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A distributed set-membership estimator for linear systems considering multi-hop subspace decomposition

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Abstract: This paper presents a distributed set-membership estimator for linear full-coupled systems affected by bounded disturbances. The estimator makes use of a recently developed multi-hop subspace decomposition of the system that allows to transform the dynamic matrix into an upper triangular matrix, decoupling the influence of the non-observable modes to the observable ones. This way, each agent has to compute sets to encompass the observable dynamics on the one hand, and the unobservable dynamics on the other. The sets are mathematically described by zonotopes. Due to the multi-hop decomposition, the agents are able to design different gains for the observable and the unobservable part, pursuing the reduction in volume of the corresponding sets. The paper presents the solution for the two-agents case. Simulations are given to compare the proposed solution with existing ones in the field.

Keywords: Distributed set-membership estimation, Multi-agent systems, Linear system, Guaranteed estimation, Zonotopes

1. INTRODUCTION

With the development of wireless networks, embedded devices and agents with processing and sensing capabilities, distributed techniques are becoming increasingly important. In particular, distributed estimation allows monitoring important variables that can not be locally measured and, this way, making it possible to operate and control the system in a decentralized way.

Different distributed estimators have been proposed depending on the model of the system, disturbances and noises (Ge et al., 2019; Ierardi et al., 2019). Most research considers different modifications of the distributed Kalman filter (Mahmoud and Khalid, 2013; Olfati-Saber, 2007), particle filter (Hlinka et al., 2013), H_∞ filtering (Dong et al., 2013; Shen et al., 2011), or Luenberger observer (del Nozal et al., 2019; Mitra and Sundaram, 2018).

In contrast to the aforementioned paradigms, distributed set-membership observers aim to bound the variables estimation under certain geometrical set like intervals (Efimov and Raïssi, 2016; Chebotarev et al., 2015; Mazenc and Bernard, 2011), ellipsoids (Zhou et al., 2013; Merhy et al., 2019), and zonotopes (Alamo et al., 2008; Le et al., 2013; Combastel, 2015; Chabane et al., 2014; Wang et al., 2017). Compared to interval descriptions or ellipsoids, zonotopes have the advantage of allowing for a trade-off between mathematical complexity and precision, through the number of generator vectors used.

The state of the art in distributed set-membership estimation is rather scarce. Ellipsoidal descriptions have been recently used in Xia et al. (2018); Ma et al. (2016); Liu et al. (2019). The authors in Xia et al. (2018) consider an asynchronous event-driven transmission policy between the agents. Ma et al. (2016) deals with sensor saturations. The work Liu et al. (2019) analyses the impact of the Round-Robin scheduling in the estimation performance.

Concerning zonotopic formulations, the authors present several contributions for the case when the communications between agents are affected by delays (García et al., 2017), multi-rate communications (Orihuela et al., 2017), and bandwidth limited networks (Orihuela et al., 2018). In addition, Wang et al. (2018); Combastel and Zolghadri (2018) proposed distributed zonotopic estimators for large scale-systems. Interestingly, the observer in Combastel and Zolghadri (2018) merges set-membership with Gaussian estimation. The most interesting feature of these zonotopic observers is that they can be designed in a decentralized way, in contrast to the aforementioned ellipsoidal ones, in which centralized optimization problems need to be solved.

It should be mentioned that most of these results (Orihuela et al., 2018; Wang et al., 2018; Combastel and Zolghadri, 2018; Xia et al., 2018) have been developed for a description of the system that consists of a set of interconnected subsystems and, then, the estimation goals are simplified to the estimation of the local subsystem's state. In Xia et al. (2018), it is proposed an additional step to combine all local estimations into an unique global one, but this requires a central unit that must merge all the information.

Hence, this paper deals with distributed set-membership estimation with global estimation objectives, as in García et al. (2017), but with several novel contributions:

- This paper makes use of a multi-hop decomposition of the state-space into the observable and unobservable subspaces developed in (del Nozal et al., 2019). This decomposition let us write the dynamic matrix of the

system with an upper triangular form, this allowing for a cascade implementation of the observer.
- Each agent has to find a different set for each subspace, instead of a unique set for all the states.
- The observer gains that minimize the size of those sets, i.e. the estimation uncertainty, can be designed in independent distributed steps.

