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Performance Preserving Integral Extension of Linear and Homogeneous State-Feedback Controllers

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Abstract: The problem of extending an existing state-feedback controller by an integrator is considered. A structural insight into the design of such controllers is presented for the linear case, which allows to preserve the performance of the given controller in a certain sense. Using this insight, a second order homogeneous state feedback controller with discontinuous integral action is proposed, which can reject arbitrary slope bounded, i.e., Lipschitz continuous, perturbations. By means of Lyapunov methods, stability conditions for the closed loop system and a bound for its finite convergence time are derived. Numerical simulations illustrate the results and provide further insight into the tuning of the proposed approach.

Keywords: state-feedback, weighted homogeneity, integral control, finite-time convergence

1. INTRODUCTION

Controllers with integral action are an important tool for achieving satisfying disturbance rejection. Typically, the integral part allows to compensate for constant or, with multiple integrators, polynomial-like disturbances. In the context of sliding mode control, discontinuous integral action allows to compensate for the much larger class of slope bounded (i.e., Lipschitz continuous) disturbances by means of a continuous control signal. Prominent examples in that regard are the super-twisting algorithm and the sub-optimal algorithm, which are proposed in Levant (1993) and Bartolini et al. (1998), respectively. Compared to first order sliding mode techniques, which reject bounded perturbations using discontinuous control signals, these algorithms reduce the so-called chattering effect introduced by the presence of actuator dynamics or time discretization, see, e.g., Shtessel et al. (2014) and Pérez-Ventura and Fridman (2019); they are limited to sliding surfaces with relative degree one, however.

Recent research has thus focused on designing controllers with integral action in ways that may be generalized to higher relative degrees. Some key results in that regard are found in Kamal et al. (2016); Laghrouche et al. (2017) and Mercado-Uribe and Moreno (2018), for example.

In this paper, a new approach for the design of homogeneous controllers with discontinuous integral action is proposed for the case of relative degree two. The idea is based on a structural insight that is first presented for the linear time invariant case. Using this idea, a homogeneous state-feedback controller is extended by an integral term that preserves the nominal performance in the unperturbed case. This allows to separate the parameters into groups corresponding to either the nominal or the integral part, which simplifies tuning. It furthermore allows to guarantee finite-time stability of the perturbed closed loop for all positive integrator gains that are sufficiently large. The proposed technique is demonstrated by extending a control law for the double integrator $\ddot{\sigma} = u$, which is discussed, e.g., in Bacciotti and Rosier (2001) and has the form

$$u = -k_1 \left|\sigma\right|^{\frac{1}{3}} \operatorname{sign}(\sigma) - k_2 \left|\dot{\sigma}\right|^{\frac{1}{2}} \operatorname{sign}(\dot{\sigma}). \qquad (1)$$

For the controller thus proposed, quantitative results for the minimum integrator gain and for the maximum convergence time are derived. Their validity and the controller's performance are illustrated by means of simulations.

The paper is structured as follows. After some preliminaries in Section 2, the principal structural insight and idea that motivates the proposed controller structure is first explained for a linear state-feedback controller in Section 3. Then, this idea's application to the second order homogeneous control law is discussed. The main result, which consists of the proposed control law along with stability conditions and a convergence time bound, is presented in Section 4. The actual stability analysis along with some insight into the closed-loop performance is given in Section 5, and Section 6 presents simulation results and discusses the tuning of the proposed controller. Section 7, finally, draws conclusions and provides a brief outlook.

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2. PRELIMINARIES

Throughout the paper, the abbreviations $\lfloor y \rceil^p = |y|^p \operatorname{sign}(y)$ and $\lfloor y \rceil^0 = \operatorname{sign}(y)$ are used. Simultaneous transposition and inversion of a matrix $\mathbf{M}$ is written as $\mathbf{M}^{-\mathrm{T}}$. The derivative of a scalar valued function $f(\mathbf{x})$ with respect to the vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by the row vector $\frac{\partial f}{\partial \mathbf{x}} = [\frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n}]$.

A system's origin is called finite-time stable, if it is asymptotically stable and the state is equal to zero after some finite time T. The smallest T, for which this is the case, as a function of the initial state is called the system's convergence time.

3. MOTIVATION – LINEAR CASE

In this section, a motivation for the approach proposed in this paper is presented by considering a *linear time invariant* plant with a *constant* disturbance. An interesting structural insight into the design of a state-feedback controller with integral part for such a system is presented. In Sections 4 and 5 it will be shown that an extension of this idea to *nonlinear homogeneous* state-feedback controllers can handle *arbitrary slope-bounded* disturbances.

Consider a linear time invariant system

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{b}(u + w) \tag{2}$$

with parameters $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^{n \times 1}$, state vector $\boldsymbol{\xi} \in \mathbb{R}^n$, scalar control input u, and constant disturbance w. Suppose that a state-feedback control law of the form

$$u = -\mathbf{k}^{\mathrm{T}}\boldsymbol{\xi} \tag{3}$$

with parameter vector $\mathbf{k}^{\mathrm{T}} \in \mathbb{R}^{1 \times n}$ is given, which exponentially stabilizes the origin in the disturbance-free case. The task of extending this control law by an integral term is considered. This extension is designed in such a way that all original closed-loop eigenvalues are retained, i.e., such that all eigenvalues of the Hurwitz matrix $\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}}$ are also eigenvalues of the final closed loop. In this sense, the performance of the given controller is thus preserved.

