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About Dissipative and Pseudo Port-Hamiltonian Formulations of Irreversible Newtonian Compressible Flows

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Abstract: In this paper we consider the physical-based modeling of 3D and 2D Newtonian fluids including thermal effects in order to cope with the first and second principles of thermodynamics. To describe the energy fluxes of non-isentropic fluids we propose a pseudo port-Hamiltonian formulation, which includes the rate of irreversible entropy creation by heat flux. For isentropic fluids, the conversion of kinetic energy into heat by viscous friction is considered as an energy dissipation associated with the rotation and compression of the fluid. Then, a dissipative port-Hamiltonian formulation is derived for this class of fluids. In the 2D case we modify the vorticity operators in order to preserve the structure of the proposed models. Moreover, we show that a description for inviscid or irrotational fluids can be derived from the proposed models under the corresponding assumptions leading to a pseudo or dissipative port-Hamiltonian structures.

Keywords: Port-Hamiltonian systems, Compressible Fluids, Entropy, Newtonian fluids, Vorticity

1. INTRODUCTION

In control theory, models are required to describe the plant dynamics with sufficient precision and simplicity. In particular, energy-based control methods, such as energy-shaping (Macchelli et al., 2017), IDA-PBC (Vu et al., 2015), observer-based control (Toledo et al., 2019), among others, require models describing the energy flux of the physical phenomena. These models are commonly formulated using the port-Hamiltonian (PH) framework.

PH systems provides useful properties for the control theory, such as passivity, stability in the Lyapunov sense and power-preserving connectivity by ports (van der Schaft and Jeltsema, 2014). For infinite-dimensional systems a PH formulation based in a Stokes-Dirac structures is proposed by Le Gorrec et al. (2005) and an extension to include dissipative effects is presented in Villegas et al. (2006). As soon as irreversible thermodynamics systems are considered, the PH formulation are not valid anymore. In this sense, Ramirez et al. (2013) proposes a new formulation way, called irreversible PH formulation, for thermodynamic finite-dimensional systems.

In the current paper we focus in the dynamics and thermodynamics of non-reactive Newtonian compressible fluids. This kind of fluids has been studied in different engineering areas, from biomedical systems, as the phono-respiratory modeling (Mora et al., 2018), to Fluid-Structure-Interaction problems (Cardoso-Ribeiro et al., 2017).

Different energy-based approaches have been presented in literature to describe Newtonian fluids. However, these approaches are constrained due to the assumptions that were considered. For example, for ideal isentropic fluids, 1D PH models are proposed by Macchelli et al. (2017) for inviscid fluids and Kotyczka (2013) with friction dissipation, for control purposes and pipe network modeling, respectively, where the vorticity effects are neglected as a consequence of the one-dimensional assumption. A Hamiltonian model based on stream functions to describe the vorticity dynamics of 2D fluid is presented in Swaters (2000), however this model is limited to potential flows. A dissipative PH model of 3D irrotational fluids is proposed by Matignon and Hélie (2013) and a general Hamiltonian model for inviscid fluid is presented by van der Schaft and Maschke (2002). For non-isentropic fluids a one-dimensional model for reactive flows is proposed by Altmann and Schulze (2017), neglecting the vorticity effects.

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In this work we present a general energy-based formulation for isentropic and non-isentropic 3D compressible fluids using a pseudo-PH framework, including the vorticity effects in the velocity field. First we develop a pseudo-PH model for non-isentropic fluids, focusing on compressible non-reactive flows. Later, we describe the treatment of the viscous tensor under an isentropic assumption for the fluid, to obtain a dissipative PH model. Finally, we describe the necessary considerations to conserve the PH structure of the models proposed for 2D and 1D fluids.

2. NON-ISENTROPIC FLUID

In this section we describe the energy-based formulation for non-insentropic fluids. Denote by ρ, $\mathbf{v}$, s and T the density, velocity field, entropy per unit of mass, and temperature of the fluid, respectively. The fluid dynamics are described by the following governing equations:

$$\partial_t \rho = -\text{div}\ \rho\mathbf{v} \tag{1a}$$
$$\rho\partial_t \mathbf{v} = -\rho\mathbf{v}\cdot\mathbf{Grad}\ \mathbf{v} - \mathbf{grad}\ p - \mathbf{Div}\ \boldsymbol{\tau} \tag{1b}$$
$$\rho T\partial_t s = -\rho T\mathbf{v}\cdot\mathbf{grad}\ s - \boldsymbol{\tau} : \mathbf{Grad}\ \mathbf{v} - \text{div}\ \mathbf{q} \tag{1c}$$

where (1a) and (1b) are the continuity and motion equations, respectively, and (1c) is the general equation of heat transfer (Landau and Lifshitz, 1987); p is the static pressure, $\boldsymbol{\tau}$ is the viscosity tensor and $\mathbf{q}$ is the heat flux. In this work, we consider non-reactive Newtonian fluids. Then, $\boldsymbol{\tau}$ and $\mathbf{q}$ are defined as:

$$\boldsymbol{\tau} = -\mu\left[\mathbf{Grad}\ \mathbf{v} + [\mathbf{Grad}\ \mathbf{v}]^{\mathrm{T}}\right] - \eta\,(\text{div}\ \mathbf{v})\, I \tag{2}$$
$$\mathbf{q} = -K\mathbf{grad}\ T \tag{3}$$

where $\eta = \frac{2}{3}\mu - \kappa$, μ and κ are the shear and dilatational viscosities, respectively, I is the identity matrix and K is a non-negative matrix that describes the thermal conductivity of the fluid (Öttinger, 2005).

Denote by $\boldsymbol{\omega} = \mathbf{curl}\ \mathbf{v}$ the fluid vorticity, that describes the tendency of the flow to rotate. Then, using the identity (A.1) the term $\mathbf{v}\cdot\mathbf{Grad}\ \mathbf{v}$ in (1b), can be rewritten as $\mathbf{v}\cdot\mathbf{Grad}\ \mathbf{v}{=}\mathbf{grad}\ \left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \boldsymbol{\omega}\times\mathbf{v}$, where $\boldsymbol{\omega}\times\mathbf{v}$, describes the power exchange between the velocity field components given by the fluid rotation.