This paper presents the analysis for the two-agents case, constituting a preliminary work of an extension for a general case. The computational and communication requirements of the proposed method are compared to other formulations in which only one zonotope must be found.

The paper is organized as follows. Section 2 presents some preliminaries. The problem formulation is detailed in Section 3. Section 4 describes the proposed estimator design. Section 5 develops the minimization of the estimation uncertainty. Section 6 presents some consideration about computational and communication costs. Simulations and examples are illustrated in Section 7. Finally, some conclusions and future works are drawn in Section 8.

Notation. Let $R \in \mathbb{R}^{n\times p}$. Then, $\|R\|_F = \sqrt{tr(R^T R)}$ is the Frobenius norm of R. Given matrices A, B of appropriate dimensions, operator $cat\{A, B\} \triangleq [A\ B]$.

2. PRELIMINARIES

2.1 The multi-hop decomposition

The multi-hop decomposition described here was introduced in del Nozal et al. (2019). The most important details are given here for the case of two agents. For a complete description, the reader is referred to that paper.

Consider a set of agents $\mathcal{V} = \{1, 2\}$, connected with a bidirectional communication link, able to measure some information from the next discrete-time LTI system:

$$x(k+1) = Ax(k), \tag{1}$$

$$y_i(k) = C_i x(k) \quad \forall i \in \mathcal{V}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, A is the system matrix, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by each agent i and $C_i \in \mathbb{R}^{m_i\times n}$ is the output matrix of agent i.

Let's define the 0-hop output matrix of agent i as its output matrix, this is $C_{i,0} = C_i$. The 1-hop output matrix of agent i, $C_{i,1}$, is a matrix that stacks the 0-hop output matrix of agent i and the 0-hop output matrix of its neighbor, that is:

$$C_{i,1} := \begin{bmatrix} C_{i,0} \\ C_{j,0} \end{bmatrix},\ j \neq i.$$

For any agent, there exists a coordinate transformation matrix $\left[\bar{V}_{i,\rho}\ V_{i,\rho}\right] \in \mathbb{R}^{n\times n}$, being ρ the $\{0,1\}$-hop, according to pair $(C_{i,\rho}, A)$ such that the change of variable $z_{i,\rho} \triangleq [\bar{V}_{i,\rho}\ V_{i,\rho}]^\top x \in \mathbb{R}^n$ transforms the original state-space representation into the observability staircase form, for $\rho = 0, 1$.

Definition 2. The ρ-hop unobservable subspace from agent i is the unobservable subspace related to pair $(C_{i,\rho}, A)$ using the above coordinate transformation:

$$\bar{\mathcal{O}}_{i,\rho} := Im(\bar{V}_{i,\rho}).$$

The orthogonal complement of $\bar{\mathcal{O}}_{i,\rho}$ is denoted ρ-hop observable subspace from agent i, $\mathcal{O}_{i,\rho} := Im(V_{i,\rho})$.

According to previous definitions, it is clear that:

$$\mathcal{O}_{i,0} \subseteq \mathcal{O}_{i,1}, \quad \forall i \in \mathcal{V}. \tag{3}$$

Then, the vectors of the "innovation" basis that generates $\mathcal{O}_{i,1} \cap (\mathcal{O}_{i,0})^\perp$ can be stacked into a matrix $W_{i,1}$ in such a way that $Im(W_{i,1}) := \mathcal{O}_{i,1} \cap (\mathcal{O}_{i,0})^\perp$. Then $Im(W_{i,1})$ corresponds to the observable modes for agent i at hop 1 that are not observable at hop 0. From these definitions it is clear that

$$Im(V_{i,1}) = Im\left([W_{i,1}\ \ V_{i,0}]\right), \tag{4}$$

$$Im(\bar{V}_{i,0}) = Im\left([W_{i,1}\ \ \bar{V}_{i,1}]\right). \tag{5}$$

Accordingly, a orthogonal transformation matrix T_i can be defined such as:

$$T_i := \left[\bar{V}_{i,1}\ W_{i,1}\ W_{i,0}\right], \tag{6}$$

where $\bar{V}_{i,1}$ corresponds to the modes that are not observable by any of the agents.

