Consider a system (1), which is drift observable on $U(x^0)$ so that the local diffeomorphism $\xi = \Phi_o(x)$ exists. If it is possible to find a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that by applying the diffeomorphism $\tilde{\xi} = P\Phi_o(x)$, with inverse $x = \Phi_o^{-1}(P^\top \tilde{\xi})$, the system (1) can be mapped to the cascaded form, i.e. the *observability normal form*

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \sum_{i=1}^{n_u} \tilde{b}_{1,i}(\tilde{\xi}_1) u_i \\ \dot{\tilde{\xi}}_2 &= \tilde{\xi}_3 + \sum_{i=1}^{n_u} \tilde{b}_{2,i}(\tilde{\xi}_1, \tilde{\xi}_2) u_i \\ &\vdots \\ \dot{\tilde{\xi}}_{n-1} &= \tilde{\xi}_n + \sum_{i=1}^{n_u} \tilde{b}_{n-1,i}(\tilde{\xi}_1, ..., \tilde{\xi}_{n-1}) u_i \\ \tilde{\xi}_1 &= y, \end{aligned} \tag{7}$$

then the system (1) is *complete uniform observable* on $U(x^0)$.◁

Obviously, (7) can be directly solved for $\xi_i, i = 1, ..., n$, yielding the explicit differential algebraic expression

$$\xi = \Psi(\bar{y}, \bar{u}) \tag{8}$$

with $\bar{y} = \mathrm{col}\left(y_1, ..., y_1^{(\bar{s}_1-1)}, ..., y_{n_y}, ..., y_{n_y}^{(\bar{s}_{n_y}-1)}\right)$, $\bar{u} = \mathrm{col}\left(u_1, ..., u_1^{(k_1)}, ..., u_{n_u}, ..., u_{n_u}^{(k_{n_u})}\right)$ and the constants $k_1 \leq$

$n-1, ..., k_{n_u} \leq n-1$. Additionally, $\dot{\xi}$ follows directly by differentiation of (8) resulting in

$$\dot{\xi} = \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\bar{y},\bar{u}) = \frac{\partial}{\partial \bar{y}}\Psi(\bar{y},\bar{u})\dot{\bar{y}} + \frac{\partial}{\partial \bar{u}}\Psi(\bar{y},\bar{u})\dot{\bar{u}} = \tilde{\Psi}(\bar{y},\dot{\bar{y}},\bar{u},\dot{\bar{u}}). \tag{9}$$

Remark 5. The property of describing ξ and $\dot{\xi}$ as an expression of $\bar{y}$ and $\bar{u}$ is analogous to the definition of *differential algebraic observability* (see, e.g., Martinez-Guerra et al. (2013)). □

In order to avoid time derivatives of the fault and the disturbance in the expression for the residual, it has to be assumed that for all $k = 1, ..., n_y$ the two conditions

$$\mathrm{L}_{g_i}\mathrm{L}_a^{j-1}c_k(x) = 0, \quad i = 1, ..., n_d,\ j = 1, ..., \bar{s}_k - 1 \tag{10a}$$

and

$$\mathrm{L}_{e_i}\mathrm{L}_a^{j-1}c_k(x) = 0, \quad i = 1, ..., n_f,\ j = 1, ..., \bar{s}_k - 1 \tag{10b}$$

are fulfilled for all $x \in U(x^0)$. This means that the relative degree of the output y_k, $k = 1, ..., n_y$, w.r.t. disturbances and faults is greater than $\bar{s}_k$, which is determined by (4).

Remark 6. Although the requirements (10) seem to be restrictive, they are met for many systems such as fully actuated rigid robots, even if elastic couplings in the joints are considered. □

For complete uniform observable systems (1) with (10) satisfied, the vector $\xi_{\bar{s}} = \mathrm{col}\left(\xi^1_{\bar{s}_1}, ..., \xi^{n_y}_{\bar{s}_{n_y}}\right) \in \mathbb{R}^{n_y}$ can be obtained by premultiplication of ξ with a selection matrix $S \in \mathbb{R}^{n_y \times n}$, i.e.