Definition 1. Let $\boldsymbol{\omega} = [\omega_1\ \omega_2\ \omega_3]^{\mathrm{T}}$ the vorticity vector of the fluid. We define the fluid Gyroscope $G_{\boldsymbol{\omega}}$ as the skew-symmetric matrix, such that $G_{\boldsymbol{\omega}}\mathbf{v} = \boldsymbol{\omega}\times\mathbf{v}$. For 3D fluids, the Gyroscope is given by:

$$G_{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4}$$

On the other hand in (1c), the term div $\mathbf{q}$ can be rewritten as $\text{div}\ \mathbf{q} = T\text{div}\ \mathbf{q}_s + \mathbf{q}_s\cdot\mathbf{grad}\ T$ where $\mathbf{q}_s$ is the entropy flux by heat conduction (Bird et al., 2015). In non-reactive fluids $\mathbf{q}_s$ is defined by $\mathbf{q}_s = -\frac{K}{T}\mathbf{grad}\,T$. Then, considering the Gibbs equation

$$du = -pd\left(\frac{1}{\rho}\right) + Tds \tag{5}$$

that describes the change of the specific internal energy u with respect to changes of ρ and s, the fluid enthalpy $h = u + p/\rho$ and the relationships $\mathbf{grad}\ \frac{1}{\rho}p = \frac{1}{\rho}\mathbf{grad}\ p + p\mathbf{grad}\ \frac{1}{\rho}$ and $T = \partial_s u$. Then, we can rewrite the fluid dynamics in terms of the state variables and the temperature, namely:

$$\partial_t \rho = -\text{div}\ \rho\mathbf{v} \tag{6a}$$
$$\partial_t \mathbf{v} = -\mathbf{grad}\ \left(\frac{\mathbf{v}\cdot\mathbf{v}}{2} + h\right) - G_{\boldsymbol{\omega}}\mathbf{v} + T\mathbf{grad}\ s - \frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau} \tag{6b}$$
$$\partial_t s = -\mathbf{v}\cdot\mathbf{grad}\ s - \frac{\boldsymbol{\tau}}{\rho T} : \mathbf{Grad}\ \mathbf{v} - \frac{\mathbf{q}_s}{\rho T}\cdot\mathbf{grad}\ T - \frac{1}{\rho}\text{div}\ \mathbf{q}_s \tag{6c}$$

Note that entropy generation, is given by the following non-negative condition (Öttinger, 2005):

$$-\frac{1}{\rho T}\boldsymbol{\tau} : \mathbf{Grad}\ \mathbf{v} - \frac{\mathbf{q}_s}{\rho T}\cdot\mathbf{grad}\ T \geq 0 \tag{7}$$

where $-\frac{1}{\rho T}\boldsymbol{\tau} : \mathbf{Grad}\ \mathbf{v}$ is the rate of entropy creation by the kinetic energy dissipated into heat by viscosity friction, and $-\frac{\mathbf{q}_s}{\rho T}\cdot\mathbf{grad}\ T$ is the rate of entropy creation by heat flux.

2.1 Pseudo port-Hamiltonian description of non-isentropic fluids

Consider the fluid domain Ω with boundary $\partial\Omega$. The total energy of the fluid described in (6) is given by:

$$\mathcal{H} = \int_\Omega \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho u\,(\rho, s) \tag{8}$$

Then, the fluid efforts $\mathbf{e} = \left[e_\rho\ \mathbf{e}_{\mathbf{v}}^{\mathrm{T}}\ e_s\right]^{\mathrm{T}}$ are given by the variational derivative of the energy, namely

$$\begin{bmatrix} e_\rho \\ \mathbf{e}_{\mathbf{v}} \\ e_s \end{bmatrix} = \begin{bmatrix} \delta_\rho\mathcal{H} \\ \delta_{\mathbf{v}}\mathcal{H} \\ \delta_s\mathcal{H} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + h \\ \rho\mathbf{v} \\ \rho T \end{bmatrix} \tag{9}$$

where $\delta_\rho\mathcal{H} = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u + \rho\partial_\rho u$. Given the relationship $p = \rho^2\partial_\rho u$, we obtain $\delta_\rho\mathcal{H} = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + h$. Using (9) in (6) we can express the fluid dynamics as:

$$\partial_t \rho = -\text{div}\ \mathbf{e}_{\mathbf{v}} \tag{10a}$$
$$\partial_t \mathbf{v} = -\mathbf{grad}\ e_\rho - \frac{G_{\boldsymbol{\omega}}}{\rho}\mathbf{e}_{\mathbf{v}} + \mathbf{grad}\ s\frac{e_s}{\rho} - \frac{1}{\rho}\mathbf{Div}\ \frac{\boldsymbol{\tau}}{\rho T}e_s \tag{10b}$$
$$\partial_t s = -\mathbf{grad}\ s\cdot\frac{\mathbf{e}_{\mathbf{v}}}{\rho} - \frac{1}{\rho T}\boldsymbol{\tau} : \mathbf{Grad}\ \frac{\mathbf{e}_{\mathbf{v}}}{\rho} + \frac{1}{\rho T}\left\|\mathbf{grad}\ \frac{e_s}{\rho}\right\|^2_{\frac{K}{T}} + \frac{1}{\rho}\text{div}\ \left[\frac{K}{T}\mathbf{grad}\ \frac{e_s}{\rho}\right] \tag{10c}$$

To obtain the pseudo-PH formulation, it is necessary to set the interconnections between the components of the fluid dynamics. In the case of the velocity field and the entropy, they are interconnected through the operator $\mathcal{J}_{\boldsymbol{\tau}}$ and the corresponding adjoint $\mathcal{J}_{\boldsymbol{\tau}}^*$ in the effort space. Similarly, the last two terms of the right-hand side in (10c) describes the rate of irreversible entropy creation and entropy diffusion by the heat flux $\mathbf{q}$, thus, these phenomena can be characterized using an operator $\mathcal{J}_{\mathbf{q}}$. The above operators are defined in the following Lemmas.