Proposition 1. For each agent i, the orthogonal similarity transformation given by T_i in (6) transforms the system matrix A into a block upper-triangular matrix in the form:

$$T_i^\top A T_i = \begin{bmatrix} \bar{V}_{i,1}^\top A\bar{V}_{i,1} & \bar{V}_{i,1}^\top AW_{i,1} & \bar{V}_{i,1}^\top AW_{i,0} \\ 0 & W_{i,1}^\top AW_{i,1} & W_{i,1}^\top AW_{i,0} \\ 0 & 0 & W_{i,0}^\top AW_{i,0} \end{bmatrix}$$

2.2 Zonotopes

A zonotope $\mathcal{X}$, denoted with calligraphic, capital letters, is a centrally symmetric, convex set determined by its center $c \in \mathbb{R}^n$, and by a matrix $H \in \mathbb{R}^{n\times p}$: $\mathcal{X} = \langle c, H\rangle = \{c + \sum_{i=1}^{p} \varsigma_i h_i : \forall i\ |\varsigma_i| \le 1\}$, where $h_i \in \mathbb{R}^n$ (columns of H) are called *generator vectors*. The *order* of a zonotope is given by the number of generator vectors, its *F-radius* is the Frobenius norm of H, and its *covariation* is defined as $P_{\mathcal{X}} = HH^T$ (see Combastel (2015)).

Let $\mathcal{X} = \langle c_x, H_x\rangle$ and $\mathcal{Y} = \langle c_y, H_y\rangle$ be two zonotopes, and let R be a matrix of appropriate dimensions. A linear transformation of a zonotope is given by $R\mathcal{X} = \langle Rc_x, RH_x\rangle$, and the Minkowski sum of two zonotopes is obtained as $\mathcal{X} \oplus \mathcal{Y} = \langle c_x + c_y, cat\{H_x, H_y\}\rangle$. Given a matrix A and any vectors such that $x \in \mathcal{X}$ and $w \in \mathcal{W}$, it holds that $y := Ax + w \in A\mathcal{X} \oplus \mathcal{W}$. The operator $red_q(\mathcal{X})$, defined as in Combastel (2015), is an order reduction of the zonotope $\mathcal{X}$ in such a way that $\mathcal{X} \subseteq red_q(\mathcal{X})$, and the order of $red_q(\mathcal{X})$ is q.

3. PROBLEM FORMULATION

In this paper, we consider the problem of set-membership state estimation for a multi-output linear system affected by bounded disturbances, observed by two agents. The system is described by the following equations:

$$x(k+1) = Ax(k) + Dw(k), \tag{7}$$

$$y_1(k) = C_1 x(k) + v_1(k), \tag{8}$$

$$y_2(k) = C_2 x(k) + v_2(k), \tag{9}$$

where $x \in \mathbb{R}^n$ is the state vector, $y_i \in \mathbb{R}^{m_i}$ is the output locally measured by each agent, w represents disturbances or unmodeled dynamics, and v_i are measurement noises, Matrices A, D, C_1, C_2 are constant matrices of appropriate dimensions.

Assumption 1. Disturbances and noises belong to bounded known sets described as zero-centered zonotopes, this is, $w(k) \in \mathcal{W} = \langle 0, Q\rangle$, $v_i(k) \in \mathcal{V}_i = \langle 0, R_i\rangle$, being Q and R_i matrices of adequate dimensions.

Assumption 2. System (7)-(9) is collectively observable. That is, the pair (C, A) is observable, where $C := cat\{C_1^\top, C_2^\top\}^\top$.

For each agent $i = 1, 2$, it is possible to find a coordinate transformation matrix $T_i \in \mathcal{R}^{n\times n}$ such that under the change of variables $z_i = T_i^\top x_i$, system (7)-(9) can be transformed into the observability staircase form:

$$z_i(k) = T_i^\top x(k) = \begin{bmatrix} W_{i,1}^\top \\ W_{i,0}^\top \end{bmatrix} x(k) = \begin{bmatrix} z_{i,1}(k) \\ z_{i,0}(k) \end{bmatrix}. \tag{10}$$