$$\xi_{\bar{s}} = S\xi. \tag{11}$$

The time derivative of $\xi_{\bar{s}}$ thus satisfies

$$\dot{\xi}_{\bar{s}} = \gamma(\xi) + \Lambda_u(\xi)u + \Lambda_d(\xi)d + \Lambda_f(\xi)f \tag{12}$$

on $U(x^0)$ with $\gamma(\xi) \in \mathbb{R}^{n_y}, \Lambda_u(\xi) \in \mathbb{R}^{n_y \times n_u}, \Lambda_d(\xi) \in \mathbb{R}^{n_y \times n_d}$ and $\Lambda_f(\xi) \in \mathbb{R}^{n_y \times n_f}$ given by

$$\gamma(\xi) = \begin{bmatrix} \mathrm{L}_a^{\bar{s}_1}c_1 \\ \vdots \\ \mathrm{L}_a^{\bar{s}_{n_y}}c_{n_y} \end{bmatrix} \circ \Phi_o^{-1}(\xi) \tag{13a}$$

$$\Lambda_u(\xi) = \begin{bmatrix} \mathrm{L}_{b_1}\mathrm{L}_a^{\bar{s}_1-1}c_1 & \cdots & \mathrm{L}_{b_{n_u}}\mathrm{L}_a^{\bar{s}_1-1}c_1 \\ \vdots & \ddots & \vdots \\ \mathrm{L}_{b_1}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} & \cdots & \mathrm{L}_{b_{n_u}}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} \end{bmatrix} \circ \Phi_o^{-1}(\xi) \tag{13b}$$

$$\Lambda_d(\xi) = \begin{bmatrix} \mathrm{L}_{g_1}\mathrm{L}_a^{\bar{s}_1-1}c_1 & \cdots & \mathrm{L}_{g_{n_d}}\mathrm{L}_a^{\bar{s}_1-1}c_1 \\ \vdots & \ddots & \vdots \\ \mathrm{L}_{g_1}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} & \cdots & \mathrm{L}_{g_{n_d}}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} \end{bmatrix} \circ \Phi_o^{-1}(\xi) \tag{13c}$$

$$\Lambda_f(\xi) = \begin{bmatrix} \mathrm{L}_{e_1}\mathrm{L}_a^{\bar{s}_1-1}c_1 & \cdots & \mathrm{L}_{e_{n_f}}\mathrm{L}_a^{\bar{s}_1-1}c_1 \\ \vdots & \ddots & \vdots \\ \mathrm{L}_{e_1}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} & \cdots & \mathrm{L}_{e_{n_f}}\mathrm{L}_a^{\bar{s}_{n_y}-1}c_{n_y} \end{bmatrix} \circ \Phi_o^{-1}(\xi). \tag{13d}$$

Subsequently, it is always assumed that a transformation $\tilde{\xi} = P\Phi_o(x)$ into the representation (12) exists, since it is used for the further analysis of fault detection and identification. For notational convenience, the dependencies of ξ in the vector $\gamma(\xi)$ and the matrices $\Lambda_u(\xi), \Lambda_d(\xi)$ and $\Lambda_f(\xi)$ are not displayed in the subsequent section.

Remark 7. Since is not trivial to obtain the representation (12) and also to test, whether the transformation $\tilde{\xi} = P\Phi_o(x)$ exists, the Algorithm 1 is provided in the appendix. This algorithm generates such a representation, if it exists, and can be implemented with computer algebra software such as the symbolic math toolbox of Matlab or Mathematica. □

By rearranging (12) the quantity

$$r_d = \dot{\xi}_{\bar{s}} - \gamma - \Lambda_u u = \Lambda_d d + \Lambda_f f \tag{14}$$

can be defined such that condition (I) is fulfilled if d is neglected. However, in order to fulfill condition (II) obviously $\mathrm{rank}\Lambda_f(\xi) = n_f$ has to hold for all $\xi \in U(\xi^0)$, with $\xi^0 = \Phi_o(x^0)$. If unknown disturbances d affect the system, it is necessary to decouple them from the residual. For this purpose, the *left-sided annihilator* $\Lambda_d^{\perp} \in \mathbb{R}^{n_y \times n_y}$ of Λ_d, i.e. $\Lambda_d^{\perp}\Lambda_d = 0$, $\forall \xi \in U(\xi^0)$, and

$$\Lambda_d^{\perp} = I - \Lambda_d\Lambda_d^{\dagger} \tag{15}$$

is introduced, in which

$$\Lambda_d^{\dagger} = (\Lambda_d^{\top}\Lambda_d)^{-1}\Lambda_d^{\top} \in \mathbb{R}^{n_d \times n} \tag{16}$$