For the state-feedback controller with integral part, consider the ansatz

$$u = -\mathbf{k}_{\mathrm{s}}^{\mathrm{T}}\boldsymbol{\xi} + k_{\mathrm{i}} v, \tag{4a}$$

$$\dot{v} = y \tag{4b}$$

with an output $y = \mathbf{g}^{\mathrm{T}}\boldsymbol{\xi}$ of the system and constant parameters $k_{\mathrm{i}} \in \mathbb{R}$ and $\mathbf{k}_{\mathrm{s}}^{\mathrm{T}}, \mathbf{g}^{\mathrm{T}} \in \mathbb{R}^{1 \times n}$. Denoting the difference of $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{k}$ by $k_{\mathrm{i}}\mathbf{h} := \mathbf{k}_{\mathrm{s}} - \mathbf{k}$ and collecting $\boldsymbol{\xi}$ and v in the extended state vector $\boldsymbol{\zeta} := [\boldsymbol{\xi}^{\mathrm{T}} \quad v]^{\mathrm{T}}$, the closed-loop system is given by $\dot{\boldsymbol{\zeta}} = (\hat{\mathbf{A}} - \hat{\mathbf{b}}\hat{\mathbf{k}}^{\mathrm{T}})\boldsymbol{\zeta}$ with

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{g}^{\mathrm{T}} & 0 \end{bmatrix}, \quad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}} = k_{\mathrm{i}} \begin{bmatrix} \mathbf{h} \\ -1 \end{bmatrix}. \tag{5}$$

Due to the block triangular form of $\hat{\mathbf{A}}$, one can see that it has one eigenvalue at zero and shares all other eigenvalues with the matrix $\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}}$. Since the latter should also be closed-loop eigenvalues, only the eigenvalue at zero needs to be reassigned. In order to achieve this, the vector $\hat{\mathbf{k}}^{\mathrm{T}}$ has to be chosen as the left eigenvector of $\hat{\mathbf{A}}$ for the zero eigenvalue, i.e., $\hat{\mathbf{k}}^{\mathrm{T}}\hat{\mathbf{A}} = \mathbf{0}$ has to hold. All such eigenvectors are given by

$$\hat{\mathbf{k}}^{\mathrm{T}} = \frac{\lambda}{\mathbf{g}^{\mathrm{T}}(\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})^{-1}\mathbf{b}} \left[\mathbf{g}^{\mathrm{T}}(\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})^{-1} \quad -1\right] \tag{6}$$

with a scalar parameter $\lambda \neq 0$. As one can see from

$$\hat{\mathbf{k}}^{\mathrm{T}}(\hat{\mathbf{A}} - \hat{\mathbf{b}}\hat{\mathbf{k}}^{\mathrm{T}}) = -(\hat{\mathbf{k}}^{\mathrm{T}}\hat{\mathbf{b}})\hat{\mathbf{k}}^{\mathrm{T}} = -\lambda\hat{\mathbf{k}}^{\mathrm{T}}, \tag{7}$$

this parameter specifies the location of the (negative) eigenvalue that is assigned to $\hat{\mathbf{A}} - \hat{\mathbf{b}}\hat{\mathbf{k}}^{\mathrm{T}}$ instead of zero. All other eigenvalues remain unchanged, since

$$(\hat{\mathbf{A}} - \hat{\mathbf{b}}\hat{\mathbf{k}}^{\mathrm{T}})\mathbf{V} = \mathbf{V}(\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}}) \tag{8}$$

holds with $\mathbf{V} = [\mathbf{I} \quad (\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})^{-\mathrm{T}}\mathbf{g}]^{\mathrm{T}}$. These considerations prove the following result.

Proposition 1. Let a positive parameter λ and a nominal state-feedback gain $\mathbf{k}^{\mathrm{T}} \in \mathbb{R}^{1 \times n}$ be given such that the matrix $\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}}$ is Hurwitz. Then, for any output vector $\mathbf{g}^{\mathrm{T}} \in \mathbb{R}^{1 \times n}$, the extended control law

$$u = -\mathbf{k}^{\mathrm{T}}\boldsymbol{\xi} - k_{\mathrm{i}}\mathbf{h}^{\mathrm{T}}\boldsymbol{\xi} + k_{\mathrm{i}} v, \tag{9a}$$

$$\dot{v} = \mathbf{g}^{\mathrm{T}}\boldsymbol{\xi} \tag{9b}$$

with

$$\mathbf{h}^{\mathrm{T}} = \mathbf{g}^{\mathrm{T}}(\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})^{-1}, \qquad k_{\mathrm{i}} = \frac{\lambda}{\mathbf{h}^{\mathrm{T}}\mathbf{b}} \tag{10}$$

asymptotically stabilizes the plant (2) for any constant disturbance w, and the closed-loop eigenvalues are given by $-\lambda$ and the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}}$.