The transformed state vector comprises two terms: the local observable modes $z_{i,0}(k) \in \mathbb{R}^{n_{i,0}}$, and the local unobservable modes that are locally observable by the other agent $z_{i,1}(k) \in \mathbb{R}^{n_{i,1}}$. Now, using $T_i^{-1} = T_i^\top$, it holds:

$$x(k) = T_i z_i(k) = W_{i,0} z_{i,0}(k) + W_{i,1} z_{i,1}(k). \tag{11}$$

Referring to equation (7), we obtain:

$$z_i(k+1) = T_i^\top x(k+1) = T_i^\top A T_i z_i(k) + T_i^\top D w(k), \tag{12}$$

with $T_i^\top A T_i$ upper block diagonal according to Proposition 1. Then, the dynamics of each hop is given by:

$$z_{i,\rho}(k+1) = \sum_{r=0}^{\rho} W_{i,\rho}^\top A W_{i,r} z_{i,r}(k) + W_{i,\rho}^\top D w(k). \tag{13}$$

Let's define the following *a priori* and *a posteriori* zonotopes for each agent and hop, using the notation in Section 2.2, intended to contain the actual state of each subspace:

- *A priori*: $\hat{\mathcal{Z}}_{i,\rho}(k+1|k) = \langle c_{i,\rho}(k+1|k), H_{i,\rho}(k+1|k)\rangle$,
- *A posteriori*: $\hat{\mathcal{Z}}_{i,\rho}(k|k) = \langle c_{i,\rho}(k|k), H_{i,\rho}(k|k)\rangle$.

Assuming that the local information measured by each agent (8)-(9) is not enough to reconstruct the whole state, they have to communicate with the other agent in order to fulfill the following objectives:

(i) Find sets, by means of zonotopes, in which the actual state of its associated subspace is continuously contained. In particular, both agents must obtain an *a priori* and an *a posteriori* sets such that $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k)$ and $z_{i,\rho}(k+1) \in \hat{\mathcal{Z}}_{i,\rho}(k+1|k)$, $\forall i, \rho, k$.

(ii) Minimize the estimation uncertainty, measured through the F-radius of the *a posteriori* zonotopes. This is, minimize $\|\hat{\mathcal{Z}}_{i,\rho}(k|k)\|_F, \forall i, \rho$.

4. PROPOSED ESTIMATOR

This section presents the distributed set-membership estimator considering the multi-hop subspace decomposition. Let's assume that each agent knows an *a priori* estimation set for hops $\rho = 0, 1$, i.e. $\hat{\mathcal{Z}}_{i,0}(k|k-1) = \langle c_{i,0}(k|k-1), H_{i,0}(k|k-1)\rangle$ and $\hat{\mathcal{Z}}_{i,1}(k|k-1) = \langle c_{i,1}(k|k-1), H_{i,1}(k|k-1)\rangle$. Those zonotopes contain the actual state of the associated subspace.

The *a posteriori* estimation set for hop $\rho = 0$ is derived using the information locally measured from the system $y_i(k)$ as follows:

$$c_{i,0}(k|k) = c_{i,0}(k|k-1) + L_i(k)\left(y_i(k) - C_i W_{i,0} c_{i,0}(k|k-1)\right), \tag{14}$$

$$H_{i,0}(k|k) = cat\{\left(I - L_i(k) C_i W_{i,0}\right) H_{i,0}(k|k-1), - L_i(k) R_i\}. \tag{15}$$

Local observer gain $L_i(k)$ is computed so that the F-radius of the *a posteriori* zonotope $\hat{\mathcal{Z}}_{i,0}(k|k)$ is minimized. This problem will be tackled in Section 5.

Now, an order reduction step is conducted $\tilde{\mathcal{Z}}_{i,0}(k|k) = \langle \tilde{c}_{i,0}(k|k), \tilde{H}_{i,0}(k|k)\rangle = red_q(\hat{\mathcal{Z}}_{i,0}(k|k))$. This set is sent through the communication network to the other agent.