is the *Moore-Penrose generalized inverse* (see, e.g., Penrose (1955)). Note that (16) always exists, because $\mathrm{rank}G = n_d$ (cf. Sec.2) and (10a) imply $\mathrm{rank}\Lambda_d = n_d$ on $U(\xi^0)$. By premultiplication of (14) with the annihilator $\Lambda_d^{\perp}$, the influence of the disturbance d on r_d is eliminated yielding the desired residual

$$r = \Lambda_d^{\perp}\left(\dot{\xi}_{\bar{s}} - \gamma - \Lambda_u u\right) = \Lambda_d^{\perp}\Lambda_f f, \quad \forall f \in \mathbb{R}^{n_f} \tag{17}$$

according to (2). It is unequal to the zero vector if any fault f is present, as long as

$$\mathrm{rank}\Lambda_d^{\perp}\Lambda_f(\xi) = n_f, \quad \xi \in U(\xi^0), \tag{18}$$

which is a sufficient condition for *fault detectability*. Then, r solves the RGP-FD and can be used for fault detection. In view of (18) also fault identification is achievable. For this, (17) is solved for f to obtain

$$f = (\Lambda_d^{\perp}\Lambda_f)^{\dagger}\Lambda_d^{\perp}\left(\dot{\xi}_{\bar{s}} - \gamma - \Lambda_u u\right). \tag{19}$$

Therein, $(\Lambda_d^{\perp}\Lambda_f)^{\dagger}$ is the Moore-Penrose generalized inverse of $\Lambda_d^{\perp}\Lambda_f$. Since (17) is satisfied for any vector $f \in \mathbb{R}^{n_f}$ the solution (19) exists uniquely. If $\mathrm{rank}(\Lambda_d^{\perp}\Lambda_f) = \tilde{n}_f < n_f$ then the solution of (17) is no longer unique and thus f cannot be determined unambiguously. However, it is possible to realize fault identification at least partially (see (Lomakin and Deutscher, 2019, Section. III D)).

The remaining problem is that the determined residual and fault estimate depend on ξ and $\dot{\xi}$ and therefore cannot be evaluated directly. In (17) and (19) the unknown variables ξ and $\dot{\xi}$ can be replaced according to the differential algebraic formulation in (8) and (9), whereby the expressions are only dependent on y and u, as well as on their time derivatives. Hence, analogous to the description in (2) the residual r and the fault f can be written as

$$r = \Phi_r(\bar{y}, \dot{\bar{y}}, \bar{u}, \dot{\bar{u}}) \in \mathbb{R}^{n_r} \tag{20}$$

and

$$f = \Phi_f(\bar{y}, \dot{\bar{y}}, \bar{u}, \dot{\bar{u}}) \in \mathbb{R}^{n_f}. \tag{21}$$

4. ALGEBRAIC FAULT DETECTION AND IDENTIFICATION

In order to implement (20), (21) the time derivatives have to be computed on the basis of u and y. For this, the methods of algebraic derivative estimation are introduced. The derivative estimation, already shown in Kiltz et al. (2012) and Kiltz and Rudolph (2013), is adapted in this paper to determine the polynomial approximation of any function by an integral transformation. This method is briefly reviewed in this section to show how the procedure can be used to substitute time derivatives successively. For a detailed explanation of the procedure, the reader is referred to the paper Lomakin and Deutscher (2019).

For the consideration of algebraic approximation approaches, first a signal $x(t)$ is defined, which is quadratically Lebesgue integrable within any finite time interval $\mathcal{I}_{t,T} = [t-T, t]\,, T > 0$, and furthermore the k-th derivative $x^{(k)}$ exists, i.e. $x \in L_2(t-T, t) \cap C^{k-1}[t-T, t]$ and $x^{(k)}\,\exists$. By defining the bijective transformation $\phi_T : \tilde{\mathcal{I}} = [-1,1] \mapsto \mathcal{I}_{t,T}$ and the corresponding inverse mapping ϕ_T^{-1}, the transformed function $\bar{x} = x \circ \phi_T$ can be defined on a Hilbert space $\mathcal{H} = L_2(-1,1)$ with the inner product

$$\langle \varphi_i, \varphi_j \rangle = \int_{-1}^{1} \varphi_i(\tau)\varphi_j(\tau) w^{(\alpha,\beta)}(\tau)\mathrm{d}\tau, \ \ \forall \varphi_i, \varphi_j \in \mathcal{H}, \tag{22}$$

and the induced norm

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}, \quad \forall \varphi \in \mathcal{H}. \tag{23}$$