Lemma 1. Let $\boldsymbol{\tau}$ be a symmetric second order tensor and $\mathcal{J}_{\boldsymbol{\tau}} = \mathbf{grad}\ s\left(\frac{\cdot}{\rho}\right) - \frac{1}{\rho}\mathbf{Div}\ \left(\frac{\boldsymbol{\tau}}{\rho T}\cdot\right)$ an operator on the entropy effort e_s. Then, the adjoint operator $\mathcal{J}_{\boldsymbol{\tau}}^*$ in the effort space of the fluid is given by $\mathcal{J}^*_{\boldsymbol{\tau}} = \mathbf{grad}\ s \cdot \left(\frac{\cdot}{\rho}\right) + \frac{\boldsymbol{\tau}}{\rho T} : \mathbf{Grad}\ \frac{\cdot}{\rho}$, such that

$$\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s \rangle_\Omega - \langle e_s, \mathcal{J}^*_{\boldsymbol{\tau}} \mathbf{e_v} \rangle_\Omega = -\int_{\partial\Omega} \boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\right] \tag{11}$$

Proof. Consider the operators $\mathcal{J}_{\boldsymbol{\tau}}$ and $\mathcal{J}^*_{\boldsymbol{\tau}}$ above described. Defining $\boldsymbol{\sigma} = \boldsymbol{\tau} e_s/\rho T$ and $\mathbf{u} = \mathbf{e_v}/\rho$, and using the identity (B.2), we can obtain that $\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s \rangle_\Omega = \langle \mathcal{J}^*_{\boldsymbol{\tau}} \mathbf{e_v}, e_s \rangle_\Omega - \int_{\partial\Omega} [\boldsymbol{\sigma} \cdot \mathbf{u}] \cdot \mathbf{n}$, where $\mathbf{n}$ is the normal outward unitary vector to the boundary $\partial\Omega$, (see (Mora et al., 2020) for details), this implies that $\mathcal{J}^*_{\boldsymbol{\tau}}$ is the formal adjoint of $\mathcal{J}_{\boldsymbol{\tau}}$. Finally, using the mathematical identity $(\boldsymbol{\tau} \cdot \mathbf{v}) \cdot \mathbf{n} = \boldsymbol{\tau} : \mathbf{v}\mathbf{n}^{\mathrm{T}}$ we obtain

$$\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s \rangle_\Omega - \langle e_s, \mathcal{J}^*_{\boldsymbol{\tau}} \mathbf{e_v} \rangle_\Omega = -\int_{\partial\Omega} \boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\right]$$

where $\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}$ is the tangential projection of $\mathbf{v}$. □

Lemma 2. Let $\mathcal{J}_{\mathbf{q}}$ be an operator on space of the entropy effort e_s, defined as

$$\mathcal{J}_{\mathbf{q}} = \mathcal{Q}_T - \mathcal{G}^*_T S_T \mathcal{G}_T, \tag{12}$$

where Q_T and $\mathcal{G}^*_T S_T \mathcal{G}_T$ describe two phenomena associated with the heat flux. $\mathcal{Q}_T = \frac{1}{\rho T}\|\mathbf{grad}\ \frac{\cdot}{\rho}\|^2_{S_T}$ describes the entropy creation, such that $\mathcal{Q}_T e_s \geq 0, \forall e_s$, and $-\mathcal{G}^*_T S_T \mathcal{G}_T$ describes the entropy diffusion, where the operator $\mathcal{G}^*_T = \frac{1}{\rho}\mathrm{div}$ is the formal adjoint of $\mathcal{G}_T = -\mathbf{grad}\ \frac{\cdot}{\rho}$ and $S_T = \frac{K}{T} \geq 0$. Then, the rate of entropy addition by heat flux can be expressed as $-\frac{1}{\rho T}\mathrm{div}\ \mathbf{q} = \mathcal{J}_{\mathbf{q}} e_s$, satisfying

$$\langle e_s, \mathcal{J}_{\mathbf{q}} e_s \rangle_\Omega = -\int_{\partial\Omega} T\,(\mathbf{q}_s \cdot \mathbf{n}) \tag{13}$$

Proof. Given that the formal adjoint of divergence is minus the gradient, it is easy to prove that $\mathcal{G}^*_T = \frac{1}{\rho}\mathrm{div}$ is the formal adjoint of operator $\mathcal{G}_T = -\mathbf{grad}\ \frac{\cdot}{\rho}$. Note that $\frac{1}{\rho T}\mathrm{div}\ \mathbf{q} = \frac{\mathbf{q}_s}{\rho T} \cdot \mathbf{grad}\ T + \frac{1}{\rho}\mathrm{div}\ \mathbf{q}_s$. Considering that $-\mathbf{q}_s \cdot \mathbf{grad}\ T = \|\mathbf{grad}\ \frac{e_s}{\rho}\|^2_{S_T} = \rho T \mathcal{Q}_T e_s$ and $-\mathrm{div}\ \mathbf{q}_s = \mathrm{div}\ [S_T \mathbf{grad}\ \frac{e_s}{\rho}]$ where $S_T = K/T$. Then, the entropy addition by heat flux can be expressed as (see Mora et al. (2020) for details): $-\frac{1}{\rho T}\mathrm{div}\ \mathbf{q} = (\mathcal{Q}_T - \mathcal{G}^*_T S_T \mathcal{G}_T)\, e_s = \mathcal{J}_{\mathbf{q}} e_s$. Moreover, the inner product in the left-hand side of (13), is given by $\langle e_s, \mathcal{J}_{\mathbf{q}} e_s \rangle_\Omega = -\int_\Omega \mathbf{q}_s \cdot \mathbf{grad}\ \frac{e_s}{\rho} + \frac{e_s}{\rho}\mathrm{div}\ \mathbf{q}_s$. Finally, using the property (B.2) we obtain (13). □

Lemma 3. Let $\mathcal{E}_s = \{\forall e_s : e_s/\rho T = 1\}$ be the space of entropy efforts. Then, the operator $\mathcal{J}_{\mathbf{q}} = \mathcal{Q}_T - \mathcal{G}^*_T S_T \mathcal{G}_T$ is skew-symmetric in $\mathcal{E}_s$.

Proof. Let be the efforts $e_{s,1} \in \mathcal{E}_s$ and $e_{s,2} \in \mathcal{E}_s$. Considering that $\mathbf{grad}\ \frac{e_{s,j}}{\rho} = \mathbf{grad}\ \frac{e_{s,i}}{\rho}, \forall\{e_{s,j}, e_{s,i}\} \in \mathcal{E}_s$ and using the identities in Appendix A, its easy to proof that for boundary conditions equal to 0, $\langle e_{s,1}, \mathcal{J}_{\mathbf{q}} e_{s,2} \rangle_\Omega = -\langle \mathcal{J}_{\mathbf{q}} e_{s,1}, e_{s,2} \rangle_\Omega$ (see (Mora et al., 2020) for details). □

Thus, using the above Lemmas, the fluid dynamics for non-isentropic fluids can be expressed as an energy-based model, as we show in the next proposition.