The *a posteriori* estimation set for $\rho = 1$, $\hat{\mathcal{Z}}_{i,1}(k|k)$, is computed as:

$$c_{i,1}(k|k) = \left(I - N_{i,j}(k) W_{j,0}^\top W_{i,1}\right) c_{i,1}(k|k-1) - N_{i,j}(k) W_{j,0}^\top W_{i,0} c_{i,0}(k|k) + N_{i,j}(k) \tilde{c}_{j,0}(k|k), \tag{16}$$

$$H_{i,1}(k|k) = cat\Big\{\left(I - N_{i,j}(k) W_{j,0}^\top W_{i,1}\right) H_{i,1}(k|k-1), - N_{i,j}(k) W_{j,0}^\top W_{i,0} H_{i,0}(k|k), N_{i,j}(k) \tilde{H}_{j,0}(k|k)\Big\}, \tag{17}$$

where $j \neq i$ represents the neighbor agent. Gain $N_{i,j}(k)$ is chosen to minimize the F-radius of the *a posteriori* zonotope $\hat{\mathcal{Z}}_{i,1}(k|k)$. See Section 5 for more details.

Having computed the *a posteriori* zonotopes for both hops, let's move to the *a priori* ones. For hop $\rho = 0$, the *a priori* estimation set is obtained as follows:

$$c_{i,0}(k+1|k) = W_{i,0}^\top A W_{i,0} c_{i,0}(k|k), \tag{18}$$

$$H_{i,0}(k+1|k) = cat\{W_{i,0}^\top A W_{i,0} H_{i,0}(k|k), W_{i,0}^\top D Q\}. \tag{19}$$

Finally, the *a priori* estimation set for $\rho = 1$ is given by:

$$c_{i,1}(k+1|k) = W_{i,1}^\top A W_{i,1} c_{i,1}(k|k) + W_{i,1}^\top A W_{i,0} c_{i,0}(k|k), \tag{20}$$

$$H_{i,1}(k+1|k) = cat\{[W_{i,1}^\top A W_{i,1} H_{i,1}(k|k), W_{i,1}^\top A W_{i,0} H_{i,0}(k|k), W_{i,1}^\top D Q\}. \tag{21}$$

Next theorem states that, no matter the value that is given to matrices $L_i(k), N_{i,j}(k)$, objective (i) is fulfilled if the previous equations are implemented.

Theorem 1. Let's assume that $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k-1), \forall i, \rho$ for some particular instant k. Then, if the distributed set-membership observer in (14)-(21) is implemented by every agent, both the *a posteriori* and *a priori* estimation sets continuously contain the associated system states.

Proof. Consider $z_{i,\rho}(k) \in \hat{\mathcal{Z}}_{i,\rho}(k|k-1)$ for all i and for $\rho = 0, 1$ at some instant k. According to (8)-(9):

$$z_{i,0}(k) = z_{i,0}(k) + L_i(k)\Big(y_i(k) - C_i W_{i,0} z_{i,0}(k) - v_i(k)\Big),$$

that, under some manipulations, yields:

$$z_{i,0}(k) = \Big(I - L_i(k)C_i W_{i,0}\Big) z_{i,0}(k) \\ + L_i(k)y_i(k) - L_i(k)v_i(k),$$

Since $z_{i,0}(k) \in \hat{\mathcal{Z}}_{i,0}(k|k-1)$, $v_i(k) \in \mathcal{V}_i$, and $y_i(k)$ is known, state $z_{i,0}(k)$ is contained in:

$$z_{i,0}(k) \in \Big(I - L_i(k)C_i W_{i,0}\Big)\hat{\mathcal{Z}}_{i,0}(k|k-1) \\ \oplus L_i(k)\langle y_i(k), 0\rangle \oplus \Big(-L_i(k)\Big)\mathcal{V}_i,$$

which is the zonotope described in (14)-(15).

Using a similar idea for the transformed states at hop $\rho = 1$, it is satisfied:

$$z_{i,1}(k) = z_{i,1}(k) + N_{i,j}(k)\Big(W_{j,0}^{\top}\hat{x}_j(k) - W_{j,0}^{\top}x(k) \\ + W_{j,0}^{\top}e_j(k)\Big),$$

being $e_j(k) = x(k) - \hat{x}_j(k)$.