Remark 2. The structure of (9) can be interpreted intuitively by relating the state functions $\mathbf{h}^{\mathrm{T}}\boldsymbol{\xi}$ and $\mathbf{g}^{\mathrm{T}}\boldsymbol{\xi}$ to the unperturbed, nominal closed loop $\dot{\boldsymbol{\xi}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})\boldsymbol{\xi}$. Along the trajectories of this system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h}^{\mathrm{T}}\boldsymbol{\xi} = \mathbf{h}^{\mathrm{T}}\dot{\boldsymbol{\xi}} = \mathbf{h}^{\mathrm{T}}(\mathbf{A} - \mathbf{b}\mathbf{k}^{\mathrm{T}})\boldsymbol{\xi} = \mathbf{g}^{\mathrm{T}}\boldsymbol{\xi} \tag{11}$$

holds, i.e., $\mathbf{h}^{\mathrm{T}}\boldsymbol{\xi}$ is a state function of the integral of $\mathbf{g}^{\mathrm{T}}\boldsymbol{\xi}$ for the *nominal* closed loop. The integrator state v subtracts the same integral computed for the *actual* closed loop. Thus, with the initial condition $v(0) = \mathbf{h}^{\mathrm{T}}\boldsymbol{\xi}(0)$ for the integrator, the nominal behavior is recovered in the disturbance-free case, because then $v(t) = \mathbf{h}^{\mathrm{T}}\boldsymbol{\xi}(t)$ holds for all t. If the integrator is initialized differently or if $w \neq 0$, then the integrals' values tend towards each other, recovering the nominal performance at least exponentially.

4. MAIN RESULT – HOMOGENEOUS CASE

The application of the presented idea to the homogeneous, finite-time stabilizing state-feedback controller (1) is now studied. Consider the second order integrator chain

$$\dot{x}_1 = x_2, \tag{12a}$$

$$\dot{x}_2 = u + w \tag{12b}$$

with a control input u and a matched disturbance w. Its state variables x_1, x_2 are aggregated in the state vector $\mathbf{x} = [x_1 \quad x_2]^{\mathrm{T}}$. The disturbance w is assumed to be Lipschitz continuous, i.e., its time derivative $\dot{w}$ is assumed to be bounded by

$$|\dot{w}| \leq L \tag{12c}$$

with some non-negative constant L.

In the *disturbance-free* case, i.e., for $w = 0$, this system may be stabilized in finite time by means of the homogeneous state-feedback control law

$$u = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}} \tag{13}$$

with positive parameters k_1 and k_2, see Bacciotti and Rosier (2001).

To obtain finite-time stability for *arbitrary disturbances* that satisfy (12c), the control law is extended by an integrator with *discontinuous* right-hand side. Motivated by the structure of the linear controller (9), the following homogeneous control law is proposed

$$u = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}} - k_3 h(x_1, x_2) + k_3 v, \qquad (14a)$$
$$\dot{v} = g(x_1, x_2) \qquad (14b)$$

with abbreviations

$$h(\mathbf{x}) = \frac{x_2}{\left[\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right]^{\frac{1}{4}}}, \qquad (15a)$$

$$g(\mathbf{x}) = \frac{\frac{k_1}{2}\lfloor x_1 \rceil^{\frac{1}{3}} |x_2|^2 + k_2 \lfloor x_2 \rceil^{\frac{5}{2}}}{\left[\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right]^{\frac{5}{4}}} - \frac{k_1 \lfloor x_1 \rceil^{\frac{1}{3}} + k_2 \lfloor x_2 \rceil^{\frac{1}{2}}}{\left[\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right]^{\frac{1}{4}}}. \qquad (15b)$$

Fig. 1 depicts the structure of this control law. Similar to (11), the function g is the time derivative of the continuous homogeneous function h along the trajectories of the nominal closed loop, i.e.,

$$g(\mathbf{x}) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}(-k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}}), \qquad (16)$$

and k_3 is a positive parameter. One may verify that h is continuous, and that g is homogeneous with degree zero; specifically, $g(\alpha^3 x_1, \alpha^2 x_2) = g(x_1, x_2)$ holds for all $\alpha > 0$. Thus, g is discontinuous (only) in the origin and solutions of the closed-loop system are understood in the sense of Filippov (1988). This means that (14b) for $\mathbf{x} = 0$ is to be read as the differential inclusion

$$\dot{v} \in [-G, G] \qquad (17)$$

with $G = \sup_{\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}} g(\mathbf{x}) = -\inf_{\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}} g(\mathbf{x})$. Since g has homogeneity degree zero, the supremum may be taken on any curve encircling the origin, e.g., the unit circle.

Contrary to the linear case, (16) does not automatically guarantee stability, and hence g can not be chosen arbitrarily. Indeed, a Lyapunov based stability analysis will reveal that, additionally, sign definiteness of $\frac{\partial h}{\partial x_2}$ is of importance.

The following main theorem gives conditions for finite-time stability of the closed-loop system and provides an upper bound for its convergence time.

Theorem 3. Let positive parameters k_1, k_2, k_3 and a nonnegative Lipschitz constant L be given. Consider the closed loop system formed by the interconnection of the perturbed integrator chain (12) and the control law (14) with functions g and h given in (15). If the conditions

$$k_1^{\frac{3}{4}} > \frac{2k_2}{3}, \qquad k_3 > \frac{L}{k_2} \frac{19 k_1^{\frac{3}{4}}}{3k_1^{\frac{3}{4}} - 2k_2} \qquad (18)$$

are satisfied, then the closed loop's origin is finite-time stable. Furthermore, if the integrator's initial condition is given by $v(0) = h(c_1, c_2)$ with $\mathbf{x}(0) = [c_1 \quad c_2]^{\mathrm{T}}$ and the initial disturbance is bounded by $|w(0)| \le W$, then the convergence time is bounded from above by

$$\overline{T} = \frac{\left[\left(\frac{3k_1}{2}|c_1|^{\frac{4}{3}} + |c_2|^2\right)^{\frac{5}{4}} + \frac{3k_2}{2} c_1 c_2\right]^{\frac{1}{5}} + \frac{19W}{15k_3}}{\frac{k_2}{5}\left(1 - \frac{2}{3} k_2 k_1^{-\frac{3}{4}}\right) - \frac{19L}{15k_3}}, \qquad (19)$$

i.e., $\mathbf{x}(t) = \mathbf{0}$ holds for all $t \ge \overline{T}$.