Proposition 1. Consider a non-isentropic Newtonian compressible fluid, whose total energy is described by (8). Then, the governing equations in (6) can be expressed as the pseudo infinite-dimensional port-Hamiltonian system

$$\partial_t \boldsymbol{x} = \mathcal{J}\mathbf{e} \tag{14}$$

where $\boldsymbol{x} = \begin{bmatrix}\rho & \mathbf{v}^{\mathrm{T}} & s\end{bmatrix}^{\mathrm{T}}$ is the state vector, $\mathbf{e} = \begin{bmatrix}e_\rho & \mathbf{e}^{\mathrm{T}}_{\mathbf{v}} & e_s\end{bmatrix}^{\mathrm{T}}$ is the fluid effort vector described in (9), $\mathbf{f}_R = \mathcal{G}_T e_s$ and $\mathbf{e}_R = S_T \mathbf{f}_R$ are the flow and effort associated with entropy diffusion, and $\mathcal{J}$ is an operator given by

$$\mathcal{J} = \begin{bmatrix} 0 & -\mathrm{div} & 0 \\ -\mathbf{grad} & -\frac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{J}_{\boldsymbol{\tau}} \\ 0 & -\mathcal{J}^*_{\boldsymbol{\tau}} & \mathcal{Q}_T - \mathcal{G}^*_T S_T \mathcal{G}_T \end{bmatrix} \tag{15}$$

satisfying

$$\dot{\mathcal{H}} = \langle \mathbf{e}_\partial, \mathbf{f}_\partial \rangle_{\partial\Omega} \tag{16}$$

where $\langle \mathbf{e}_\partial, \mathbf{f}_\partial \rangle_{\partial\Omega}$ is the power supplied through the boundary $\partial\Omega$ and the boundary flows $\mathbf{f}_\partial$ and efforts $\mathbf{e}_\partial$ are given by

$$\mathbf{f}_\partial = \begin{bmatrix} -(\mathbf{e_v} \cdot \mathbf{n})|_{\partial\Omega} \\ -\mathbf{v}\mathbf{n}^{\mathrm{T}}|_{\partial\Omega} \\ -(\mathbf{q}_s \cdot \mathbf{n})|_{\partial\Omega} \end{bmatrix} \text{ and } \mathbf{e}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \boldsymbol{\tau}|_{\partial\Omega} \\ T|_{\partial\Omega} \end{bmatrix}.$$

Proof. The fluid governing equations in (6) can be rewritten as function of the fluid efforts described in (9), as shown in (10). Then, using the operators defined in Lemmas 1 and 2 we obtain

$$\underbrace{\begin{bmatrix} \partial_t \rho \\ \partial_t \mathbf{v} \\ \partial_t s \end{bmatrix}}_{\partial_t \boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 & -\mathrm{div} & 0 \\ -\mathbf{grad} & -\frac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{J}_{\boldsymbol{\tau}} \\ 0 & -\mathcal{J}^*_{\boldsymbol{\tau}} & \mathcal{Q}_T - \mathcal{G}^*_T S_T \mathcal{G}_T \end{bmatrix}}_{\mathcal{J}} \underbrace{\begin{bmatrix} e_\rho \\ \mathbf{e_v} \\ e_s \end{bmatrix}}_{\mathbf{e}} \tag{17}$$

and the energy balance for this system is given by:

$$\begin{aligned}\dot{\mathcal{H}} = &= -\int_\Omega e_\rho \mathrm{div}\ \mathbf{e_v} + \mathbf{e_v} \cdot \mathbf{grad}\ e_\rho - \frac{\mathbf{e_v}}{\rho} \cdot G_{\boldsymbol{\omega}} \mathbf{e_v} \\ &+ \langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s \rangle_\Omega - \langle e_s, \mathcal{J}^*_{\boldsymbol{\tau}} \mathbf{e_v} \rangle_\Omega + \langle e_s, \mathcal{J}_{\mathbf{q}} e_s \rangle_\Omega \end{aligned} \tag{18}$$

Note that given the skew-symmetry property of the gyroscope $\frac{\mathbf{e_v}}{\rho} G_{\boldsymbol{\omega}} \mathbf{e_v} = 0$. Then, using (11) and (13), equation (18) can be rewritten as

$$\dot{\mathcal{H}} = -\int_{\partial\Omega} e_\rho\,(\mathbf{e_v} \cdot \mathbf{n}) + \boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\right] + T\,[\mathbf{q}_s \cdot \mathbf{n}] \tag{19}$$

Defining the boundary flows and efforts as

$$\mathbf{f}_\partial = \begin{bmatrix} -(\mathbf{e_v} \cdot \mathbf{n})|_{\partial\Omega} \\ -\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\Big|_{\partial\Omega} \\ -(\mathbf{q}_s \cdot \mathbf{n})|_{\partial\Omega} \end{bmatrix} \text{ and } \mathbf{e}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \boldsymbol{\tau}|_{\partial\Omega} \\ \frac{e_s}{\rho}\Big|_{\partial\Omega} \end{bmatrix},$$

where $\mathbf{e_v} \cdot \mathbf{n}$ is the normal projection of the momentum density, $\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}$ is the tangential projection of the velocity field ans $e_s/\rho = T$ is the temperature. Then, the rate of change of the total energy is given by $\dot{\mathcal{H}} = \langle e_\partial, f_\partial \rangle_{\partial\Omega}$. □

Remark 1. System (14) looks like a Stokes-Dirac structure because of the skew symmetry of the operators involved, and the associated power balance (16) with the appropriate boundary efforts and flows. However, since the coefficients of the operator depend explicitly on the effort variable $e_s = \rho T$, and not only on the energy variables $(\rho, \mathbf{v}, s)$, then, the Jacobi identities are not satisfied and (14)-(16) does not define a Dirac structure. □

Remark 2. Using some simple mathematical operation the term $\boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\right]$ in (19) can be rewritten as

$$\boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^{\mathrm{T}}\right] = \frac{\mathbf{e_v}}{\rho} \cdot [\boldsymbol{\tau} \cdot \mathbf{n}]$$

