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Laplacian Controllability of a Class of Non-Simple Ring Graphs ⋆

Ping-Yen Yang * **Shun-Pin Hsu** **

* *National Chung-Hsing University, Taichung, Taiwan (e-mail: youngbinyeah@gmail.com)*
** *National Chung-Hsing University, Taichung, Taiwan (e-mail: shsu@nchu.edu.tw)*

Abstract: The Laplacian controllability of a family of graphs that are non-simple is studied in the paper. Without the regular assumption that the adjacency matrix is binary, the authors consider more flexible weighting parameters to represent the practical connection strength between nodes. Suppose the node states of the graphs evolve according to the Laplacian dynamics. The Laplacian eigenspaces of a class of ring graphs are explored, by which a sufficient condition to render the graphs controllable with the minimum number of input is proposed. Numerical examples are provided to illustrate the theoretical results.

Keywords: Laplacian Controllability, Multiagent Network, Ring Graphs, Linear Systems

1. INTRODUCTION

Motivated by the fundamental importance in effective operations of networked systems, many researchers have addressed the controllability issues of connected network whose node states evolve according to some specific dynamics (Egerstedt et al. (2012)). Over the past decade, one of the major research themes has been on the condition that determines the controllability from the network topology composed of leader nodes and follower nodes, in which the leader nodes serve as the sources of input signals (Rahmani et al. (2009)). Recently, this topic is considered from the viewpoint of linear systems evolving on the Laplacian dynamics. In this model the control input does not appear as the node in the network but as simply exterior input sources. As shown in Aguilar and Gharesifard (2015), this setting gains its wide popularity since it integrates the study of multi-input network into a similar formulation. By exploring the symmetric connection structure in a very general sense, several simple and verifiable conditions for determining the *uncontrollability* of the network were proposed. However, it is not yet clear how to transform the algebraic methods such as Kalman's rank test or Popov-Belevitch-Hautus's (PBH) rank test introduced in, e.g., Chen (1999), into their graph-theoretic counterparts. Though some result independent of the connecting type was proposed by Hsu (2019b), in general the condition to ensure the controllability of a network by the given controller(s) is available only for specific types of connection structures such as the paths, grids, circulant graphs, complete graphs, multi-chain, threshold graphs, and so on (see Parlangeli and Notarstefano (2012), Hsu and Yang (2019), Notarstefano and Parlangeli (2013), Nabi-Abdolyousefi and Mesbahi (2013), Zhang et al. (2011), Cao et al. (2013), Hsu (2017), and Hsu (2019a) for the related results). Among these uncontrollability or controllability conditions, the network connection is assumed to form a simple and connected graph. That means the adjacency matrix is binary; its (i, j)th entry is 1 if node i and j is connected and is 0 otherwise. This model cannot reflect the interacting strength between nodes, and restricts its applicability to the case in which negative weighting parameters can better describe the interaction of two adversarial nodes (Altafini (2013); Sun et al. (2017)). In this note we address this issue and consider a class of flexible weighting parameters. To simplify the analysis we focus on the ring graphs that satisfy some condition of periodic constant product. Under this condition we explore the Laplacian eigenspace properties of the graphs. These properties shed light on the method to use the minimum number of controllers to maneuver each node state of the graph.

The rest of the paper is organized as follows. The second section is a review of essential graph-theoretical notations, concepts, and related control theories. Our main results on the controllability properties of ring graphs satisfying the periodic product rule are presented in the third section.The paper is concluded in Section 4, where potential extensions to more generalized network are discussed.

2. PRELIMINARIES

Let $\mathbf{0}$ and $\mathbf{1}$ be the column vectors of 0's and 1's respectively, and $\boldsymbol{e}_i$ the ith column of the identity matrix $\mathbf{I}$, with appropriate size. Let $\mathbb{Z}$ be the set of integer numbers, and

$$\mathbb{Z} \setminus n\mathbb{Z} = \{z \in \mathbb{Z} : z \neq kn,\ k \in \mathbb{Z}\}\,. \tag{1}$$

The product of $s_1, s_2, \cdots, s_n$ is denoted by $\prod_{i=1}^{n} s_i$, and the determinant of matrix A is $|A|$ or $\det(A)$. Let $\mathbb{G} = (V, E, W)$ represent a directed network or graph, where $V = \mathbb{I}_n := \{1, 2, \cdots, n\}$ is the set of nodes or vertices and $E \subseteq V \times V$ the set of arrows. In addition, $W : E \to \mathbb{R}$ is a mapping of the arrows to the real numbers. An ordered pair $(i, j) \in E$ is called the arrow of the graph $\mathbb{G}$. We call