Proof. Given in Section 5.2.

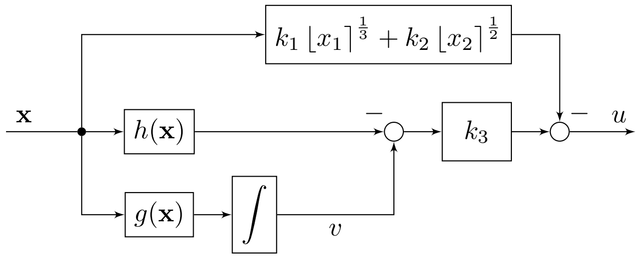

Fig. 1. Structure of the proposed control law with positive parameters k_1, k_2, k_3, and continuous function h and discontinuous function g defined in (15).

Remark 4. When tuning the controller, k_3 may be increased to handle larger Lipschitz constants L, as can be seen in (18), and to reduce the convergence time bound $\overline{T}$, given in (19), down to a minimum value that is determined by the nominal controller's parameters k_1 and k_2.

Remark 5. If $x_2(0) = 0$ and either $w = 0$ or $k_3 \to \infty$, then the convergence time bound simplifies to

$$\overline{T} = \frac{1}{k_2^{\frac{2}{3}}} \frac{5(\frac{3}{2})^{\frac{1}{4}} |x_1(0)|^{\frac{1}{3}}}{\left(k_2 k_1^{-\frac{3}{4}}\right)^{\frac{1}{3}} \left(1 - \frac{2}{3} k_2 k_1^{-\frac{3}{4}}\right)}. \qquad (20)$$

One can see here as well as in (18) that the ratio $k_2 k_1^{-3/4}$ crucially influences the closed-loop's stability properties and convergence time. When this ratio is fixed, increasing k_2 reduces the smallest achievable convergence time, while $k_2 k_3$ determines the actual gain of the integral term, i.e., the value of $k_3 G$ with G as in (17). This gain and therefore—as also reflected in (18)—the value of $k_2 k_3$ are responsible for dominating the disturbance's time derivative $\dot{w}$.

5. PERFORMANCE AND STABILITY ANALYSIS

Closed-loop performance and stability with the proposed controller may be analyzed by introducing the variable $z := -k_3 h(\mathbf{x}) + k_3 v + w$. Using (16), the closed-loop dynamics of the interconnection of (12) and (14) are obtained as

$$\dot{x}_1 = x_2, \qquad (21a)$$
$$\dot{x}_2 = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}} + z, \qquad (21b)$$
$$\dot{z} = -k_3 \frac{\partial h}{\partial x_2} z + \dot{w}. \qquad (21c)$$

Note that h is everywhere differentiable except in the origin, where $\frac{\partial h}{\partial x_2}$ has a singularity. Therefore, these differential equations are valid only for $\mathbf{x} \neq \mathbf{0}$. Nevertheless, they provide valuable insight into the closed-loop behavior, and they will be useful for computing the time derivative of a Lyapunov function candidate for $\mathbf{x} \neq \mathbf{0}$ later on. A Filippov inclusion for the closed loop, which is well-defined everywhere but provides less insight, is obtained by considering the state variable $q = k_3 v + w$ instead of z.

The dynamics (21) show that the original closed-loop performance with controller (13) is recovered, if $w = 0$ and $z(0) = 0$, i.e., if there is no disturbance and the integrator's initial condition is chosen as $v(0) = h(\mathbf{x}(0))$. They furthermore suggest that obtaining stability guarantees may be possible, if $k_3 \frac{\partial h}{\partial x_2}$ is positive definite; indeed, it will be shown that this intuition is correct.

The stability analysis is divided into two parts. First, a Lyapunov function for the nominal closed loop, i.e., for (21a)–(21b) with $z = 0$, is constructed based on a family of such Lyapunov functions proposed in Cruz-Zavala et al. (2018). Then, this result is used to construct a Lyapunov function for the actual closed loop, which allows to prove the main result.

5.1 Lyapunov Function for the Nominal Closed Loop

The nominal closed loop obtained from (21) for $z = 0$ is

$$\dot{x}_1 = x_2, \tag{22a}$$

$$\dot{x}_2 = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}}. \tag{22b}$$

These dynamics are equivalently obtained when applying the nominal control law (13) to the plant (12) without perturbation, i.e., with $w = 0$.

In Cruz-Zavala et al. (2018), a family of strict Lyapunov functions for this system is proposed. For their construction, the following lemma is used, which is a special case of (Cruz-Zavala et al., 2018, Lemma 7).