The weight function $w^{(\alpha,\beta)}$, which allows to consider Jacobi polynomials as an orthonormal basis for $\mathcal{H}$, is given by

$$w^{(\alpha,\beta)}(\tau) = \begin{cases} (1-\tau)^\alpha (1+\tau)^\beta, & \tau \in [-1,1] \\ 0, & \tau \notin [-1,1]\,, \end{cases} \tag{24}$$

with real exponential coefficients $\alpha, \beta > -1$ as a degree of freedom. Then, it is possible to introduce an orthonormal basis $P_i^{(\alpha,\beta)}, i \in \mathbb{N}_0$ for $\mathcal{H}$ by the normalized Jacobi polynomials $P_i^{(\alpha,\beta)}$ (see (Szegö, 1959, Sec. 4.3)). Then, according to the projection theorem (see, e.g. Luenberger (1997)) the best fitting (in the least squares sense) approximation of N-th order $\hat{x} \in \mathcal{H}$ of $\bar{x}$ always exists unambiguously, and can be calculated by

$$\hat{x}(\tau) = \sum_{i=0}^{N} \langle \bar{x}, P_i^{(\alpha,\beta)} \rangle P_i^{(\alpha,\beta)}(\tau), \qquad \tau \in [-1,1]\,. \tag{25}$$

The approximation (25) can be evaluated at any time $t' \in \mathcal{I}_{t,T}$. In order to reduce the approximation error by 1 (see Mboup et al. (2009)) a delay $t_d \geq 0$ is added, which is selected as zero $p_{N+1}^{(\alpha,\beta)}$ of the Jacobi polynomial $P_{N+1}^{(\alpha,\beta)}$. The delayed polynomial approximation of x based on (25) can thus be written as

$$\hat{x}(t - t_d) = \langle x \circ \phi_T, R_{N,t_d}^{(\alpha,\beta)} \rangle \tag{26}$$

with

$$R_{N,t_d}^{(\alpha,\beta)}(\tau) = \sum_{i=0}^{N} P_i^{(\alpha,\beta)}(\tau)\,(P_i^{(\alpha,\beta)} \circ \phi_T^{-1}(t - t_d)). \tag{27}$$

The definition of the inner product (22) and also the substitution $\bar{\tau} = t - \phi_T(\tau)$ can be used to represent $\hat{x}(t-t_d)$ by the integral

$$\hat{x}(t - t_d) = \int_0^T x(t-\bar{\tau}) g_{N,t_d}(\bar{\tau})\mathrm{d}\bar{\tau} =: \mathcal{P}_{N,t_d}\{x\}(t), \tag{28}$$

within the original time window $\mathcal{I}_{t,T}$ and with the kernel

$$g_{N,t_d}(\bar{\tau}) = \frac{2}{T}(R_{N,t_d}^{(\alpha,\beta)}\, w^{(\alpha,\beta)}) \circ \phi_T^{-1}(t - \bar{\tau})\,, \tag{29}$$

which is independent of t, since $\phi_T^{-1}(t-\bar{\tau}) = 1 - \frac{2}{T}\bar{\tau}$. Because the kernel (29) and its $k-1$ derivatives have a compact support due to (24) in $[0, T]$ for $\alpha, \beta \geq k$, the polynomial approximation $x^{(k)}$ can be calculated by successive application of integration by parts yielding

$$\begin{aligned} \mathcal{P}_{N,t_d}\{x^{(k)}\}(t) &= \int_0^T x^{(k)}(t-\tau) g_{N,t_d}(\tau)\mathrm{d}\tau \\ &= \int_0^T x(t-\tau) g_{N,t_d}^{(k)}(\tau)\mathrm{d}\tau =: \mathcal{P}_{N,t_d}^{(k)}\{x\}(t) \end{aligned} \tag{30}$$

with the derivative of the kernel given by

$$g_{N,t_d}^{(k)}(\tau) = (-1)^k \frac{2}{T}(R_{N,t_d}^{(\alpha,\beta)} w^{(\alpha,\beta)})^{(k)} \circ \phi_T^{-1}(t-\tau). \tag{31}$$

The polynomial approximation of $x^{(k)}$ can thus be calculated by applying of the differentiation approximation operator $\mathcal{P}_{N,t_d}^{(k)}\{\cdot\}$ to x, i.e., only evaluating integrals.

Furthermore, the polynomial approximation operator $\mathcal{P}_{N,t_d}\{\cdot\}$ is linear and the other properties defined in Lomakin and Deutscher (2019) apply accordingly.

