Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Sampled-data tracking under model predictive control and multi-rate planning*

Mohamed Elobaid *,** **Mattia Mattioni** * **Salvatore Monaco** *
Dorothée Normand-Cyrot **

* *Dipartimento di Ingegneria Informatica, Automatica e Gestionale (La Sapienza University of Rome) Rome, 00185 Italy (e-mail: {mohamed.elobaid, mattia.mattioni, salvatore.monaco}@uniroma1.it).*
** *Laboratoire de Signaux et Systèmes (L2S, CNRS); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: {mohamed.elobaid, dorothee.normand-cyrot}@centralesupelec.fr)*

Abstract: In this paper, a new control scheme for sampled-data nonlinear model predictive control is proposed making use of a multi-rate based trajectory planning for designing admissible references over the prediction horizon. The proposed controller is compared with existing reference generators for model predictive control through simulations over a benchmark example.

Keywords: Digital implementation; Nonlinear predictive control; Tracking

1. INTRODUCTION

Several control problems involve the cancellation of the so called zero-dynamics. This issue is even more evident in the nonlinear context when solving input-output feedback linearization or tracking problems (see Isidori (2013)).
The problem becomes more delicate under sampling and, in particular, when direct digital control approaches are employed to design the controller. In that case in fact, the minimum-phase property of the system can even be lost for small sampling periods, due to the appearance of the unstable so-called sampling zeros (Åström et al. (1984); Monaco and Normand-Cyrot (1988) for linear and nonlinear cases).
The idea of making use of a piecewise constant control over sub-intervals of the sampling interval, that is multi-rate control, has been properly introduced in the nonlinear context to overcome the aforementioned pathologies (Monaco and Normand-Cyrot (1992)). The possibility of varying the control several times in the sampling interval confers more degrees of freedom to the control action (Monaco and Normand-Cyrot (1991, 2001)).
A major limitation in the use of multi-rate control stands in its intrinsic dependence on the model so implying lack of robustness. Moreover, since the input signal changes over sub-intervals of the sampling period, the control works in open loop at all those sub-intervals since the corresponding measures of the states are not available.
On the other hand, model predictive control (MPC) represents a powerful and effective design technique that relies upon the solution of a constrained optimization problem subject to the system dynamics plus possible additional requirements over a finite time horizon. Indeed, it has now become the mainstay for regulation and tracking in the industry for constrained multi-input/multi-output (MIMO) systems (see Camacho and Alba (2013); García et al. (1989)). Additionally, under suitable assumptions, it has been proved that the feedback generated by a nominal nonlinear model predictive problem is *inherently robust* with respect to different perturbing actions and model uncertainties (e.g., Picasso et al. (2010); Grimm et al. (2004); Magni et al. (2009)). Still, as the intuition suggests, some difficulties may arise when discrete-time models are used by the MPC designer to ensure a prefixed reference profile for the output. As sampling induces unstable zero-dynamics, a naive implementation of MPC feedback might yield unboundedness of the internal trajectories and thus of the feedback. To overcome this issue, standard MPC implementations introduce further penalizing weights on the objective function and/or constraints (typically LMI) over the optimization problem (e.g., Bemporad and Morari (1999); Byun (1988)). As a drawback, such a stratagem is not based on the understanding of the source of unstability so that ad-hoc tuning is needed.
Motivated by these reasons, in Elobaid et al. (2019a) a first attempt to handle those aspects has been proposed by directly designing MPC utilizing multi-rate inputs and the equivalent multi-rate sampled-data model for prediction. Under small penalties on the input, this approach has been shown to be effective as no further modifications of the optimization problem are required for preserving boundedness of the closed loop; however it requires huger capabilities of the sample and hold devices.
In this respect, this work is aimed to weaken this demand by leaving the holding and sampling devices (i.e., actuators and sensors) synchronous. The new proposed control scheme combines a classical single-rate MPC controller with a multi-rate planner that computes, starting from samples of the desired output profile, a suitable admissible reference trajectory to be fed to the MPC. The inner MPC controller working on the fast sampling rate will now guarantee the prefixed boundedness of the internal behaviour of the overall system with no need of introducing further constraints. Roughly speaking, in the proposed control

* Partially funded by *Université Franco-Italienne/Università Italo-Francese* (UFI/UIF) through the Vinci program.

scheme, MPC is used to robustify multi-rate design, and multi-rate planning is used to improve MPC.

This work is organized as follows. In Section 2 recalls on single and multi-rate sampling and model predictive control are given. In Section 3 the main ideas and motivations behind multi-rate planning for MPC control are introduced. Section 4 is devoted to the proposed MPC-MR control scheme which is applied to the simplified model of a planar vertical take-off and landing in Section 5. Concluding remarks end the paper.