Lemma 6. (Cruz-Zavala et al., 2018). For any positive constants α, β, μ, and $\nu > 1$, the function

$$V(x_1, x_2) = \left(\alpha |x_1|^{2\mu} + \beta |x_2|^2\right)^{\frac{\nu}{2}} + \delta \lfloor x_1 \rceil^{(\nu-1)\mu} x_2 \tag{23}$$

is positive semidefinite but not positive definite if and only if

$$\delta^2 = \beta\nu \left(\frac{\alpha\nu}{\nu - 1}\right)^{\nu-1} \tag{24}$$

holds.

The following immediate corollary will be useful later on.

Corollary 7. Let α, β, μ be positive, let $\nu > 1$, define

$$\tilde{V}(x_1, x_2) = \alpha |x_1|^{2\mu} + \beta |x_2|^2, \tag{25}$$

and let $\delta > 0$ satisfy (24). Then,

$$\inf_{\substack{x_1, x_2 \in \mathbb{R} \\ \tilde{V}(x_1,x_2)=1}} \lfloor x_1 \rceil^{(\nu-1)\mu} x_2 = -\frac{1}{\delta} = -\frac{1}{\sqrt{\beta\nu}} \left(\frac{\nu - 1}{\alpha\nu}\right)^{\frac{\nu-1}{2}}. \tag{26}$$

Proof. One may write V in (23) as

$$V(x_1, x_2) = \tilde{V}(x_1, x_2)^{\frac{\nu}{2}} + \delta \lfloor x_1 \rceil^{(\nu-1)\mu} x_2, \tag{27}$$

which is is only positive semidefinite according to Lemma 6. Thus, one has

$$0 = \inf_{\tilde{V}=1} V(x_1, x_2) = 1 + \delta \inf_{\tilde{V}=1} \lfloor x_1 \rceil^{(\nu-1)\mu} x_2, \tag{28}$$

which yields the claimed result.

Using these relations, a strict Lyapunov function for (22) of the form

$$V(\mathbf{x}) = \tilde{V}(\mathbf{x})^{\frac{5}{4}} + \varepsilon x_1 x_2 \tag{29}$$

is proposed in Cruz-Zavala et al. (2018), where $\tilde{V}$ is a weak Lyapunov function for the system and ε is a positive constant. It is shown that V is positive definite and its time derivative along the trajectories of (22) is negative definite if ε is sufficiently small.

The following proposition, which is proven in the appendix, extends this result by providing a quantitative analysis.

Proposition 8. Let k_1, k_2 be positive parameters satisfying

$$3k_1^{\frac{3}{4}} > 2k_2 \tag{30}$$

and consider the function

$$V(x_1, x_2) = \left(\frac{3k_1}{2} |x_1|^{\frac{4}{3}} + |x_2|^2\right)^{\frac{5}{4}} + \frac{3k_2}{2} x_1 x_2. \tag{31}$$

Then, V is positive definite and its time derivative $\dot{V}$ along the trajectories of (22) satisfies $\dot{V} \leq -CV^{\frac{4}{5}}$ with

$$C = \left(1 - \frac{2k_2}{3k_1^{\frac{3}{4}}}\right) k_2. \tag{32}$$

Proof. Given in the appendix.

In order to use this result for the construction of a Lyapunov function for the actual closed loop (21), the following auxiliary result is needed. It uses the positive definiteness of $\frac{\partial h}{\partial x_2}$ to establish a relation between the partial derivatives of V and h.

Lemma 9. Let k_1, k_2 be positive parameters and consider functions h and V defined in (15a) and (31), respectively. Suppose that (30) is fulfilled. Then,

$$\left|\frac{\partial V}{\partial x_2}\right| \leq F V^{\frac{4}{5}} \frac{\partial h}{\partial x_2} \tag{33}$$

holds with $F = \frac{19}{3}$.

Proof. Given in the appendix.

5.2 Lyapunov Function and Proof for the Main Theorem

The stability result and the convergence time bound in Theorem 3 can now be proven. To that end, consider the positive definite function

$$\overline{V}(x_1, x_2, z) = V(x_1, x_2)^{\frac{1}{5}} + \beta |z| \tag{34}$$

with V given in (31) and a positive constant β to be chosen later. If $\mathbf{x} \neq 0$ and $z \neq 0$, then this function is differentiable, and its time derivative along the trajectories of the closed-loop system formed by (12) and (14), or equivalently system (21), is

$$\dot{\overline{V}} = \frac{1}{5} V^{-\frac{4}{5}} (Q + \frac{\partial V}{\partial x_2} z) - k_3 \beta \frac{\partial h}{\partial x_2} |z| + \beta \lfloor z \rceil^0 \dot{w}, \tag{35}$$

where the function $Q(x_1, x_2)$ denotes the time derivative of V along the trajectories of the *nominal* closed-loop (22), i.e.,

$$Q(\mathbf{x}) = \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2} (-k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}}). \tag{36}$$