Thus, for computational purposes the boundary flows and efforts can be expressed as

$$\mathbf{f}_\partial = \begin{bmatrix} -\left(\mathbf{e_v}\cdot\mathbf{n}\right)|_{\partial\Omega} \\ -\mathbf{v}|_{\partial\Omega} \\ -\left(\mathbf{q}_s\cdot\mathbf{n}\right)|_{\partial\Omega} \end{bmatrix} \text{ and } \mathbf{e}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \left[\boldsymbol{\tau}\cdot\mathbf{n}\right]|_{\partial\Omega} \\ T|_{\partial\Omega} \end{bmatrix},$$

respectively. □

Now, consider $\mathbf{f}_d = \mathcal{G}_T e_s$ and $\mathbf{e}_d = S_T\mathbf{f}_d$ as the flow and effort associated with the entropy diffusion by heat flux. Thus, an alternative representation for the model described in (14) is given by:

$$\begin{bmatrix} \partial_t\rho \\ \partial_t\mathbf{v} \\ \partial_t s \\ \mathbf{f}_d \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{div} & 0 & 0 \\ -\mathbf{grad} & -\dfrac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{J}_{\boldsymbol{\tau}} & 0 \\ 0 & -\mathcal{J}^*_{\boldsymbol{\tau}} & \mathcal{Q}_T & -\mathcal{G}^*_T \\ 0 & 0 & \mathcal{G}_T & 0 \end{bmatrix} \begin{bmatrix} e_\rho \\ \mathbf{e_v} \\ e_s \\ \mathbf{e}_d \end{bmatrix} \tag{20}$$

where the boundary flow and effort are given by $\mathbf{f}_\partial = \begin{bmatrix} -\left(\mathbf{e_v}\cdot\mathbf{n}\right)|_{\partial\Omega} \\ -\mathbf{v}\mathbf{n}^{\mathrm{T}}\big|_{\partial\Omega} \\ -\left(\mathbf{e}_d\cdot\mathbf{n}\right)|_{\partial\Omega} \end{bmatrix}$ and $\mathbf{e}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \boldsymbol{\tau}|_{\partial\Omega} \\ T|_{\partial\Omega} \end{bmatrix}$, respectively, satisfying the power balance in (16).

3. ISENTROPIC FLUID

In this section we describe the dissipative port-Hamiltonian formulation for ideal isentropic fluids. The governing equations are reduced to the continuity and motion equations described in (1a) and (1b), respectively. Similarly, the Gibbs equation reduce to

$$du = -pd\left(\frac{1}{\rho}\right) \tag{21}$$

In isentropic fluids the internal energy is a function that depends only on the density, as shown in (21). Then, the total energy is described as

$$\mathcal{H} = \int_\Omega \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho u\left(\rho\right) \tag{22}$$

and the fluid efforts $\mathbf{e} = \left[e_\rho\ \mathbf{e}^{\mathrm{T}}_{\mathbf{v}}\right]^{\mathrm{T}}$ are given by

$$\begin{bmatrix} e_\rho \\ \mathbf{e_v} \end{bmatrix} = \begin{bmatrix} \delta_\rho\mathcal{H} \\ \delta_{\mathbf{v}}\mathcal{H} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + h \\ \rho\mathbf{v} \end{bmatrix} \tag{23}$$

Then, the fluid dynamics can be expressed as

$$\partial_t\rho = -\mathrm{div}\ \mathbf{e_v} \tag{24a}$$

$$\partial_t\mathbf{v} = -\mathbf{grad}\ e_\rho - \frac{1}{\rho}G_{\boldsymbol{\omega}}\mathbf{e_v} - \frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau} \tag{24b}$$

where the term $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau}$ represents the viscous friction effects over the velocity field. In previous section, the velocity field and the entropy of the fluid are interconnected through the heat generated by this friction, by means of the operators $\mathcal{J}_{\boldsymbol{\tau}}$ and $\mathcal{J}^*_{\boldsymbol{\tau}}$. In this case, given the isentropic assumption, we can interpret $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau}$ as the dissipation associated with heat generation as a consequence of the viscous friction. According to Villegas et al. (2006), in infinite-dimensional port-Hamiltonian systems the dissipative terms are expressed as $\mathcal{G}^*S\mathcal{G}\mathbf{e}$ where $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$, and $S = S^{\mathrm{T}} \geq 0$. Then, $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau}$ can be expressed as a PH dissipation term, as shown in the following Lemma. .

Lemma 4. Define the operators $\mathcal{G}_r = \mathbf{curl}\ \frac{\cdot}{\rho}$ and $\mathcal{G}_d = \mathrm{div}\ \frac{\cdot}{\rho}$ and the corresponding adjoints $\mathcal{G}^*_r = \frac{1}{\rho}\mathbf{curl}$ and $\mathcal{G}^*_d = -\frac{1}{\rho}\mathbf{grad}$. Then, for a Newtonian fluid, $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau}$ can be expressed as a dissipative port-Hamiltonian terms associated with the velocity effort, namely,

$$\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau} = \mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}}\mathbf{e_v} \tag{25}$$

where $\mathcal{G}^*_{\boldsymbol{\tau}} = [\mathcal{G}^*_r\ \mathcal{G}^*_d]$, $S_{\boldsymbol{\tau}} = \begin{bmatrix} \mu & 0 \\ 0 & \hat{\mu} \end{bmatrix}$ and $\mathcal{G}_{\boldsymbol{\tau}} = \begin{bmatrix} \mathcal{G}_r \\ \mathcal{G}_d \end{bmatrix}$, with $\hat{\mu} = \frac{4}{3}\mu + \kappa$.