The polynomial approximation can now be applied to the previously defined residual and fault estimate. In order to evaluate (20) and (21), all polynomial approximations of y and u, as well as their time derivatives, must be calculated and then substituted into (20) and (21) such that

$$\hat{r} \approx \Phi_r(\mathcal{P}_{N,t_d}\{\bar{y}\}, \mathcal{P}_{N,t_d}\{\dot{\bar{y}}\}, \mathcal{P}_{N,t_d}\{\bar{u}\}, \mathcal{P}_{N,t_d}\{\dot{\bar{u}}\}), \tag{32}$$

and

$$\hat{f} \approx \Phi_f(\mathcal{P}_{N,t_d}\{\bar{y}\}, \mathcal{P}_{N,t_d}\{\dot{\bar{y}}\}, \mathcal{P}_{N,t_d}\{\bar{u}\}, \mathcal{P}_{N,t_d}\{\dot{\bar{u}}\}) \tag{33}$$

result. Thus, (I) and (II) of the RGP-FDI can be fulfilled and fault detection and identification can be performed solely from the measurable variables u and y.

The application of the polynomial approximation however does not affect the properties of the residual w.r.t. the decoupling of the input u and the disturbance d and can be determined sufficiently well according to the choice of the polynomial degree N. By applying the operator $\mathcal{P}_{N,t_d}\{\cdot\}$ to (17), using its linearity and by substitution of ξ and $\dot{\xi}_{\bar{s}}$ according to (8), (9) and (11), the polynomial approximation of r follows from

$$\begin{aligned} \hat{r} &= \mathcal{P}_{N,t_d}\{r\} = \mathcal{P}_{N,t_d}\{\Lambda_d^{\perp}\left(\dot{\xi}_{\bar{s}} - \gamma - \Lambda_u u\right)\} \\ &= \mathcal{P}_{N,t_d}\{(\Lambda_d^{\perp} \circ \Psi(\bar{y}, \bar{u}))S\tilde{\Psi}(\bar{y}, \dot{\bar{y}}, \bar{u}, \dot{\bar{u}})\} \\ &\quad - \mathcal{P}_{N,t_d}\{(\Lambda_d^{\perp}\gamma) \circ \Psi(\bar{y}, \bar{u})\} - \mathcal{P}_{N,t_d}\{(\Lambda_d^{\perp}\Lambda_u) \circ \Psi(\bar{y}, \bar{u})u\} \end{aligned} \tag{34}$$

Then, every single element in $\bar{y}, \dot{\bar{y}}, \bar{u}$ and $\dot{\bar{u}}$ can be determined individually with the aid the differentiation approximation operator (see Lomakin and Deutscher (2019)). Similarly, the operator can be applied to (19) to obtain the reconstructed fault $\hat{f}$ yielding

$$\begin{aligned} \hat{f} &= \mathcal{P}_{N,t_d}\{f\} = \mathcal{P}_{N,t_d}\{(\Lambda_d^{\perp}\Lambda_f)^{\dagger}\Lambda_d^{\perp}\left(\dot{\xi}_{\bar{s}} - \gamma - \Lambda_u u\right)\} \\ &= \mathcal{P}_{N,t_d}\{((\Lambda_d^{\perp}\Lambda_f)^{\dagger}\Lambda_d^{\perp}) \circ \Psi(\bar{y}, \bar{u})S\tilde{\Psi}(\bar{y}, \dot{\bar{y}}, \bar{u}, \dot{\bar{u}})\} \\ &\quad - \mathcal{P}_{N,t_d}\{((\Lambda_d^{\perp}\Lambda_f)^{\dagger}\Lambda_d^{\perp}\gamma) \circ \Psi(\bar{y}, \bar{u})\} \\ &\quad - \mathcal{P}_{N,t_d}\{((\Lambda_d^{\perp}\Lambda_f)^{\dagger}\Lambda_d^{\perp}\Lambda_u) \circ \Psi(\bar{y}, \bar{u})u\}. \end{aligned} \tag{35}$$

Remark 8. According to the calculation rules of the polynomial approximation operator described in Lomakin and Deutscher (2019), e.g., the partial approximation, it may be advisable to apply them to the polynomial approximation of the residual (34) or the reconstructed fault (35), instead of substituting the derivatives in $\bar{y}, \dot{\bar{y}}, \bar{u}$ and $\dot{\bar{u}}$, to further decompose the terms and make the approximation more accurate. □

5. EXAMPLE

In this section, the presented method is applied to a nonlinear example taken from the literature in order to highlight the advantages of the new fault diagnosis approach in comparison.