Notations: All functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. In the paper T will be the length of the sampling period, and any suitable sub-interval will be denoted by δ. M_U denotes the space of measurable and locally bounded functions $u : \mathbb{R} \to U$ with $U \subseteq \mathbb{R}$. $\mathcal{U}_s \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of fixed length $s \in]0, T^*[$ and T^* small enough; i.e. $\mathcal{U}_s = \{u \in M_U \text{ s.t. } u(t) = u_k, \forall t \in [ks, (k+1)s[; k \geq 0\}$. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^{n} f_i(\cdot)\frac{\partial}{\partial x_i}$. The Lie exponential operator is denoted as e^{L_f} and defined as $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i \geq 1} \frac{\mathrm{L}_f^i}{i!}$ with I being the identity operator. Finally, $\|x\|_P = x^\top P x$ denotes the seminorm of $x \in \mathbb{R}^n$ for some $P \geq 0$. A function $\beta(\cdot) : [0, \infty) \to [0, \infty)$ that is zero in zero and strictly increasing and unbounded is said to be of class κ_∞. A function $R(x, \delta) = O(\delta^p)$ is said to be of order $\delta^p, p \geq 1$ if, whenever it is defined, it can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$ and there exists a function $\beta(\delta) \in \kappa_\infty$ and $\delta^\star > 0$ s.t. $\forall \delta \leq \delta^\star, |\tilde{R}(x, \delta)| \leq \beta(\delta)$

2. PRELIMINARIES AND RECALLS

2.1 Sampled-data systems

Single-rate (SR) and multi-rate (MR) sampled-data (SD) equivalent models of a continuous-time process are recalled in the sequel (Monaco and Normand-Cyrot (2001)). Consider a nonlinear continuous-time input-affine system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{1}$$

and let $u(t) \in \mathcal{U}_T$ and $y(t) = y_k$ for $t \in [kT, (k+1)T[$ (with $T \geq 0$ being the sampling period). Then, denoting $x_k := x(kT)$, $y_k := y(kT)$, $u_k := u(kT)$ for $k \geq 0$, the evolutions of (1) at the sampling instants $t = kT$ with $T \geq 0$, are described by its *single-rate* (SR) *sampled-data equivalent model*

$$x_{k+1} = F^T(x_k, u_k), \quad y_k = h(x_k) \tag{2}$$

where the mapping $F^T(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ admits the following series expansion in powers of T

$$F^T(x_k, u_k) = e^{T(\mathrm{L}_f + u_k \mathrm{L}_g)} x\big|_{x_k} = x_k + \sum_{i>0} \frac{T^i}{i!}(\mathrm{L}_f + u_k \mathrm{L}_g)^i\big|_{x_k}. \tag{3}$$

Since a closed-form expression for the state and output evolutions of (1) does not exist in general, only approximated expansions in power of T can be computed. In this respect, if the series expansion (3) is characterized by a finite number of terms, then system (1) is said to be *finitely discretizable* (Monaco and Normand-Cyrot (2001)).

It is a matter of computations to verify that if (1) has well defined relative degree, say $r \leq n$, the relative degree of the sampled-data equivalent model always falls to $r_d = 1$; namely, one has

$$y_{k+1} = h(x_k) + \sum_{i=1}^{r} \frac{T^i}{i!} \mathrm{L}_f^i h(x)\big|_{x_k} + \frac{T^r}{r!} u_k \mathrm{L}_g \mathrm{L}_f^{r-1} h(x)\big|_{x_k} + O(T^{r+1})$$

so that $\frac{\partial y_{k+1}}{\partial u_k} \neq 0$. As a consequence, whenever $r > 1$, the sampling process induces a further zero-dynamics of dimension $r - 1$ (the *sampling zero-dynamics*, Monaco and Normand-Cyrot (1988)) that is in general unstable for $r > 1$. In that case inversion-like techniques via single-rate sampling cannot be achieved while guaranteeing stability of the internal dynamics.

To overcome this pathology multi-rate (MR) sampling, corresponding to sample the state and output at lower frequency with respect to changes of the piecewise constant control, has been introduced in Monaco and Normand-Cyrot (1991). Accordingly, denoting by $u_k^i(t)$, $x_k^i(t)$ and $y_k^i(t)$ the input, state and output variables at any $t = kT + (i-1)\delta$ for $i = 1, \ldots, m$ (with $u_k = u_k^1$, $x_k = x_k^1$ and $y_k = y_k^1$), the multi-rate equivalent model of order m of (1) gets the form

$$x_{k+1} = F_m^T(x_k, u_k^1, \ldots, u_k^m) \tag{4}$$

with $T = m\delta$, $u(t) \in \mathcal{U}_\delta$ and

$$F_m^T(x_k, u_k^1, \ldots, u_k^m) = e^{\delta(\mathrm{L}_f + u_k^1 \mathrm{L}_g)} \cdots e^{\delta(\mathrm{L}_f + u_k^m \mathrm{L}_g)} x\big|_{x_k} = F^\delta(\cdot, u_k^m) \circ \cdots \circ F^\delta(x_k, u_k^1).$$

It has been proved by the authors that, with $m = r$, the MR sampled-data model has vector relative degree $r^\delta = (1, \ldots, 1)$ under the choice of a suitable output vector specified by the output itself and its first $(r-1)$ derivatives. Accordingly, the corresponding zero-dynamics inherits the zero-dynamics stability properties of (1) (see Monaco and Normand-Cyrot (1988)). In addition in Mattioni et al. (2017), it has been shown that the minimum-phase condition can be relaxed by increasing the MR order.