Proof. Consider the viscosity tensor described in (2). Then, applying the identities (A.4)-(A.6) we obtain $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau} = \frac{1}{\rho}\mathbf{curl}\left[\mu\mathbf{curl}\ \frac{\mathbf{e_v}}{\rho}\right] - \frac{1}{\rho}\mathbf{grad}\left(\hat{\mu}\mathrm{div}\ \frac{\mathbf{e_v}}{\rho}\right)$. Given that the curl operator is self-adjoint and the adjoint of divergence is minus the gradient, then, it is easy to check that $\mathcal{G}^*_r = \frac{1}{\rho}\mathbf{curl}$ is the formal adjoint of $\mathcal{G}_r = \mathbf{curl}\ \frac{\cdot}{\rho}$ and $\mathcal{G}^*_d = -\frac{1}{\rho}\mathbf{grad}$ is the adjoint of $\mathcal{G}_d = \mathrm{div}\ \frac{\cdot}{\rho}$. Thus, $\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau}$ can be expressed as the sum of 2 dissipative terms, namely

$$\frac{1}{\rho}\mathbf{Div}\ \boldsymbol{\tau} = \mathcal{G}^*_r\mu\mathcal{G}_r\mathbf{e_v} + \mathcal{G}^*_d\hat{\mu}\mathcal{G}_d\mathbf{e_v} = \underbrace{\begin{bmatrix} \mathcal{G}^*_r & \mathcal{G}^*_d \end{bmatrix}}_{\mathcal{G}^*_{\boldsymbol{\tau}}}\underbrace{\begin{bmatrix} \mu & 0 \\ 0 & \hat{\mu} \end{bmatrix}}_{S_{\boldsymbol{\tau}}}\underbrace{\begin{bmatrix} \mathcal{G}_r \\ \mathcal{G}_d \end{bmatrix}}_{\mathcal{G}_{\boldsymbol{\tau}}}\mathbf{e_v} \tag{26}$$

where $S_{\boldsymbol{\tau}}$ satisfies the positive condition $S_{\boldsymbol{\tau}} = S^{\mathrm{T}}_{\boldsymbol{\tau}} \geq 0$. □

Note that $\mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}}\mathbf{e_v}$ can be expressed as the sum of two dissipations, as shown in (26). The first dissipation, $\mathcal{G}^*_r\mu\mathcal{G}_r\mathbf{e_v}$, describes the losses associated with the fluid rotation or vorticity, and it is equal to 0 under an irrotational assumption. The second dissipation, $\mathcal{G}^*_d\hat{\mu}\mathcal{G}_d\mathbf{e_v}$, describes the losses associated with the dilatation or compression of the fluid, and it is equal to 0 under incompressible assumption.

Proposition 2. Consider an isentropic Newtonian fluid in a domain Ω with boundary $\partial\Omega$. Considering the vorticity as a strictly internal phenomena, the governing equations can be expressed as the following port-Hamiltonian system with dissipation:

$$\partial_t\boldsymbol{x} = \left(\mathcal{J} - \mathcal{G}^*S\mathcal{G}\right)\mathbf{e} \tag{27}$$

where $\boldsymbol{x} = \left[\rho\ \mathbf{v}^{\mathrm{T}}\right]^{\mathrm{T}}$ is the state vector, $\mathbf{e} = \left[e_\rho\ \mathbf{e}^{\mathrm{T}}_{\mathbf{v}}\right]^{\mathrm{T}}$ are the fluid efforts, and

$$\mathcal{J} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\mathbf{grad} & -\dfrac{G_{\boldsymbol{\omega}}}{\rho} \end{bmatrix}, \mathcal{G}^* = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{G}^*_{\boldsymbol{\tau}} \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 0 & S_{\boldsymbol{\tau}} \end{bmatrix}, \mathcal{G} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix}$$

Satisfying the following relationship for the rate of change of the energy: $\frac{d\mathcal{H}}{dt} \leq \int_{\partial\Omega}\mathbf{f}_\partial\cdot\mathbf{e}_\partial$, where $\mathbf{f}_\partial = \left(e_\rho - e_d\right)|_{\partial\Omega}$ and $\mathbf{e}_\partial = -\left(\mathbf{e_v}\cdot\mathbf{n}\right)|_{\partial\Omega}$ are the boundary flow and effort, respectively, with e_d the effort associated with the dissipation by dilatation and $\mathbf{n}$ the normal unitary outward vector to the boundary.

Proof. Considering the Lemma 4, the governing equations can be expressed as

$$\partial_t\begin{bmatrix} \rho \\ \mathbf{v} \end{bmatrix} = \left(\begin{bmatrix} 0 & -\mathrm{div} \\ -\mathbf{grad} & -\dfrac{G_{\boldsymbol{\omega}}}{\rho} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix}\right)\begin{bmatrix} e_\rho \\ \mathbf{e_v} \end{bmatrix}$$

Rewriting the term $\begin{bmatrix} 0 \\ \mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix}$ we obtain the dissipative port-Hamiltonian formulation described in (27).

On the other hand, for the rate of change of the total energy we obtain

$$\frac{d\mathcal{H}}{dt} = -\int_{\partial\Omega} e_\rho\left(\mathbf{e_v}\cdot\mathbf{n}\right) - \int_\Omega \mathbf{e}\cdot\mathcal{G}^*S\mathcal{G}\mathbf{e} \tag{28}$$

Defining $\mathbf{f}_R = \left[\mathbf{f}_r^{\mathrm{T}}\ f_d\right]^{\mathrm{T}}$ and $\mathbf{e}_R = \left[\mathbf{e}_r^{\mathrm{T}}\ e_d\right]^{\mathrm{T}}$ as the flows and efforts associated with the dissipations, where $\mathbf{f}_r = \mathcal{G}_r \mathbf{e}_\mathbf{v}$, $f_d = \mathcal{G}_d \mathbf{e}_\mathbf{v}$, $\mathbf{e}_r = \mu \mathbf{f}_r$ and $e_d = \hat{\mu} f_d$, we obtain $\mathbf{e} \cdot \mathcal{G}^* S \mathcal{G} \mathbf{e} = \mathbf{e}_\mathbf{v} \cdot \mathcal{G}_r^* \mathbf{e}_r + \mathbf{e}_\mathbf{v} \cdot \mathcal{G}_d^* e_d$. Then, considering the vorticity equal to 0 in the boundaries, the equation (28) can be rewritten as $\frac{d\mathcal{H}}{dt} = -S_{\boldsymbol{\tau}} \left\langle \mathbf{f}_R, \mathbf{f}_R \right\rangle_\Omega - \int_{\partial\Omega} \left(e_\rho - \frac{e_d}{\rho}\right) (\mathbf{e}_\mathbf{v} \cdot \mathbf{n})$. Given that $S_{\boldsymbol{\tau}} \geq 0$, then we obtain the inequality $\frac{d}{dt}\mathcal{H} \leq \int_{\partial\Omega} \mathbf{f}_\partial \cdot \mathbf{e}_\partial$, where $\mathbf{f}_\partial = (e_\rho - \frac{e_d}{\rho})|_{\partial\Omega}$ and $\mathbf{e}_\partial = -\left(\mathbf{e}_\mathbf{v} \cdot \mathbf{n}\right)|_{\partial\Omega}$. □

Note that, considering different assumptions the fluid model proposed in (27), corresponds to PH models of isentropic fluids described in previous works. For example, under an irrotational assumption operators $G_{\boldsymbol{\omega}}$, $\mathcal{G}_r$ and $\mathcal{G}_r^*$ disappear, leading to the fluid model described by Matignon and Hélie (2013). On the other hand, for inviscid fluids, the operator $\mathcal{G}^* S \mathcal{G}$ is equal to 0, and then, the port-Hamiltonian system (27) is equivalent to the model proposed in van der Schaft and Maschke (2002).