Note that the term $W_{j,0}^{\top}\hat{x}_j(k)$ plays here the role of $y_i(k)$ in the previous hop. This is the information that agent i receives and uses to compute the estimation set. This signal $W_{j,0}^{\top}\hat{x}_j(k)$, representing the states that the neighbour locally observes, is affected by the uncertainty of the *a posteriori* zonotope $\hat{\mathcal{Z}}_{i,0}(k|k)$. Then, from the fact

$$W_{j,0}^{\top}x(k) = W_{j,0}^{\top}\Big(W_{i,0}z_{i,0}(k) + W_{i,1}z_{i,1}(k)\Big),$$

we finally obtain

$$z_{i,1}(k) = \Big(I - N_{i,j}(k)W_{j,0}^{\top}W_{i,1}\Big)z_{i,1}(k) \\ - N_{i,j}(k)W_{j,0}^{\top}W_{i,0}z_{i,0}(k) + N_{i,j}(k)W_{j,0}^{\top}\Big(\hat{x}_j(k) - e_j(k)\Big).$$

Therefore, state $z_{i,1}(k)$ is ensured to belong to the set:

$$z_{i,1}(k) \in \Big(I - N_{i,j}(k)W_{j,0}^{\top}W_{i,1}\Big)\hat{\mathcal{Z}}_{i,1}(k|k-1) \\ \oplus \Big(-N_{i,j}(k)W_{j,0}^{\top}W_{i,0}\Big)\hat{\mathcal{Z}}_{i,0}(k|k) \oplus N_{i,j}(k)\hat{\mathcal{Z}}_{j,0}(k|k).$$

Since $\hat{\mathcal{Z}}_{j,0}(k|k) \subseteq \tilde{\mathcal{Z}}_{j,0}(k|k)$, $z_{i,1}(k)$ is proven to be contained in a zonotope described by equations (16)-(17).

Regarding the *a priori* sets, taking into account (7) and Proposition 1, it is possible to write:

$$z_{i,0}(k+1) \in W_{i,0}^{\top}AW_{i,0}\hat{\mathcal{Z}}_{i,0}(k|k) \oplus W_{i,0}^{\top}D\mathcal{W},$$

$$z_{i,1}(k+1) \in W_{i,1}^{\top}AW_{i,1}\hat{\mathcal{Z}}_{i,1}(k|k) \oplus W_{i,1}^{\top}AW_{i,0}\hat{\mathcal{Z}}_{i,0}(k|k) \\ \oplus W_{i,1}^{\top}D\mathcal{W}.$$

Previous sets can be computed with (18)-(21). □

Assuming a sufficiently large initial zonotope, in such a way that $z_{i,0}(0) \in \hat{\mathcal{Z}}_{i,0}(0|-1), z_{i,1}(0) \in \hat{\mathcal{Z}}_{i,1}(0|-1)$, Theorem 1 ensures the satisfaction of objective (i).

5. OPTIMAL ESTIMATOR

The second objective of this work is to synthesize matrices $L_i(k)$ and $N_{i,j}(k)$ to minimize the F-radius of zonotopes $\hat{\mathcal{Z}}_{i,0}(k|k)$ and $\hat{\mathcal{Z}}_{i,1}(k|k)$, respectively.

Theorem 2. The observer gains $L_i(k)$ and $N_{i,j}$(k) that minimize the F-radius of $\hat{\mathcal{Z}}_{i,\rho}(k|k)$, for $\rho = 0, 1$ are:

$$L_i(k) = P_{\hat{\mathcal{Z}}_{i,0}(k|k-1)}W_{i,0}^{\top}C_i^{\top} \times \\ \Big(C_iW_{i,0}P_{\hat{\mathcal{Z}}_{i,0}(k|k-1)}W_{i,0}^{\top}C_i^{\top} + P_{\mathcal{V}_i}\Big)^{-1}, \quad (22)$$

$$N_{i,j}(k) = P_{\hat{\mathcal{Z}}_{i,1}(k|k-1)}W_{i,1}^{\top}W_{j,0}\Big(W_{j,0}^{\top}(W_{i,1}P_{\hat{\mathcal{Z}}_{i,1}(k|k-1)}W_{i,1}^{\top} \\ + W_{i,0}P_{\hat{\mathcal{Z}}_{i,0}(k|k)}W_{i,0}^{\top})W_{j,0} + P_{\tilde{\mathcal{Z}}_{j,0}(k|k)}\Big)^{-1}, \quad (23)$$

where $P_{\mathcal{X}}$ denote the covariation of zonotope $\mathcal{X}$, defined in Section 2.2.