As example the faulty satellite from De Persis and Isidori (2001) is considered, which was slightly adapted by a changed, angle-dependent effect of the disturbance and is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{\theta_1}{x_1^2} \\ x_4 \\ -\frac{2x_2x_4}{x_1}\end{bmatrix} + \begin{bmatrix}0&0\\ \theta_2&0\\ 0&0\\ 0&\frac{\theta_2}{x_1}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}0\\ 2+\cos x_3\\ 0\\ \frac{\theta_2}{x_1}\end{bmatrix} d$$
$$+ \begin{bmatrix}0&0&0&\frac{\theta_2}{x_1}\end{bmatrix}^\top f, \quad x_1 \neq 0 \tag{36a}$$
$$y = \begin{bmatrix}x_1 & x_3 & x_4\end{bmatrix}^\top, \tag{36b}$$

wherein the position $(\rho, \phi) = (x_1, x_3)$ is given in polar coordinates and the radial and angular velocities v and ω are represented by x_2 and x_4, respectively. The inputs u_1 and u_2 are the radial and tangential thrust, where the fault f is an actuator fault for u_2 and d is the disturbance. The parameters θ_1 and θ_2 are assumed to be known and nonzero. Though it can be shown that the algorithm of De Persis and Isidori (2001) is applicable to (36), its evaluation becomes very tedious. For the presented method, however, the expressions for the reconstruction of the fault and the residual can be directly determined with the presented algebraic methods.

In order to check drift observability for (36), it is verified that the observability map (4) is a local diffeomorphism for $\bar{s}_1 = 2, \bar{s}_2 = 1$ and $\bar{s}_3 = 1$. Then, by application of the Algorithm 1, which was implemented in Matlab with the Symbolic Math Toolbox, to the system (36) the representation according to (12) yields

$$\begin{bmatrix}\dot{\xi}^1_2\\ \dot{\xi}^2_1\\ \dot{\xi}^3_1\end{bmatrix} = \underbrace{\begin{bmatrix}\xi^1_1(\xi^3_1)^2 - \frac{\theta_1}{(\xi^1_1)^2}\\ \xi^3_1\\ -\frac{2\xi^1_2\xi^3_1}{\xi^1_1}\end{bmatrix}}_{\gamma(\xi)} + \underbrace{\begin{bmatrix}\theta_2&0\\ 0&0\\ 0&\frac{\theta_2}{\xi^1_1}\end{bmatrix}}_{\Lambda_u(\xi)}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix}2+\cos\xi^2_1\\ 0\\ \frac{\theta_2}{\xi^1_1}\end{bmatrix}}_{\Lambda_d(\xi)} d + \underbrace{\begin{bmatrix}0\\ 0\\ \frac{\theta_2}{\xi^1_1}\end{bmatrix}}_{\Lambda_f(\xi)} f, \quad \xi^1_1 \neq 0. \tag{37}$$

Then, the annihilator $\Lambda_d^\perp$ results in

$$\Lambda_d^\perp(\xi) = \left(\frac{\theta_2^2}{(\xi^1_1)^2} + \left(\cos\xi^2_1 + 2\right)^2\right)$$
$$\cdot \begin{bmatrix}\frac{\theta_2^2}{(\xi^1_1)^2} & 0 & \frac{-\theta_2\left(\cos\xi^2_1+2\right)}{\xi^1_1}\\ 0 & \frac{\theta_2^2 + (\xi^1_1)^2\left(\cos\xi^2_1+2\right)^2}{(\xi^1_1)^2} & 0\\ \frac{-\theta_2\left(\cos\xi^2_1+2\right)}{\xi^1_1} & 0 & \left(\cos\xi^2_1+2\right)^2\end{bmatrix} \tag{38}$$

and correspondingly

$$(\Lambda_d^\perp\Lambda_f)^\dagger(\xi) = \begin{bmatrix}-\frac{1}{\cos\xi^2_1+2} & 0 & \frac{\xi^1_1}{\theta_2}\end{bmatrix}. \tag{39}$$

Since $\operatorname{rank}\Lambda_d^\perp\Lambda_f(\xi) = 1$ is valid for all $\{\xi \in \mathbb{R}^n | \xi^1_1 \neq 0\}$, the condition (18) is fulfilled in any neighborhood $U(\xi^0)$ of ξ^0