4. TWO-DIMENSIONAL FLUIDS

The cross product and the curl are three-dimensional mathematical operators. Thus, for two-dimensional fluids we need to properly define the terms associated with these operators.

Let us denote by $\{x_1, x_2\}$ the variables associated with the axes of a two-dimensional velocity field $\mathbf{v} = [v_1\ v_2]^{\mathrm{T}}$. The vorticity ω is a scalar defined as $\omega = -\partial_{x_2} v_1 + \partial_{x_1} v_2$. For convenience we rewrite ω as $\omega = -\mathrm{div}\ [W\mathbf{v}]$, where $W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation matrix. Then, the Gyroscope in a two-dimensional velocity field is defined as (Carodo-Ribeiro, 2016):

$$G_{\boldsymbol{\omega}} = \omega W = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \tag{29}$$

On the other hand, with respect to the dissipative terms of the viscosity tensor, the operators $\mathcal{G}_r$ and $\mathcal{G}_r^*$ for two-dimensional fluids are defined as:

$$\mathcal{G}_r = -\mathrm{div}\left[W \frac{\cdot}{\rho}\right], \text{ and } \mathcal{G}_r^* = \frac{1}{\rho} W^{\mathrm{T}} \mathbf{grad} \tag{30}$$

Thus, given the operator definitions in (29)-(30), the port-Hamiltonian formulations in Propositions 1 and 2 can be used to describe non-isentropic and isentropic two-dimensional fluids, respectively.

In the case of 1D fluids, all terms associated with the vorticity disappear and div $=$ **grad** $= \partial_{x_1}$. Thus, the fluid model (14) is equivalent to the model described in Altmann and Schulze (2017), neglecting the reactive part. Similarly, the model described in Proposition 2 correspond to the model used in Kotyczka (2013) and Macchelli et al. (2017).

5. CONCLUSION

A pseudo-PH formulation for 3D non-isentropic Newtonian fluids was presented for non-reactive flows. This model describes the fluid dynamics, including the vorticity effects on the velocity fields, and the thermodynamics, including the irreversible entropy creation by heat flux. Similarly, under an isentropic assumption, the transformation of kinetic energy into heat by viscosity friction is described as dissipative terms associated with fluid rotation and compression, obtaining a dissipative-PH model for three-dimensional isentropic fluids. These models present a general formulation for non-reactive compressible flows, i.e., a description for inviscid or irrotational fluids can be derived from the proposed models under the corresponding assumptions in the energy-based structure. Moreover, we have described the necessary considerations on the operators used in the proposed models for the case of two-dimensional and one-dimensional fluids, obtaining formulations for fluids models equivalent to those found in the literature.

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Appendix A. NOMENCLATURE AND USEFUL IDENTITIES

The nomenclature used in this paper is summarized in the next Table.

Table A.1. Nomenclature

Symbol	Description
$\mathbf{v}\cdot\mathbf{u}$	Scalar product between 2 vectors, $\mathbf{v}^{\mathrm{T}}\mathbf{u}$.
$\mathbf{v}\times\mathbf{u}$	Cross product
$\boldsymbol{\tau}:\boldsymbol{\sigma}$	Scalar product between 2 tensors, $Tr\,(\boldsymbol{\tau}\boldsymbol{\sigma})=\sum_{i,j}\tau_{ij}\sigma_{ji}$ (Bird et al., 2015).
$\operatorname{div}\mathbf{u}$	Divergence of vector $\mathbf{u}$.
$\mathbf{grad}\, f$	Gradient of scalar f.
$\mathbf{curl}\,\mathbf{u}$	Curl or rotational of $\mathbf{u}$.
$\mathbf{Grad}\,\mathbf{u}$	Gradient of vector $\mathbf{u}$.
$\mathbf{Div}\,\boldsymbol{\sigma}$	Divergence of tensor $\boldsymbol{\sigma}$.
$\\|\mathbf{v}\\|_X^2$	Square of the weighted Euclidean norm, $\mathbf{v}^TX^T\mathbf{v}$.
$\int_\Omega f$	Integral in domain Ω, $\int_\Omega f d\Omega$
$\int_{\partial\Omega} f$	Integral in boundary $\partial\Omega$, $\int_{\partial\Omega} f\partial\Omega$

Additionally, the set of mathematical identities (Bird et al., 2015, Appendix A) used in this work are described below:

$$\mathbf{u}\cdot\mathbf{Grad}\,\mathbf{u}=\mathbf{grad}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+[\mathbf{curl}\,\mathbf{u}]\times\mathbf{u} \tag{A.1}$$

$$\boldsymbol{\sigma}:\mathbf{Grad}\,\mathbf{u}=\operatorname{div}\,[\boldsymbol{\sigma}\cdot\mathbf{u}]-\mathbf{u}\cdot\mathbf{Div}\,\boldsymbol{\sigma} \tag{A.2}$$

$$\operatorname{div}\,[f\mathbf{u}]=[\mathbf{grad}\,f]\cdot\mathbf{u}+f\operatorname{div}\mathbf{u} \tag{A.3}$$

$$\mathbf{Div}\,[\mathbf{Grad}\,\mathbf{u}]=\mathbf{grad}\,(\operatorname{div}\mathbf{u})-\mathbf{curl}\,[\mathbf{curl}\,\mathbf{u}] \tag{A.4}$$

$$\mathbf{Div}\,\left[\mathbf{Grad}\,\mathbf{u}^{\mathrm{T}}\right]=\mathbf{grad}\,(\operatorname{div}\mathbf{u}) \tag{A.5}$$

$$\mathbf{Div}\,[(\operatorname{div}\mathbf{u})\,I]=\mathbf{grad}\,(\operatorname{div}\mathbf{u}) \tag{A.6}$$

where f is a scalar, $\mathbf{u}$ is a vector and $\boldsymbol{\sigma}$ is a symmetric second order tensor.