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i the parent vertex of j and j the child vertex of i. The neighbor set $\mathcal{N}_i$ of vertex i is the collection of the parent vertices of i. That is,

$$\mathcal{N}_i = \{j \in V : (j,i) \in E\}. \tag{2}$$

Suppose W maps the arrow (i,j) to w_{ij}, such that $(j,i) \in E$ if and only if $w_{ij} \neq 0$. The adjacency matrix $\mathcal{W}$ of graph $\mathbb{G}$ is an $n \times n$ matrix whose (i,j)th entry is w_{ij}. To simplify the analysis, we assume that there is no self-loop in the graph; i.e., $w_{ii} = 0$, $\forall i \in V$. The degree d_i of vertex i is defined as $d_i = \sum_{j \in \mathcal{N}_i} |w_{ij}|$. The Laplacian matrix $\mathcal{L}$ of $\mathbb{G}$ is

$$\mathcal{L} := \mathcal{D} - \mathcal{W}, \tag{3}$$

where $\mathcal{D}$ is a diagonal matrix with its ith diagonal term being d_i. Thus, the (i,j)th entry of the Laplacian matrix can be written as

$$\ell_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -w_{ij}, & \text{o.w.} \end{cases}. \tag{4}$$

Suppose the evolution of node states x_i of the graph $\mathbb{G}$ follows the updating law:

$$\dot{x}_i = -d_i x_i + \sum_{j \in \mathcal{N}_i} w_{ij} x_j, \tag{5}$$

or in matrix form,

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x}, \tag{6}$$

where $\boldsymbol{x}^T = [x_1 \cdots x_n]$ and $\mathcal{L}$ is the Laplacian matrix corresponding to the graph $\mathbb{G}$. Suppose the autonomous system in (6) is driven by p controllers. The controlled graph becomes a standard linear and time invariant system

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x} + B\boldsymbol{u}(t), \tag{7}$$

where $\boldsymbol{u}^T(t) = [u_1(t) \cdots u_p(t)]$ and B is a binary matrix whose (i,j)th component is 1 if node i is connected to controller j and is 0 otherwise. Throughout the paper, we say a ring graph is (Laplacian) controllable if its corresponding LTI system in (7) is controllable.

In this paper, we focus on the controllability of a class of ring graphs. As shown in Parlangeli and Notarstefano (2012), if the graph is simple, meaning that the adjacency matrix of the ring is binary, two controllers connected to two neighboring nodes suffice to render the graph Laplacian controllable. In the subsequent section we will show that this result can be extended to any non-simple ring graph. Furthermore, we will show that one controller is enough to maneuver each node state of the ring graph if some condition of periodic constant product is satisfied. The controllability result is based on the Laplacian eigenspace analysis of the graph and the application of the Popov-Belevitch-Hautus theorem (Chen (1999)):

Theorem 1. Let $\boldsymbol{v}$ be the left-eigenvector of $\mathcal{L}$ corresponding to λ; i.e., $\boldsymbol{v}\mathcal{L} = \lambda\boldsymbol{v}$, $\boldsymbol{v} \neq \boldsymbol{0}$. The system $(\mathcal{L}, B)$ in (7) is controllable if and only if there is no left-eigenvector satisfying $\boldsymbol{v}B = \boldsymbol{0}$.

3. MAIN RESULTS

Thoughout this paper, we use n as the number of nodes in the graph. Let the Laplacian matrix of a ring graph be written as

$$\mathcal{L}(\gamma,\delta) = \begin{bmatrix} b & c_1 & & \delta \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ \gamma & & a_{n-1} & b \end{bmatrix}^T, \tag{8}$$

We first present a controllability result that is applicable to general ring graphs.

Lemma 2. Any connected non-simple ring graph is Laplacian controllable by two controllers connected to two neighboring nodes respectively.

Proof. In a connected ring, the parameters in (8) have nonzero γ, δ, and a_i, c_i for $i \in \{1, 2, \cdots, n-1\}$. Thus the rank of the matrix $\mathcal{L}(\gamma,\delta) - \lambda I$ is at least $n-2$ and thus the geometric multiplicity of any eigenvalue of $\mathcal{L}(\gamma,\delta)$ is at most 2. Furthermore, if two adjacent components of any eigenvector of $\mathcal{L}(\gamma,\delta)$ are zero, then the eigenvector must be a zero vector, a contradiction. We conclude by Theorem 1 that the ring graph is controllable by two controllers connected to two neighboring nodes respectively.