According to Proposition 8 and Lemma 9, the inequalities

$$Q(\mathbf{x}) \leq -CV(\mathbf{x})^{\frac{4}{5}}, \qquad \left|\frac{\partial V}{\partial x_2}\right| \leq F \frac{\partial h}{\partial x_2} V(\mathbf{x})^{\frac{4}{5}} \tag{37}$$

hold with C given in (32) and $F = \frac{19}{3}$. Thus, choosing $\beta = \frac{F}{5k_3} = \frac{19}{15k_3}$, one has

$$\begin{aligned}
\dot{\overline{V}} &\leq -\frac{C}{5} + \frac{1}{5} V^{-\frac{4}{5}} \frac{\partial V}{\partial x_2} z - \frac{F}{5} \frac{\partial h}{\partial x_2} |z| + \frac{F}{5k_3} \lfloor z \rceil^0 \dot{w} \\
&\leq -\frac{C - F\frac{L}{k_3}}{5} + \frac{1}{5} V^{-\frac{4}{5}} \left|\frac{\partial V}{\partial x_2}\right| |z| - \frac{F}{5} \frac{\partial h}{\partial x_2} |z| \\
&\leq -\frac{C - F\frac{L}{k_3}}{5} = -\left(1 - \frac{2k_2}{3k_1^{\frac{3}{4}}}\right) \frac{k_2}{5} + \frac{19L}{15k_3} < 0,
\end{aligned} \tag{38}$$

where the upper bound is negative due to (18).

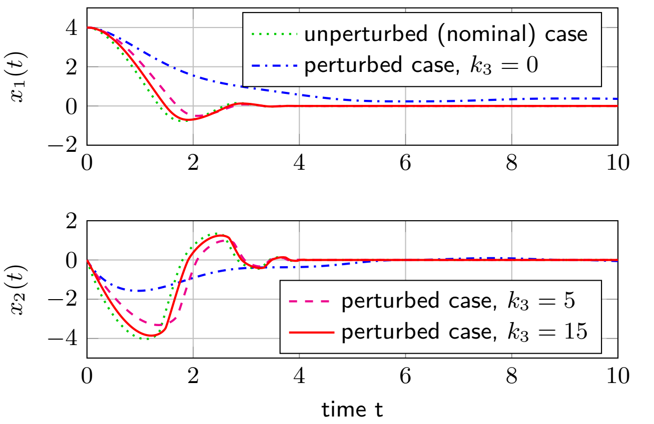

Fig. 2. Simulation results illustrating the influence of the parameter k_3 for $k_1 = 6$, $k_2 = 3$, initial condition $v(0) = h(\mathbf{x}(0)) = 0$, and disturbance $w(t) = 4 + \sin t$.

The cases where $\overline{V}$ is not differentiable are now considered: For $z \neq 0$, $\mathbf{x}(t) = \mathbf{0}$ can occur only at isolated time instants t, because then $\dot{x}_2 = z$, and thus x_2 immediately becomes non-zero again. For z this is not the case, however, because one can have $z(t) = 0$ on time intervals with non-zero length even if $\mathbf{x} \neq \mathbf{0}$. In this case, the closed-loop dynamics and the Lyapunov function both reduce to the nominal case, however, i.e., dynamics are given by (22) and $\overline{V} = V^{\frac{1}{5}}$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}\overline{V} \leq -5^{-1}C$ holds according to Proposition 8, which again implies the inequality (38).

From these considerations one concludes that the differential inequality (38) holds everywhere except at isolated time instants and in the equilibrium $\mathbf{x} = \mathbf{0}$, $z = 0$. Therefore, it may be integrated to establish finite-time stability and show that the convergence time T for $v(0) = h(\mathbf{x}(0))$, i.e., for $z(0) = w(0)$ is bounded by

$$T \leq \frac{5\overline{V}(\mathbf{x}(0), z(0))}{C - F\frac{L}{k_3}} = \frac{5V(\mathbf{x}(0))^{\frac{1}{5}} + \frac{F}{k_3}|w(0)|}{C - F\frac{L}{k_3}} = \overline{T}, \quad (39)$$

which proves the theorem.

6. SIMULATION RESULTS

The proposed controller (14) is applied to the plant (12) with parameter values $k_1 = 6$, $k_2 = 3$, initial condition $v(0) = h(\mathbf{x}(0))$, and disturbance $w(t) = 4 + \sin t$. Simulation results for this setup are obtained using forward Euler discretization with a step size of 10^{-4}.

Fig. 2 illustrates the influence of the tuning parameter k_3 on the closed-loop performance. One can see that the pure (nominal) state-feedback controller (13) without integral action, i.e., (14) with $k_3 = 0$, can not reject the disturbance and converges to zero only in the unperturbed case. With values $k_3 = 5$ and $k_3 = 15$, chosen using Theorem 3, convergence to the origin is achieved again and performance can be seen to approach the nominal case with increasing values of k_3.

Fig. 3 shows that Theorem 3 also yields reasonable convergence time bounds, provided that the parameter k_3 is not too close to the lower limit in (18). One can see that for $k_3 = 20$ the upper bound $\overline{T} \approx 13.7$ obtained from (19)

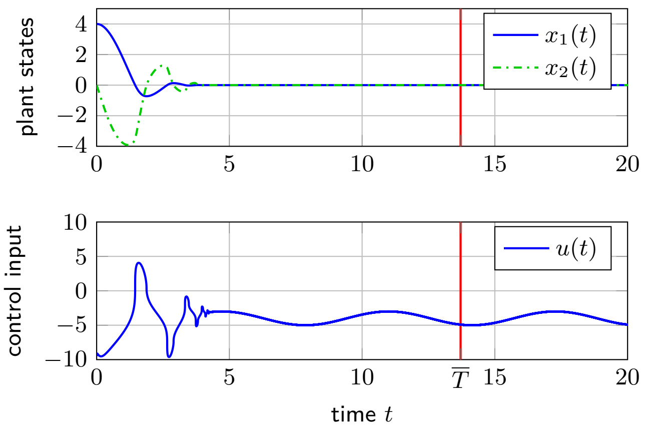

Fig. 3. Simulation results with parameters $k_1 = 6$, $k_2 = 3$, $k_3 = 20$, initial condition $v(0) = 0$, and disturbance $w(t) = 4 + \sin t$ and convergence time bound $\overline{T} \approx 13.7$ obtained from (19) with $W = 5$, $L = 1$.

with $W = 5$, $L = 1$ overestimates the actual convergence time $T \approx 4.3$ only by a factor of three, approximately.