Proof. The arguments $L_i(k)$ and $N_{i,j}(k)$ that minimize the F-radius of $\hat{\mathcal{Z}}_{i,\rho}(k|k)$ are the same arguments that minimize $J_{i,\rho}(k) \triangleq \|\hat{\mathcal{Z}}_{i,\rho}(k|k)\|_F^2 = tr(P_{\hat{\mathcal{Z}}_{i,\rho}(k|k)})$, for $\rho = 1, 2$. These Frobenius norms are convex with respect to $L_i(k)$ and $N_{i,j}(k)$, so the gains that minimize $J_{i,\rho}$ hold:

$$L_i^*(k) = \arg\left(\frac{\partial J_{i,0}(k)}{\partial L_i(k)} = 0\right), \quad (24)$$

$$N_{i,j}^*(k) = \arg\left(\frac{\partial J_{i,1}(k)}{\partial N_{i,j}(k)} = 0\right). \quad (25)$$

Solving (24)-(25), the optimal observer gains in (22),(23) are obtained. Intermediate steps has been omitted due to space restrictions. □

Note that both observer gains can be computed using only local information available for each agent at instant k. For the sake of clarity and ease of implementation, the iterative procedure to be implemented in each agent is presented in Table 1. Steps 0 and 4 indicate the available and received information needed to implement the complete algorithm.

Table 1. Algorithm 1. Estimation loop

0. Initial zonotopes	$\hat{\mathcal{Z}}_{i,0}(k|k-1)$, $\hat{\mathcal{Z}}_{i,1}(k|k-1)$
1. Measurement	$y_i(k)$ with (8)-(9)
2. *A posteriori* estimation	
2.1 Local gain	$L_i(k)$ with (22)
2.1 Zonotope	$\hat{\mathcal{Z}}_{i,0}(k|k)$ with (14)-(15)
3. Order reduction	$red_q(\hat{\mathcal{Z}}_{i,0}(k|k))$
4. Communication	
4.1 Send	$red_q(\hat{\mathcal{Z}}_{i,0}(k|k))$
4.2 Receive	$red_q(\hat{\mathcal{Z}}_{j,0}(k|k))$
5. *A posteriori* estimation	
5.1 Neighbour gain	$N_{i,j}(k)$ with (23)
5.2 Zonotope	$\hat{\mathcal{Z}}_{i,1}(k|k)$ with (16)-(17)
6. *A priori* estimation	$\hat{\mathcal{Z}}_{i,0}(k+1|k)$ with (18)-(19), $\hat{\mathcal{Z}}_{i,1}(k+1|k)$ with (20)-(21)

6. COMPUTATIONAL AND COMMUNICATION REQUIREMENTS

Table 2 lists computational and communication requirements for the proposed algorithm and other distributed set-membership estimation algorithms in the literature valid for global estimation. It is denoted by $O(n_1, n_2)$ when a matrix of dimension $n_1 \times n_2$ must be computed or transmitted. Letters n and q are reserved for the dimension of the state vector and the order chosen after the reduction step, respectively. For all the algorithms listed in Table 2, only the *a posteriori* requirements are included.

Table 2. Comput.&Commun. requirements

Algorithm	Communication	Computation
Table 1	$O(n_{i,0}, q)$	$O(n_{i,0}, q) + O(n_{i,1}, q)$
García et al. (2017)	$O(n, q)$	$O(n, q)$
Xia et al. (2018)	$O(n, n)$	$O(n, n)$

Since $n_{i,0} + n_{i,1} = n$, the computational requirements of the proposed method and the ones for García et al. (2017) are equivalent, communication requirements are reduced. This table also illustrates the trade-off between complexity and precision mentioned in the introduction. By introducing additional generator vectors, less uncertainty is introduced in the order reduction step. This is made at a cost of higher computational and communication requirements. This does not appear in ellipsoidal observers.

Remark. We have intendedly used the word requirement instead of cost, since each paper propose a completely different way to find the sets. We have focused here in the size of the matrices that must be computed, but not on how they are computed.