Appendix B. USEFUL PROPERTIES

In this section we describe some useful properties used in this paper.

Theorem 5. (Gauss Divergence Theorem). Let be a close domain Ω, enclosed by the boundary surface $\partial\Omega$, then

$$\int_\Omega \operatorname{div}\mathbf{u}=\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n} \tag{B.1}$$

Proof. See (Bird et al., 2015, p. 704). □

Theorem 6. (Adjoint of div). Let be the Hilbert space of the square integrable scalar functions, denoted by $\mathscr{H}_0=L^2(\Omega,\mathbb{R})$, and the Hilbert space of the square integrable vector functions, denoted by $\mathscr{H}_1=L^2(\Omega,\mathbb{R}^n)$. Given the operators $\operatorname{div}:\mathscr{H}_1\to\mathscr{H}_0$ and $\mathbf{grad}:\mathscr{H}_0\to\mathscr{H}_1$, where $-\mathbf{grad}$ is the formal adjoint of div , then,

$$\int_\Omega f\operatorname{div}\mathbf{u}+\int_\Omega\mathbf{grad}\,f\cdot\mathbf{u}=\int_{\partial\Omega}f\,(\mathbf{u}\cdot\mathbf{n}) \tag{B.2}$$

Proof. Denote by $\langle f_1,f_2\rangle_{\mathscr{H}_0}=\int_\Omega f_1f_2$ and $\langle\mathbf{u}_1,\mathbf{u}_2\rangle_{\mathscr{H}_1}=\int_\Omega\mathbf{u}_1\cdot\mathbf{u}_2$ the inner products in $\mathscr{H}_0$ and $\mathscr{H}_1$, respectively. Then, using identity (A.3) and Theorem 5 we obtain

$$\langle f,\operatorname{div}\mathbf{u}\rangle_{\mathscr{H}_0}+\langle\mathbf{grad}\,f,\mathbf{u}\rangle_{\mathscr{H}_1}=\int_\Omega\operatorname{div}f\mathbf{u}=\int_{\partial\Omega}f\,(\mathbf{u}\cdot\mathbf{n})$$

where for boundary conditions equal to 0, the relationship $\langle f,\operatorname{div}\mathbf{u}\rangle_{\mathscr{H}_2}=\langle-\mathbf{grad}\,f,\mathbf{u}\rangle_{\mathscr{H}_1}$ is obtained. □

Theorem 7. Consider the Hilbert space of the square integrable vector functions $\mathscr{H}_1$, and the Hilbert space of square integrable second order tensors $\mathscr{H}_2:=L^2(\Omega,\mathbb{R}^{n\times n})$. Given the operators $\mathbf{Div}:\mathscr{H}_2\to\mathscr{H}_1$ and $\mathbf{Grad}:\mathscr{H}_1\to\mathscr{H}_2$. Then, the formal adjoint of $\mathbf{Div}$ is $-\mathbf{Grad}$ and for any symmetric tensor $\boldsymbol{\sigma}\in\mathscr{H}_2$ and vector $\mathbf{u}\in\mathscr{H}_1$, the following relationship is satisfied

$$\int_\Omega[\mathbf{Div}\,\boldsymbol{\sigma}]\cdot\mathbf{u}+\int_\Omega\boldsymbol{\sigma}:\mathbf{Grad}\,\mathbf{u}=\int_{\partial\Omega}[\boldsymbol{\sigma}\cdot\mathbf{u}]\cdot\mathbf{n} \tag{B.3}$$

Proof. Denote by $\langle\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\rangle_{\mathscr{H}_2}=\int_\Omega Tr(\boldsymbol{\sigma}_1^{\mathrm{T}}\boldsymbol{\sigma}_2)$ inner product in $\mathscr{H}_2$ and considering $\mathbf{Div}\,\boldsymbol{\sigma}=[\operatorname{div}\sigma_1\cdots\operatorname{div}\sigma_n]^T$ where $\sigma_i=[\sigma_{1i}\cdots\sigma_{ni}]^{\mathrm{T}}$ is the i-th column of tensor $\boldsymbol{\sigma}$. Then, we obtain:

$$\begin{aligned}\langle\mathbf{Div}\,\boldsymbol{\sigma},\mathbf{u}\rangle_{\mathscr{H}_1}&=\int_\Omega[\mathbf{Div}\,\boldsymbol{\sigma}]\cdot\mathbf{u}=\int_\Omega\sum_i u_i\operatorname{div}\sigma_i\\&=-\int_\Omega Tr\left(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{Grad}\,\mathbf{u}\right)+\sum_i\int_{\partial\Omega}u_i\sigma_i\cdot\mathbf{n}\\&=-\langle\boldsymbol{\sigma},\mathbf{Grad}\,\mathbf{u}\rangle_{\mathscr{H}_2}+\int_{\partial\Omega}\mathbf{u}\cdot\left[\boldsymbol{\sigma}^T\cdot\mathbf{n}\right]\end{aligned}$$

Thus, for boundary conditions equal to 0 we obtain $\langle\mathbf{Div}\,\boldsymbol{\sigma},\mathbf{u}\rangle_{\mathscr{H}_1}=\langle\boldsymbol{\sigma},-\mathbf{Grad}\,\mathbf{u}\rangle_{\mathscr{H}_2}$, i.e., $-\mathbf{Grad}$ is the formal adjoint of $\mathbf{Div}$.

Now, considering that $\boldsymbol{\sigma}$ is a symmetric tensor in $\mathscr{H}_2$ and $\mathbf{u}\cdot\left[\boldsymbol{\sigma}^T\cdot\mathbf{n}\right]=\mathbf{u}^{\mathrm{T}}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}=[\boldsymbol{\sigma}\cdot\mathbf{u}]\cdot\mathbf{n}$, we obtain

$$\int_\Omega[\mathbf{Div}\,\boldsymbol{\sigma}]\cdot\mathbf{u}+\int_\Omega\boldsymbol{\sigma}:\mathbf{Grad}\,\mathbf{u}=\int_{\partial\Omega}[\boldsymbol{\sigma}\cdot\mathbf{u}]\cdot\mathbf{n}$$