In the following we focus on a family of ring graphs satisfying the following condition of periodic constant product.

Assumption 1. The ring graph has even number of nodes and parameters in its Laplacian matrix (8) satisfy

$$a_i c_i = \begin{cases} d_1^2, & \text{if } i \text{ is odd}, \\ d_2^2, & \text{o.w.} \end{cases} \tag{9}$$

and d_1, d_2 are nonzero.

For the possible values of γ and δ in $\mathcal{L}(\gamma,\delta)$, we define

$$\Gamma := \frac{(d_1 d_2)^{\frac{n}{2}}}{\prod_{i=1}^{n-1} c_i}, \quad \Delta := \frac{(d_1 d_2)^{\frac{n}{2}}}{\prod_{i=1}^{n-1} a_i},$$

and

$$\begin{aligned} \mathcal{L}_0 &:= \mathcal{L}(0,0), \\ \mathcal{L}_1 &:= \mathcal{L}(\Gamma, -\Delta), \\ \mathcal{L}_2 &:= \mathcal{L}(-\Gamma, \Delta). \end{aligned} \tag{10}$$

In the case of nonzero d_1, d_2 and complex θ, it is shown in Zill et al. (2006) that for each constant λ there is an θ such that

$$(\lambda - b)^2 = {d_1}^2 + {d_2}^2 + 2d_1 d_2 \cos 2\theta. \tag{11}$$

Using this identity Kouachi (2006) derived the eigenvalues of $\mathcal{L}_0$ by proposing the following result:

Lemma 3. If n is odd, the characteristic polynomial of $\mathcal{L}_0$ is

$$\det(\lambda\mathbf{I} - \mathcal{L}_0) = (d_1 d_2)^{\frac{n-1}{2}} \frac{(\lambda - b)\sin(n+1)\theta}{\sin 2\theta}, \tag{12}$$

otherwise,

$$\det(\lambda\mathbf{I} - \mathcal{L}_0) = (d_1 d_2)^{\frac{n}{2}} \frac{\sin(n+2)\theta + \frac{d_2}{d_1}\sin n\theta}{\sin 2\theta}, \tag{13}$$

where λ and θ satisfy (11).

Now we consider the cases of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 4. Let n be even and $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $\mathcal{L}_1$ or $\mathcal{L}_2$. If $k \in \left\{1, \cdots, \frac{n}{2} - 1\right\}$,

$$\lambda_k = b + \sqrt{{d_1}^2 + {d_2}^2 + 2d_1 d_2 \cos \frac{2k\pi}{n}}. \tag{14}$$

If $k \in \left\{\frac{n}{2}+1, \cdots, n-1\right\}$,

$$\lambda_k = b - \sqrt{{d_1}^2 + {d_2}^2 + 2d_1 d_2 \cos\frac{2k\pi}{n}}. \tag{15}$$

Otherwise,

$$\lambda_k = \begin{cases} b + \sqrt{{d_1}^2 - {d_2}^2}, & k = \dfrac{n}{2}, \\ b - \sqrt{{d_1}^2 - {d_2}^2}, & k = n. \end{cases} \tag{16}$$

Proof. We prove the case of $\mathcal{L}_1$ and write

$$\mathcal{E}[i:j] := \left| \lambda \mathbf{I} - \begin{bmatrix} b & c_i & & \\ a_i & \ddots & \ddots & \\ & \ddots & \ddots & c_{j-1} \\ & & a_{j-1} & b \end{bmatrix} \right|. \tag{17}$$

Clearly, $\mathcal{E}[1:n] = \det(\lambda\mathbf{I} - \mathcal{L}_0)$. Apply the Laplacian expansion on the last column of $\det(\lambda\mathbf{I} - \mathcal{L}_1)$ to yield

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathcal{L}_1) &= (\lambda - b)\mathcal{E}[1:n-1] \\ &\quad - {d_1}^2\mathcal{E}[1:n-2] + {d_2}^2\mathcal{E}[2:n-1]. \end{aligned} \tag{18}$$

By Lemma 3, we can write (18) as

$$(d_1 d_2)^{\frac{n-2}{2}} \frac{\left((\lambda - b)^2 - {d_1}^2 + {d_2}^2\right)\sin n\theta}{\sin 2\theta}.$$