7. SIMULATIONS

In this section a simulation example is presented in order to show the effectiveness of the proposed observer. We have chosen a discrete system with three states (see Millán et al. (2017)), in which the first state x_1 has an unstable dynamic and it is decoupled from the others states x_2 and x_3, which correspond to a pair of conjugated imaginary poles:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1.05 & 0 & 0 \\ 0 & 0.9954 & -0.08757 \\ 0 & 0.1248 & 0.9945 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \quad (26)$$

The two agents have access to plan outputs $y_1 = x_1$ and $y_2 = x_3$, respectively. Therefore, agent 1 can locally observe the first state, while agent 2 can locally observe remains states. However, neither of them can estimate the whole state without communicating with the other one.

The basis vectors of the observable and unobservable subspaces of both agents are given by:

$$W_{1,0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, W_{1,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, W_{2,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, W_{2,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The initial condition for the simulation is $x(0) = [0, 1, 1]^\top$. The maximum order of the *a posteriori* zonotopes is $q = 100$. Disturbances and noises are uniformly-distributed random signals described by $R_1 = 0.02, R_2 = 0.02, Q = 0.02I_{3\times3}$.

Figure 1 shows the estimation performance of agent 1, through the *a posteriori* zonotope. Dashed lines have been used for the actual states, and solid lines for the bounds of the zonotopes. It can be seen that the agent is able to estimate the overall plant states. Better performance is obtained for those states directly measured, as expected.

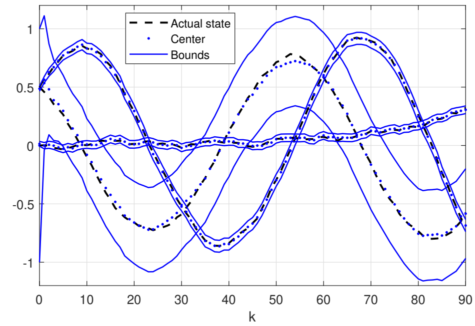

Fig. 1. Evolution of the states and estimates of agent 1.

Finally, the proposed method is compared with the one in García et al. (2017). As explained in the introduction, this is, from the authors' best knowledge, the unique available zonotopic distributed set-membership estimator dealing with the observation of the whole state vector. In that paper, they propose the use of intersections between zonotopes to include the information of neighboring agents.

We define the next indexes for a time horizon of N steps:

$$I_a = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{2}\sum_{i=1}^{2}\sqrt{||\hat{\mathcal{Z}}_{i,0}(k|k)||_F^2 + ||\hat{\mathcal{Z}}_{i,1}(k|k)||_F^2},$$

$$I_b = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{2}\sum_{i=1}^{2}||\hat{\mathcal{Z}}_{i}(k|k)||_F,$$

where index I_a is adequate for Algorithm 1 that computes two zonotopes per agent, and I_b for the method in García et al. (2017) that only computes one zonotope per agent at each sampling instant.

Figure 2 draws the different values of the indexes for different values of the maximum order of the zonotopes q. For small values of q (smaller than 200), the proposed method outperforms that in García et al. (2017) in terms of uncertainty minimization. However, if the order is large, the intersection-based algorithm proposed in García et al. (2017) is able to get smaller sets.

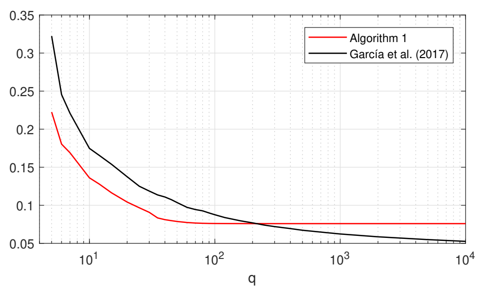

Fig. 2. Index I_a obtained with Algorithm 1 in Table 1 and index I_b with García et al. (2017).

8. CONCLUSIONS AND FUTURE WORKS

A distributed set-membership estimator for linear full-coupled systems affected by bounded disturbances has been presented in this paper. The multi-hop decomposition has allowed to propose the design the observers in such a way that the volumes of the estimation zonotope can be reduced with respect to previous works.

This paper constitutes a preliminary work and the mathematical basis to move forward to more complex problems, considering multiple agents.

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