This MR-SD equivalent model can then be used to design tracking controllers over the sampling interval T as developed in Section 3.

2.2 Sampled-data unconstrained MPC

The problem of tracking a reference $v(t)$ at the sampling instants is typically addressed within the framework of nonlinear model predictive control. Roughly speaking, the feedback is computed to minimize a quadratic cost function of the form

$$J = \sum_{i=1}^{n_p} \left(\|e_{k+i}\|_Q + \|u_{k+i-1}\|_R \right) \tag{5}$$

with $e_k(y_k, v_k)$ a suitable tracking error, $Q > 0, R \geq 0$ being appropriate penalizing weights and n_p, n_c the prediction and control horizons respectively. The functional cost (5) is repeatedly optimized at each sampling instant $t = k\delta$ over a finite horizon while the feedback is applied via a *receding horizon* implementation.

As well know, tracking control implicitly requires zero-dynamics cancellation. In this respect, as the above problem is formulated in the sampled-data context, instability of the closed loop system unavoidably arises due to the sampling zero-dynamics. To overcome this, the *uncon-*

strained MPC problem is typically enriched by redefining the cost function as

$$\mathcal{J} = \sum_{i=1}^{n_p - 1} \left(\|e_{k+i}\|_Q + \|u_{k+i-1}\|_R \right) + V_{n_p}$$
$$s.t. \;\; x_{k+1} = F^T(x_k, u_k), \quad x_{n_p} \in \mathcal{X}_f$$

with V_{n_p} and $\mathcal{X}_f$ being the terminal cost and constraints set suitably constructed for keeping the state bounded (Camacho and Alba, 2013). This new formulation gives rise to possible feasibility issues so that solution might not exist for a given set of initial conditions.

On the other hand, reformulation of the tracking MPC problem under multi-rate sampling allows to overcome those unstability and feasibility issues while avoiding unnecessary complications. Indeed, it was proved in Elobaid et al. (2019a) that, given (1) with well-defined relative degree and the MPC problem over (5) subject to the MR-SD model (4), the optimal SD feedback always exists and is uniquely defined as a formal series in powers of δ for all $n_p = n_c \geq 1$. Thus using MR control at the implementation level of MPC allows to handle the issues arising due to sampling, without resorting to penalizing terminal costs and constraints. However, it places a greater burden on the sampling and hold devices, assumes a cheap control, and does not address the issue of working in open loop over intervals of length T; this motivates the proposed control scheme.

3. MULTI-RATE PLANNING FOR MPC CONTROL

For system (1), it is required to design a single rate piecewise constant state feedback (with sampling period δ) that tracks samples of suitable reference $v(\cdot)$ at prefixed sampling instants (that is $v_k = v(kT)$, $T = m\delta$) by minimizing the cost functional (5) at all $t = k\delta$, $k \geq 0$ and ensuring boundedness of the closed loop trajectories. In the nominal multi-rate scenario and when $R = 0$ and $Q = \mathrm{I}$, this can be achieved by considering (4) with $m \geq 1$ and solving in $(u_k^1, u_k^2, .., u_k^m)$ the following system of nonlinear algebraic equations

$$H(F_m^T(x_k, u_k^1, \ldots, u_k^m)) = \mathbf{V}_{k+1} \tag{6}$$

where $H : \mathbb{R}^n \to \mathbb{R}^m$ and the vector $\mathbf{V}$ are respectively suitable augmented output and reference vectors so as to guarantee that, by invoking the implicit function theorem, the solution exists.

As already commented, major limits of this stand in its lack of robustness with respect to model uncertainties and sampling approximations and to the fact it works in open loop over time intervals of length T. How to improve its effectiveness? We propose to design a sampled-data single-rate control law, acting at all $t = k\delta$ and based on the corresponding sampled measures of the state, through a sampled-data MPC procedure that makes use of the intermediate reference output values y_k^i resulting from the application of the nominal control sequence computed from (6) to the MR-SD model (4). This reference is said to be admissible in the sense of the definition below.

Definition 3.1. A sampled-data reference sequence $\{v_k, k \geq 0\}$ is said to be admissible for (1) from $x_0 \in \mathbb{R}^n$ if, for a suitable integer $m \leq n$, equality (6) has a solution $\{u_k^i, i = 1, \ldots, m\}$ which remains bounded. It will be said SR or MR admissible if $m = 1$ or $m > 1$, respectively.

Note that, from Definition 3.1 a MR admissible sequence can be suitably enriched to be SR admissible, as suggested by the following result.