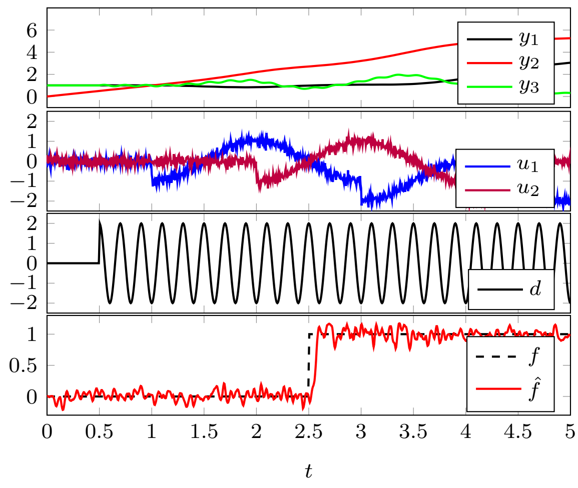

Fig. 1. Simulation results for the measured values as well as the corresponding input signals, the applied disturbance d and the obtained fault f with its estimate $\hat{f}$, in the presence of measurement and process noise of the point-mass satellite.

with $\xi^1_1 \neq 0$ and thus fault identification is possible. Therefore, the fault can be identified according to (19) and is given by

$$f = \frac{y_1\dot{y}_3}{\theta_2} - u_2 + \frac{2\,\dot{y}_1\,y_3}{\theta_2} + \frac{\theta_2 u_1 - \ddot{y}_1 + y_1\,y_3^2 - \frac{\theta_1}{y_1^2}}{\cos y_2 + 2}. \tag{40}$$

To eliminate the remaining time derivatives of y in (40), the polynomial approximation operator $\mathcal{P}_{N,t_d}\{\cdot\}$ is applied according to (35) with the aid of the partial approximation (see Lomakin and Deutscher (2019)) and yields to

$$\hat{f} = \frac{1}{\theta_2}\sum_{i=0}^{N^*}\mathcal{P}_{N,0,i}\{y_1\}\tilde{\mathcal{P}}^{(1)}_{N,t_d,i}\{y_3\} + \mathcal{P}_{N,t_d}\{\frac{y_1y_3^2 - \frac{\theta_1}{y_1^2}}{2+\cos y_2}\}$$
$$+ \mathcal{P}_{N,t_d}\{\frac{\theta_2 u_1}{2+\cos y_2} - u_2\} + \frac{2}{\theta_2}\mathcal{P}^{(1)}_{N,t_d}\{y_1\}\mathcal{P}_{N,t_d}\{y_3\}$$
$$- \sum_{i=0}^{N^*}\mathcal{P}_{N,0,i}\{\frac{1}{2+\cos y_2}\}\tilde{\mathcal{P}}^{(2)}_{N,t_d,i}\{y_1\}. \tag{41}$$

The parameters θ_1 and θ_2 in (36) are chosen to be $\theta_1 = \theta_2 = 1$. For the polynomial approximation the parameters are $\alpha = \beta = 3$ and $N = 1$, i.e., first order Jacobi polynomials are imployed. Furthermore, for the partial approximation (see Lomakin and Deutscher (2019)) the parameter N^* is set to 2, i.e., second order Jacobi polynomials are considered therein. In order to improve the approximation accuracy, the delay t_d was selected as the zero of the Jacobi polynomial of second order at $t_d = (LT_s)/3 = 0.033$s and correspondingly for discrete-time implementation of the simulation the sampling time of $T_s = 0.005$s and $L = 20$ were set according to the discretization in (Lomakin and Deutscher, 2019, Sec. III B).

The following Fig. 1 depicts the simulation results for sinusoidal input signals and disturbances with jumps and frequencies 0.5s^{-1} and 5s^{-1}. Furthermore, an additive gaussian distributed measurement noise $\bar{\omega}_y$, $\mathcal{N}_{\bar{\omega}_y}(0, 10^{-10})$ and a process noise $\bar{\omega}_u$, $\mathcal{N}_{\bar{\omega}_u}(0, 10^{-10})$ for the variables y and u, respectively, were added for the simulation to verify the robustness of the method against noise. The steplike fault f jumps to the value $f_\infty = 1$ at the time $t_f = 2.5$s.