7. CONCLUSION AND OUTLOOK

A new homogeneous state-feedback control law with discontinuous integral action for the perturbed double integrator was proposed. It is based on an interesting structural insight obtained in the linear case, which enables a performance preserving integral extension of a given state-feedback controller. Main features of the proposed approach are its easy tuning, the structurally simple stability condition, and the extendability of the idea to arbitrary order systems. In the future, these extensions and their stability properties may be studied.

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Appendix A. PROOFS

A.1 Proof of Proposition 8

Introducing the abbreviations $\varepsilon = \frac{3k_2}{2}$ and

$$\tilde{V}(x_1, x_2) = \frac{3k_1}{2} |x_1|^{\frac{4}{3}} + |x_2|^2 \tag{A.1}$$

one may write the function V as $V = \tilde{V}^{\frac{5}{4}} + \varepsilon x_1 x_2$. From Corollary 7 with $\alpha = \frac{3k_1}{2}$, $\beta = 1$, $\mu = \frac{2}{3}$, $\nu = \frac{5}{2}$ one finds

$$\begin{aligned}\inf_{\tilde{V}=1} V(x_1, x_2) &= 1 + \varepsilon \inf_{\tilde{V}=1} x_1 x_2 = 1 - \frac{\varepsilon}{k_1^{\frac{3}{4}}} \left(\frac{2}{5}\right)^{\frac{5}{4}} \\ &= 1 - \frac{3k_2}{2k_1^{\frac{3}{4}}} \left(\frac{2}{5}\right)^{\frac{5}{4}} > 1 - \frac{2k_2}{3k_1^{\frac{3}{4}}} > 0 \end{aligned} \tag{A.2}$$

due to (30), which shows that V is positive definite.

The time derivative of $\tilde{V}$ along the trajectories of (22) is $\dot{\tilde{V}} = -2k_2 |x_2|^{\frac{3}{2}}$. One thus obtains for $\dot{V}$ the upper bound

$$\begin{aligned}\dot{V} &= \frac{5}{4} \tilde{V}^{\frac{1}{4}} \dot{\tilde{V}} + \varepsilon(|x_2|^2 - k_1 |x_1|^{\frac{4}{3}} - k_2 x_1 \lfloor x_2 \rceil^{\frac{1}{2}}) \\ &= -\frac{5k_2}{2} \tilde{V}^{\frac{1}{4}} |x_2|^{\frac{3}{2}} + \varepsilon(|x_2|^2 - k_1 |x_1|^{\frac{4}{3}} - k_2 x_1 \lfloor x_2 \rceil^{\frac{1}{2}}) \\ &\le -\frac{5k_2 - 2\varepsilon}{2} |x_2|^2 - \varepsilon k_1 |x_1|^{\frac{4}{3}} - \varepsilon k_2 x_1 \lfloor x_2 \rceil^{\frac{1}{2}} \\ &= -k_2 \Big(\tilde{V} + \frac{3k_2}{2} x_1 \lfloor x_2 \rceil^{\frac{1}{2}}\Big). \end{aligned} \tag{A.3}$$

Consider now the ratio

$$\frac{\dot{V}(x_1, x_2)}{V(x_1, x_2)^{\frac{4}{5}}} \le -k_2 \frac{\tilde{V}(x_1, x_2) + \frac{3k_2}{2} x_1 \lfloor x_2 \rceil^{\frac{1}{2}}}{\left(\tilde{V}(x_1, x_2)^{\frac{5}{4}} + \frac{3k_2}{2} x_1 x_2\right)^{\frac{4}{5}}} =: c(x_1, x_2), \tag{A.4}$$

whose upper bound is denoted by the function c. This function is homogeneous of degree zero, i.e., one has $c(\alpha^3 x_1, \alpha^2 x_2) = c(x_1, x_2)$ for all $\alpha > 0$. For finding a uniform upper bound for it, it is thus sufficient to consider states x_1, x_2 that satisfy $\tilde{V}(x_1, x_2) = 1$. Thus, the expression to be bounded becomes

$$c(x_1, x_2)|_{\tilde{V}(x_1, x_2)=1} = -k_2 \frac{1 + \frac{3k_2}{2} x_1 \lfloor x_2 \rceil^{\frac{1}{2}}}{(1 + \frac{3k_2}{2} x_1 x_2)^{\frac{4}{5}}}. \tag{A.5}$$