Theorem 3.1. Consider the system (1) and let $v(t)$ be a reference signal to be tracked at $t = kT$ for $k \to \infty$. Denote by $\{v_k = v(kT), k \geq 0\}$ the sequence of samples of the reference that is assumed to be MR admissible for a suitable $m > 1$ under the input sequence $\{\hat{u}_k^i, \quad k \geq 0, i = 1, \ldots, m\}$ solution to (6) for all $k \geq 0$. Let $\{\hat{y}_k^i, \quad k \geq 0, i = 1, \ldots, m\}$ be the augmented reference generated by $\hat{y}_k^1 = v_k$, and for, $i = 2, \ldots, m$, $\hat{y}_k^i = h(\hat{x}_k^i)$ with $\hat{x}_k^1 = x(kT)$ $\hat{x}_k^i = F^\delta(\hat{x}_k^{i-1}, \hat{u}_k^{i-1})$. Then, the unconstrained MPC problem defined via the cost (5) with $e_{k+i} = \hat{y}_k^i - y_k^i$ subject to the SR-SD model (2) admits a solution which is bounded for $n_p = n_c \geq m$ and $R = 0$.

Proof: Whenever $\{v_k, k \geq 0\}$ is admissible, then there exists a solution sequence $\{\hat{u}_k^i, \quad k \geq 0, i = 1, \ldots, m\}$ such that (6) is solved and s.t $y_{k+1} = v_{k+1}, k \geq 0$. Then applying the solution to (2), one gets $\hat{x}_k^i = F^\delta(\hat{x}_k^{i-1}, \hat{u}_k^{i-1})$ and correspondingly the intermediate output values $\hat{y}_k^i = h(\hat{x}_k^i)$. Setting $\delta = \frac{T}{m}$ this sequence is SR admissible by construction, and by Definition 3.1 there exists a sequence of controls such that $y(kT) = \hat{v}_k$, so implying feasibility. To show that the MPC optimization problem recovers this solution, one sets $n_p = n_c$ and the proof proceeds along the lines of (Elobaid et al., 2019a, Th.2, Prop.7). $\triangleleft$

Note that the statement above does not assume that the reference can be tracked in continuous time, but merely that it is MR admissible. Given a reference $v(t)$ that system (1) can exactly track (in the sense of Isidori (2013)[Chapter 4]), then a fast sampling of this reference v_k is MR admissible, namely one can define $H(x) = (h(x), L_f h(x), \ldots, L_f^{r-1} h(x))^\top, \mathbf{V} = (\mathbf{v}, \dot{\mathbf{v}}, \ldots, \mathbf{v}^{(r-1)})^\top$ as in (Monaco and Normand-Cyrot, 1991) corresponding to which which (6) admits a solution. The following result can be hence given.

Corollary 3.1. Suppose system (1) is minimum-phase and has a well defined relative degree $r \leq n$, then equality (6) always admits a solution with a multi-rate of order $m \geq r$.

Proof: Since (1) has a well defined relative degree, then the input-output feedback linearization problem is solvable so that, under feedback and change of coordinates, (1) reads

$$\dot{z} = Az + Bv, \quad \dot{\eta} = q(z, \eta) + p(z, \eta)v$$

with $A, B, q(\cdot), p(\cdot)$ as in Isidori (2013)[Chapter 4]. Accordingly, the MR equivalent model of order $m \geq r$ is given by

$$z_{k+1} = A_m^T z_k + B_m^T \underline{v}_k, \quad \eta_{k+1} = \tilde{F}_m^T(z_k, \eta_k, \underline{v}_k) \tag{7}$$

with $A_m^T = (e^{A\delta})^m, B_m^T = \left[A_{sr}^{m-1} B_{sr} \ \ldots \ B_{sr}\right]$ and $A_{sr} = e^{A\delta}, B_{sr} = \int_0^\delta e^{A\delta} u(\tau) d\tau$ and $\tilde{F}_m^T(z_k, \eta_k, \underline{v}_k) = e^{\delta(\mathrm{L}_q + v_k^1 \mathrm{L}_p)} \ldots e^{\delta(\mathrm{L}_q + v_k^m \mathrm{L}_p)} x\big|_{(z_k, \eta_k)}$. In this setting, z_k corresponds to H so that (6) reduces to a linear map on the multi-rate inputs $\underline{v}_k$. Because B_m^δ is full rank by construction and because the relative degree is invariant under feedback, one gets the solution $\hat{\underline{v}}_k = (B_m^T)^{-1}(\mathbf{V}_{k+1} - A_m^\delta z_k)$. Moreover, by the fact that $\dot{\eta} = q(z, \eta) + p(z, \eta)v$ is stable, and combined with Theorem 3.1 and the arguments in Monaco and Normand-Cyrot (1988), one gets that the planned trajectories are admissible for the overall dynamics over the small interval $\delta = \frac{T}{m}$. ◁

Theorem 3.1 applies to finitely discritizable systems.

4. PLANNING AND CONTROL ALGORITHM

In this section, and referring to the discussion above, we present in a detailed manner the control scheme for designing a sampled-data feedback $u_k = u(x_k)$ ensuring tracking of a given output profile at the sampling instants $t = kT$ for all $k \geq 0$ by exploiting a planned *admissible* trajectory generated via the MR model (4).

We assume that the dynamics (1) is finitely descritizable with $F^T_m(\cdot, \hat{u}^1_k, \dots, \hat{u}^r_k)$ denoting the corresponding multi-rate finite model of order m (possibily computed under coordinate change and preliminary feedback).