The simulation results show that the obtained value $\hat{f}$ of the fault f is independent of the corresponding input u and the disturbance d, as well as their characteristics. Although both inputs u and d cannot be represented polynomially within an interval $\mathcal{I}_{t,T}$ due to jumping signals, the obtained signal of $\hat{f}$ is not affected. The reconstruction of the fault thus depends solely on the parameters N and t_d of the polynomial approximation. The simulations verify that the fault is reconstructed with a delay of 0.033s, which corresponds to the set delay t_d of the polynomial approximation. The reconstruction of the fault can thus be realized independently of all inputs and the RGP-FDI is thus solved.

6. CONCLUDING REMARKS

As illustrated in this paper, faults acting on nonlinear affine input MIMO systems can be detected and identified independently of each other by the introduced derivative estimation using a polynomial approximation and decoupled from the disturbance d. It should also be noted that the presented method can be extended to a more general class of nonlinear systems such as not complete uniform observable systems, which will be presented in further research work.

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Appendix A. TRANSFORMATION INTO OBSERVABILITY NORMAL FORM

For a given drift observable system (1), one has to determine the n_y constants $\bar{s}_1, ..., \bar{s}_{n_y}$ such that $\sum_{i=1}^{n_y} \bar{s}_i = n$ and $\xi = \Phi_o(x)$ is a local diffeomorphism in $U(x^0)$. For this, the Algorithm 1, which is depicted in Fig. A.1 can be applied. If it succeeds, then it is possible to express ξ and $\dot{\xi}_{\bar{s}}$ by $\bar{y}$, $\dot{\bar{y}}$, $\bar{u}$ and $\dot{\bar{u}}$ according to (8) and (9) and to define the equation (12) for (1). If Algorithm 1 fails, either the differential degree regarding the disturbance and fault is insufficient (see line 11 in Algorithm 1), or the system is not complete uniformly observable according to Definition 3 (see line 21 in Algorithm 1). In both cases, however, the presented method of fault detection and identification is not applicable.

Initialization:

1: **for all** $i \in \{1, ..., n_y\}$ **do**
2: $\quad s_i = 0;$ ▷ Initialize all s_i with zero
3: $\quad \xi_1^i = c_i(x);$ ▷ Initialize all ξ_1^i with the outputs y_i
4: **end for**
5: $\xi = \{\xi_1^1, ..., \xi_1^{n_y}\};$ ▷ Define set of known variables

Iteration:

6: **while** $\sum_{i=1}^{n_y} s_i < n$ **do**
7: $\quad s_{ref} = \sum_{i=1}^{n_y} s_i$ ▷ Reference to avoid infinite loops
8: $\quad$ **for** $j = 1$ **to** n_y **do**
9: $\quad\quad$ **if** $s_j < \bar{s}_j$ **then**
10: $\quad\quad\quad$ **if** $\exists k \in \{1, ..., n_d\}$ s.t. $\mathrm{L}_{g_k}\mathrm{L}_a^{s_j} c_j(x) \neq 0$ **or** $\exists k \in \{1, ..., n_f\}$ s.t. $\mathrm{L}_{e_k}\mathrm{L}_a^{s_j} c_j(x) \neq 0$ **then**
11: $\quad\quad\quad\quad$ **break;** ▷ d or f acts to early
12: $\quad\quad\quad$ **end if**
13: $\quad\quad\quad$ **if** $\nexists k \in \{1, ..., n_u\}$ s.t. $\mathrm{dL}_{b_k}\mathrm{L}_a^{s_j} c_j(x) \notin \mathrm{span}\{\mathrm{d}\xi\}$ **then**
14: $\quad\quad\quad\quad s_j = s_j + 1;$
15: $\quad\quad\quad\quad \xi_{s_j}^j = \dot{\xi}_{s_j-1}^j - \sum_{k=1}^{n_u} \mathrm{L}_{b_k}\mathrm{L}_a^{s_j-1} c_j \circ \Phi^{-1}(\xi) u_k;$
16: $\quad\quad\quad\quad \xi = \{\xi, \xi_{s_j}^j\};$ ▷ Add new variable $\xi_{s_j}^j$ to set
17: $\quad\quad\quad$ **end if**
18: $\quad\quad$ **end if**
19: $\quad$ **end for**
20: $\quad$ **if** $s_{ref} = \sum_{i=1}^{n_y} s_i$ **then**
21: $\quad\quad$ **break;** ▷ System (1) is not CUO
22: $\quad$ **end if**
23: **end while**

Fig. A.1. Transformation of the system to the observability normal form.

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