Consider abbreviations $y_1 = x_1 \lfloor x_2 \rceil^{\frac{1}{2}}$ and $y_2 = |x_2|^{\frac{1}{2}}$; one may note that $\tilde{V} = 1$ implies $|y_1| \le Y$ and $y_2 \in [0, 1]$ with

$$Y = \sqrt{-\inf_{\tilde{V}=1} \lfloor x_1 \rceil^2 x_2} = \left(\frac{1}{2}\right)^{\frac{5}{4}} k_1^{-\frac{3}{4}} \tag{A.6}$$

obtained from Corollary 7 with $\nu = 4$ and α, β, μ as before. Thus, the function c is bounded by

$$c(x_1, x_2) \le -k_2 \inf_{\substack{|y_1| \le Y \\ y_2 \in [0,1]}} \frac{1 + \frac{3k_2}{2} y_1}{(1 + \frac{3k_2}{2} y_1 y_2)^{\frac{4}{5}}}. \tag{A.7}$$

By computing the derivative of the expression to be minimized, one can verify using (30) that it is non-decreasing with respect to y_1 for all admissible values of y_1, y_2. Thus, the infimum is obtained for $y_1 = -Y$ and $y_2 = 0$, and therefore

$$c(x_1, x_2) \le -\left(1 - \left(\frac{1}{2}\right)^{\frac{5}{4}} \frac{3k_2}{2k_1^{\frac{3}{4}}}\right) k_2 \le -C \tag{A.8}$$

holds with C in (32), which completes the proof.

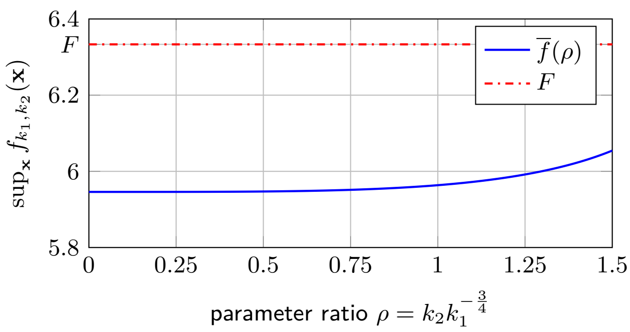

Fig. A.1. Graphical proof of Lemma 9, demonstrating that $f(\mathbf{x})$ defined in (A.11) is bounded by $f(\mathbf{x}) \le F$ for all values of $\mathbf{x}$ and all k_1, k_2 satisfying (18).

A.2 Proof of Lemma 9

Using the abbreviation $\tilde{V}$ defined in (A.1), one may write $h = x_2(\tilde{V} + |x_2|^2)^{-\frac{1}{4}}$ and $V = \tilde{V}^{\frac{5}{4}} + \frac{3k_2}{2} x_1 x_2$. One thus obtains the partial derivatives

$$\frac{\partial h}{\partial x_2} = \frac{\tilde{V}}{(\tilde{V} + |x_2|^2)^{\frac{5}{4}}}, \tag{A.9}$$

$$\frac{\partial V}{\partial x_2} = \frac{5}{2} \tilde{V}^{\frac{1}{4}} x_2 + \frac{3k_2}{2} x_1. \tag{A.10}$$

Since $\frac{\partial h}{\partial x_2}$ is positive definite, it needs to be shown that the function f defined as

$$f(\mathbf{x}) := \frac{\left|\frac{\partial V}{\partial x_2}\right|}{V^{\frac{4}{5}} \frac{\partial h}{\partial x_2}} = \frac{\left|5\tilde{V}(\mathbf{x})^{\frac{1}{4}} x_2 + 3k_2 x_1\right| (\tilde{V}(\mathbf{x}) + |x_2|^2)^{\frac{5}{4}}}{2(\tilde{V}(\mathbf{x})^{\frac{5}{4}} + \frac{3k_2}{2} x_1 x_2)^{\frac{4}{5}} \tilde{V}(\mathbf{x})} \tag{A.11}$$

satisfies $f(\mathbf{x}) \le F$ for all $\mathbf{x}$ and all admissible values of k_1, k_2. This function is homogeneous with homogeneity degree zero, i.e., it fulfills $f(\alpha^3 x_1, \alpha^2 x_2) = f(x_1, x_2)$ for all $\alpha > 0$. Therefore, considerations can be constrained to values on a circular arc in the state space given by $x_1 = \cos(\varphi)$, $x_2 = \sin(\varphi)$ with $\varphi \in [-\pi, \pi]$. Furthermore, denoting the dependence of f on the parameters k_1, k_2 explicitly as f_{k_1,k_2}, one can verify that it fulfills

$$f_{\alpha^{\frac{4}{3}} k_1, \alpha k_2}(\alpha^{-1} x_1, x_2) = f_{k_1,k_2}(x_1, x_2). \tag{A.12}$$

Thus, the maximum of f with respect to φ is only a function of the positive parameter ratio $\rho := k_2 k_1^{-3/4}$, which according to (30) is bounded by $\rho < \frac{3}{2}$. Introducing the function

$$\overline{f}(\rho) = \sup_{|\varphi| \le \pi} f_{1,\rho}(\cos\varphi, \sin\varphi), \tag{A.13}$$

it needs to be shown that $\overline{f}(\rho) \le F$ holds for all $\rho \in (0, \frac{3}{2})$. Fig. A.1 depicts this function obtained from a numerical evaluation of the supremum in (A.13) on an evenly spaced grid of 1000 values for $\varphi \in [-\pi, \pi]$; one can see graphically that the claimed inequality indeed holds for all $\rho \in (0, \frac{3}{2})$, which concludes the proof.