The following algorithm is proposed by using the admissible sequence $\{\hat{y}^i_k, \quad i = 1, \dots, m, \ k \geq 0\}$ defined in Theorem 3.1 as a reference trajectory for the MPC with $n_p = m$. Such a trajectory is computed and updated at all $t = kT$ based on the nominal multi-rate solution defined through (4). Thus, for all $t \in \{kT, kT + \delta, \dots, kT + (m-1)\delta\}$ the planned reference sequence is fed to the MPC for computing the optimizing controller which is guaranteed to exist for $R \geq 0$ small enough by virtue of Theorem 3.1. Specifically, the algorithm works over the steps depicted in Algorithm 1.

Algorithm 1 Planning and control algorithm

Initialization:

$$\mathbf{V}_a \leftarrow (v_{k+1}, v_{k+2})^\top$$
$$x_k \leftarrow x(kT), Q \leftarrow Q, R \leftarrow R, m \leftarrow m$$

while $t \geq 0$ **do**
  if $t = (k+j)T, j \in \mathbb{Z}_{\geq 0}$ **then**
    $k \leftarrow k + j$
    $(\hat{y}_k, \hat{y}) = \text{Planning}(x_k)$
    $u_k = \text{Control}(m, Q, R)$
  else
    for $t = kT + i\delta, i = 1, \dots, m-1$ **do**
      $u^i_k = \text{Control}(m, Q, R)$

procedure $(\hat{y}_k, \hat{y}_{k+1}) = \text{PLANNING}(x_k, \mathbf{V}_a)$
  $\hat{x}^1_k \leftarrow x_k$

$$\begin{pmatrix} \hat{u}^1_k \\ \vdots \\ \hat{u}^m_k \\ \vdots \\ \hat{u}^m_{k+1} \end{pmatrix} = \begin{pmatrix} (h \circ F^T_m)^{-1}(\hat{x}_k, v_{k+1}) \\ (h \circ F^T_m)^{-1}(\cdot, v_{k+2}) \circ (F^T_m)^{-1}(\hat{x}_k, v_{k+1}) \end{pmatrix} \quad (8)$$

  for $j = 0 : 1$ **do**
    for $i = 1 : m$ **do**

$$\hat{x}^{i+1}_{k+j} = F^\delta(\hat{x}^i_{k+j}, \hat{u}^i_{k+j}), \quad \hat{y}^i_{k+j} = h(\hat{x}^i_{k+j}).$$

procedure $u^\star_k = \text{CONTROL}(m, Q, R, \hat{y}^1_k, \dots, \hat{y}^{n_p}_k)$
  $n_p \leftarrow m$

$$u_k = \text{argmin}_{u_k} \quad \sum_{i=1}^{n_p} \left(\|\hat{y}^i_k - y^i_k\|_Q + \|u^{i-1}_k\|_R \right)$$

  $u^\star_k = u^1_k$

Remark 4.1. From the previous arguments, we only exploit the samples of the reference over two big steps (that is v_{k+1}, v_{k+2} and correspondingly setting in the MR planner (6) an augmented output vector $H_a(x) = (y_k, y_{k+1})^\top$). This is due to the fact that in the implementation of a receding horizon algorithm, we will need explicitly the values of the desired reference sequence over $n_p = m$ steps, and writing the second iteration of the MPC (i.e., at time $t = kT + \delta$), one notes the explicit dependence of the (optimal) control on values of the desired output trajectories at $\hat{y}^{m+1}_k = \hat{y}^1_{k+1}$.

Remark 4.2. The design the planner can be worked out on a simplified sampled-data model so to reduce the computational burden related to solve equality (8). When the conditions of Corollary 3.1 are met, the computations associated with the planner are simply the inversion of a matrix B^T_m which is full rank by construction. Consequently, as shown in the case study, one in principle uses a simplified finite model for the planner, while a more exhaustive one is employed by MPC for prediction.

Remark 4.3. The proposed control scheme inherits the nominal robustness properties of MPC (e.g. Grimm et al. (2007); Picasso et al. (2010); Grimm et al. (2004)). This will be further demonstrated in the following case study.

5. THE PVTOL AS A CASE STUDY

Let the model of the PVTOL (planar vertical take-off and landing) aircraft take the form:

$$\begin{aligned} \ddot{x} &= -sin(\theta)v_1 + \epsilon cos(\theta)v_2 \\ \ddot{z} &= cos(\theta)v_1 - 1 + \epsilon sin(\theta)v_2 \\ \ddot{\theta} &= v_2 \end{aligned} \quad (9)$$

with output $y = h(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}) = (x, z)^\top$.

5.1 Construction of the simplified MR planner model

To use our control scheme, we first define the multi-rate sampled-data model of the PVTOL. To this end, it is known (e.g. Di Giamberardino and Djemai (1994)) that (9) is feedback equivalent to a finitely discretizable system, by setting

$$v = \begin{pmatrix} \frac{1}{cos\theta} + \varepsilon\dot{\theta}^2 \\ -2\dot{\theta}^2 \tan\theta \end{pmatrix} + \begin{pmatrix} \frac{1}{cos\theta} & 0 \\ 0 & \cos^2\theta \end{pmatrix} u$$

together with the coordinates change

$$\zeta = \varphi(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}) = \epsilon\varphi_1(\cdot) + \varphi_2(\cdot)$$

with $\varphi_1(\cdot) = \begin{pmatrix} \cos\theta & -\dot{\theta}\sin\theta & 0 & -\sin\theta & 0 & -\dot{\theta}\cos\theta \end{pmatrix}^\top$, $\varphi_2(\cdot) = \begin{pmatrix} z & \dot{z} & \tan\theta & x & \frac{\dot{\theta}}{\cos^2\theta} & \dot{x} \end{pmatrix}^\top$, thus obtaining

$$\dot{\zeta} = \tilde{f}(\zeta) + \tilde{g}_1(\zeta)u_1 + \tilde{g}_2(\zeta)u_2$$
$$\tilde{f}(\zeta) = (\zeta_2 \ 0 \ \zeta_5 \ \zeta_6 \ 0 \ -\zeta_3)^\top, \tilde{g}_1(\zeta) = (0 \ 1 \ 0 \ 0 \ 0 \ -\zeta_3)^\top$$
$$\tilde{g}_2(\zeta) = (0 \ 0 \ 0 \ 0 \ 1 \ 0)^\top$$

A multi-rate of order 2 on u_1 and 4 on u_2 can be employed to ensure the invertibility of the sampled-data dynamics. Explicitly writing

$$u_1(t) = u^j_1(k), \, t \in [(k + \frac{j-1}{2})T, (k + \frac{j}{2})T[, \ j = 1, 2$$
$$u_2(t) = u^j_2(k), \, t \in [(k + \frac{j-1}{4})T, (k + \frac{j}{4})T[, \ j = 1, 2, 3, 4.$$

and developing the calculations and rearranging the terms, we obtain the MR SD equivalent model of the PVTOL as:

$$\begin{aligned}\zeta_{k+1} &= (A^{\delta} + B_1^{\delta}(u_1^2))^2(A^{\delta} + B_1^{\delta}(u_1^1))^2\zeta_k \\ &+ (A^{\delta} + B_1^{\delta}(u_1^2))^2(I + B_1^{\delta}(u_1^1))B_0^{\delta}(u_1^1, u_2^1) \\ &+ (A^{\delta} + B_1^{\delta}(u_1^2))^2 B_0^{\delta}(u_1^1, u_2^2) \\ &+ (I + B_1^{\delta}(u_1^1))B_0^{\delta}(u_1^2, u_2^3) + B_0^{\delta}(u_1^2, u_2^4)\end{aligned} \tag{10}$$

with $\zeta_k = \varphi(x_k, \dot{x}_k, z_k, \dot{z}_k, \theta_k, \dot{\theta}_k)$ for all $k \geq 0$ with

$$A^{\delta} = \begin{pmatrix} 1 & \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \delta & 0 \\ 0 & 0 & -\frac{\delta^2}{2} & 1 & -\frac{\delta^3}{6} & \delta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 0 & -\frac{\delta^2}{2} & 1 \end{pmatrix}, B_0^{\delta}(v, w) = \begin{pmatrix} \frac{\delta^2 v}{2} \\ \delta v \\ \frac{\delta^2 w}{2} \\ \frac{-\delta^4(1+v)w}{24} \\ \delta w \\ \frac{-\delta^3(1+v)w}{6} \end{pmatrix}$$

$$B_1^{\delta}(v) = \begin{pmatrix} & & \mathbf{0}_{3\times 6} & & \\ & -\frac{\delta^2}{2}v & 0 & -\frac{\delta^3}{6}v & 0 \\ \mathbf{0}_{3\times 2} & 0 & 0 & 0 & 0 \\ & -\delta v & 0 & -\frac{\delta^2}{2}v & 0 \end{pmatrix}$$

with $\delta \geq 0$ being the sampling period and $T = 4\delta$. The model (10) will be used for planning the admissible reference trajectories.

5.2 Planning and control

For all $t = kT$, planning of the intermediate output references is made on the basis of the simplified equivalent model (10). Hence one gets for all $t = kT + (i-1)\delta$, an admissible sequence $(\hat{x}^i_{k+j}, \dot{\hat{x}}^i_{k+j}, \hat{z}^i_{k+j}, \dot{\hat{z}}^i_{k+j}, \hat{\theta}^i_{k+j}, \dot{\hat{\theta}}^i_{k+j})^\top = \varphi^{-1}(\hat{\zeta}^i_k)$, and thus the intermediate reference output values $\{(\hat{x}^i_{k+j}, \hat{z}^i_{k+j}), \quad i = 1, 2, 3, 4 \text{ and } j = 0, 1\}$ for system (9). Consequently, for all $t = kT + (i-1)\delta$, the MPC computes the feedback u^i_k (for $i = 1, \ldots, 4$) with the sampled-data SR model of the PVTOL (in the form (2)) used for prediction. This feedback is then applied to the simulation model of system (9) while recomputing the reference for all $t = kT$.

5.3 Simulations

In the following we will compare the proposed control algorithm (denoted MR-MPC) to the stand-alone MPC with the trajectory planner proposed by Luigi Biagiotti (2019) (denoted FIR-MPC). Figure 1 depicts the nominal scenario (with no external perturbations nor model uncertainties), while in Figure 2 actuator disturbances as well as uncertainties on the parameter ϵ are considered. In all simulations $\delta = 1$ seconds while $n_p = n_c = 4$ seconds and $Q = \mathrm{I}$. Both schemes will utilize a sequential quadratic programming SQP based optimization solver. Time varying references are fixed on both the lateral and vertical displacements (that is x, z respectively) as a ramp signal (shown in black) with velocity $v_0 = 1$ m/s to be tracked at $t = kT, T = 4\delta$.
It results that the proposed control algorithm favourably compares to the FIR-MPC.
In the nominal case (Figure 1), we set $R = 0$. Contrarily to the MR-MPC scheme, the FIR-MPC is unable to follow the reference over the sampling steps δ and an off-set is evident. While the number of iterations in each instant to compute the minimizer is slightly more in the MR-MPC case, the minimum obtained is lower, namely it is $V_{n_p} = 0$ compared to 0.5 obtained by the FIR-MPC.
In Figure 2, $R > 0$ and a white noise is added on the actuation signal. In addition the parameter $\epsilon = 0.5$ in the simulation model, while the nominal value $\epsilon = 0.8$ is used to construct the planner and the prediction model for the MPC. In this case as well, the proposed algorithm is able to track the reference over the big intervals T, despite the effect of uncertainties being evident (although acceptable i.e. $|\theta| \leq 0.1$ rads) in the internal dynamics. Note also that while the number of iterations required to obtain the minimum is comparable between the two algorithms, the value function was still lower $V_{n_p} = 0.08$ for MR-MPC compared to 0.1 for FIR-MPC. Further simulations are reported in Elobaid et al. (2019b) comparing the MR-MPC with FIR-MPC and stand-alone MR control in different situations.

6. CONCLUSIONS

We establish an intuitive implementation of sampled-data tracking control through multi-rate planning utilizing discrete time nonlinear single-rate MPC for reference tracking. This is done by highlighting the role played by ensuring the availability of good and admissible references to the MPC controller so to improve the effectiveness of the two stand-alone design methodologies. Future works concern the study of the effect of incorporating constraints on the control, on the benefits incurred by the proposed scheme.

REFERENCES

Åström, K., Hagander, P., and Sternby, J. (1984). Zeros of sampled systems. *Automatica*, 20(1), 31 – 38.

Bemporad, A. and Morari, M. (1999). Robust model predictive control: A survey. In *Robustness in identification and control*, 207–226. Springer.

Byun, D.G. (1988). Predictive control: A review and some new stability results. *IFAC Proceedings Volumes*, 21(4), 81 – 87. IFAC Workshop on Model Based Process Control, Atlanta, GA, USA, 13-14 June.

Camacho, E.F. and Alba, C.B. (2013). *Model predictive control*. Springer Science & Business Media.

Di Giamberardino, P. and Djemai, M. (1994). Multirate digital control of a pvtol. In *Proc. 33th CDC*, 3844–3849.

Elobaid, M., Mattioni, M., Monaco, S., and Normand-Cyrot, D. (2019a). On unconstrained mpc through multirate sampling. *IFAC-PapersOnLine*, 52(16), 388–393.

Elobaid, M., Mattioni, M., Monaco, S., and Normand-Cyrot, D. (2019b). Sampled-data tracking under model predictive control and multirate planning: further simulations and remarks. Technical report, DIAG, Università degli Studi di Roma La Sapienza. URL `https://hal.archives-ouvertes.fr/hal-02365409`.

García, C.E., Prett, D.M., and Morari, M. (1989). Model predictive control: Theory and practice—a survey. *Automatica*, 25(3), 335 – 348.

Grimm, G., J. Messina, M., E. Tuna, S., and R. Teel, A. (2004). Examples when model predictive control is non-robust. *Automatica*, 40, 1729–1738.

Grimm, G., Messina, M.J., Tuna, S.E., and Teel, A.R. (2007). Nominally robust model predictive control with state constraints. *IEEE Transactions on Automatic Control*, 52(10), 1856–1870.

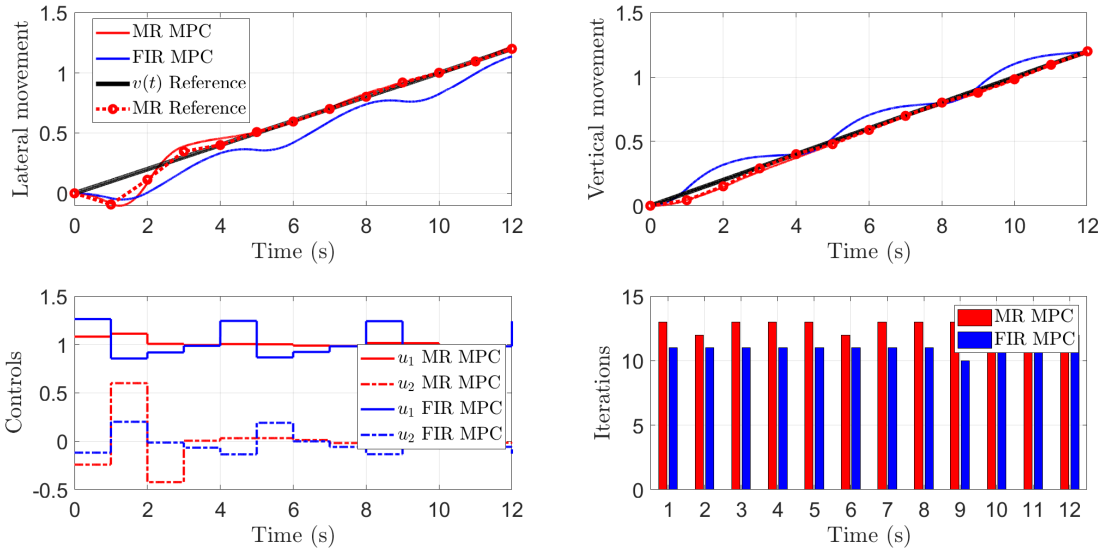

Fig. 1. Nominal tracking, $R = 0, Q = \mathrm{I}$

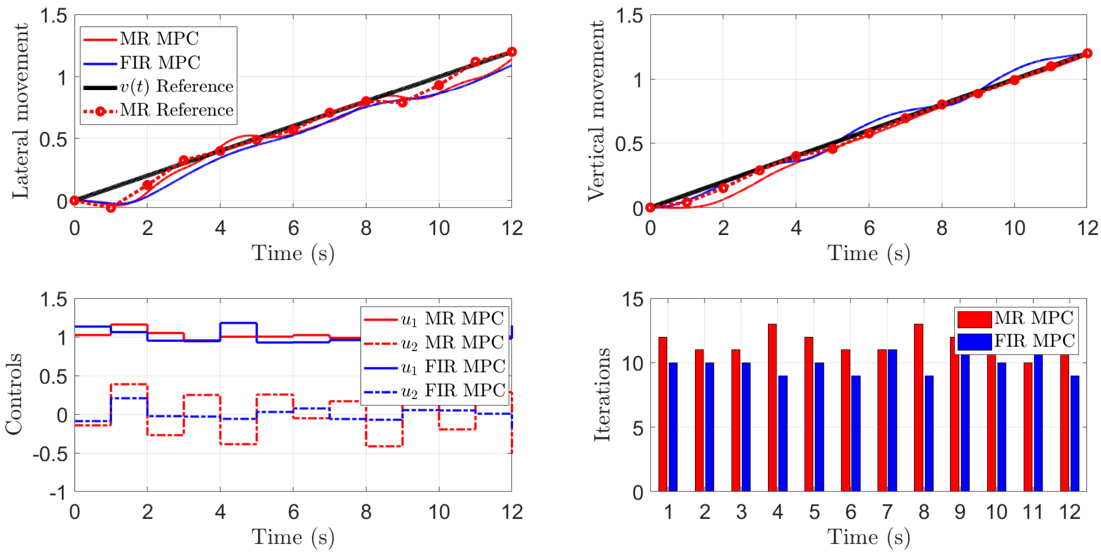

Fig. 2. Perturbed tracking, $\epsilon = 0.5, R = 0.5\mathrm{I}, Q = \mathrm{I}$ and white actuation noise with mean 10^{-3}

Isidori, A. (2013). *Nonlinear control systems*. Springer Science & Business Media.

Luigi Biagiotti, C.M. (2019). Trajectory generation via fir filters: A procedure for time-optimization under kinematic and frequency constraints. *Control Engineering Practice*, 87, 43 – 58.

Magni, L., Raimondo, D.M., and Allgöwer, F. (2009). Nonlinear model predictive control. *Lecture Notes in Control and Information Sciences*, (384).

Mattioni, M., Hassan, M., Monaco, S., and Normand-Cyrot, D. (2017). On partially minimum phase systems and nonlinear sampled-data control. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 6101–6106.

Monaco, S. and Normand-Cyrot, D. (1988). Zero dynamics of sampled nonlinear systems. *Systems & control letters*, 11(3), 229–234.

Monaco, S. and Normand-Cyrot, D. (1991). Multirate sampling and zero dynamics: from linear to nonlinear. In *Nonlinear Synthesis*, 200–213. C. Byrnes, A. Isidori Eds., Birkauser.

Monaco, S. and Normand-Cyrot, D. (2001). Issues on nonlinear digital systems. *Semi-Plenary Conference, European Journal of Control*, 7(2-3), 160–178.

Monaco, S. and Normand-Cyrot, D. (1992). An introduction to motion planning using multirate digital control. 1780 – 1785 vol.2.

Picasso, B., Desiderio, D., and Scattolini, R. (2010). Robustness analysis of nominal model predictive control for nonlinear discrete-time systems. *IFAC Proceedings Volumes*, 43(14), 214 – 219. 8th IFAC Symposium on Nonlinear Control Systems.