As a result, the eigenvalue of $\mathcal{L}_1$ can be written as

$$\lambda_+ = b + \sqrt{{d_1}^2 + {d_2}^2 + 2d_1 d_2 \cos 2\theta}, \tag{19}$$

$$\lambda_- = b - \sqrt{{d_1}^2 + {d_2}^2 + 2d_1 d_2 \cos 2\theta}, \tag{20}$$

where $\theta = \frac{k\pi}{n}$, $k \in \mathbb{Z} \setminus n\mathbb{Z}$, and

$$\lambda_+ = b + \sqrt{{d_1}^2 - {d_2}^2}, \tag{21}$$

$$\lambda_- = b - \sqrt{{d_1}^2 - {d_2}^2}. \tag{22}$$

We thus complete the proof. The case of $\mathcal{L}_2$ can be shown in a similar way and is skipped.

Now we analyze the Laplacian eigenspace. To simplify the notations, we let

$$\theta_k := \begin{cases} \dfrac{k\pi}{n}, & k \notin \left\{\dfrac{n}{2}, n\right\}, \\ \dfrac{1}{2}\cos^{-1}\left(-\dfrac{d_2}{d_1}\right), & k \in \left\{\dfrac{n}{2}, n\right\}. \end{cases} \tag{23}$$

and

$$\sigma_j := \begin{cases} \prod_{i=1}^{j-1} a_i, & j > 1, \\ 1, & j = 1. \end{cases} \tag{24}$$

Lemma 5. Suppose in Assumption 1, $d_1^2 \neq d_2^2$. Let $\lambda_1, \cdots, \lambda_n$ be defined in Lemma 4 and $\boldsymbol{v}^{(k)}$ the left-eigenvector of $\mathcal{L}_1$ corresponding to λ_k. The components $v_1^{(k)}, \cdots, v_n^{(k)}$ of $\boldsymbol{v}^{(k)}$ are given as follows:

(1) When $k \in \{1, 3, 5, \cdots, n-1\} \setminus \left\{\frac{n}{2}\right\}$,
 (a) if j is odd,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j+1}{2}} (\lambda_k - b)\sin(j-1)\theta_k, \tag{25}$$

 (b) if j is even,

$$\begin{aligned} v_j^{(k)} &= \sigma_j (d_1 d_2)^{-\frac{j}{2}} \sin(j-2)\theta_k \\ &\quad + \sigma_j (d_1 d_2)^{-\frac{j}{2}} \frac{d_2}{d_1} \sin j\theta_k. \end{aligned} \tag{26}$$

(2) When $k \in \{2, 4, 6, \cdots, n-2\} \setminus \left\{\frac{n}{2}\right\}$,
 (a) if j is odd,

$$\begin{aligned} v_j^{(k)} &= \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \sin(j-1)\theta_k \\ &\quad + \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \frac{d_1}{d_2} \sin(j+1)\theta_k, \end{aligned} \tag{27}$$

 (b) if j is even,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j}{2}} (\lambda_k - b)\sin j\theta_k. \tag{28}$$

(3) When $k \in \left\{\frac{n}{2}, n\right\}$,
 (a) if j is odd,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} (\lambda_k - b)\cos\left(\frac{n}{2} - j + 1\right)\theta_k, \tag{29}$$

 (b) if j is even,

$$v_j^{(k)} = -\sigma_j (d_1 d_2)^{-\frac{j-2}{2}} \sin\left(\frac{n}{2} - j\right)\theta_k \sin 2\theta_k. \tag{30}$$

Proof. Suppose $k \notin \left\{\frac{n}{2}, n\right\}$. For each $j \in \{1, \cdots, n-2\}$,

$$-a_j v_j^{(k)} + (\lambda_k - b)v_{j+1}^{(k)} - c_{j+1} v_{j+2}^{(k)} = 0. \tag{31}$$

In a matrix form we have

$$\begin{bmatrix} \lambda_k - b & -c_2 & & \\ -a_2 & \ddots & \ddots & \\ & \ddots & \ddots & -c_{n-2} \\ & & -a_{n-2} & \lambda_k - b \end{bmatrix} \begin{bmatrix} v_2^{(k)} \\ \vdots \\ \vdots \\ v_{n-1}^{(k)} \end{bmatrix} = \begin{bmatrix} a_1 v_1^{(k)} \\ \\ \\ c_{n-1} v_n^{(k)} \end{bmatrix}. \tag{32}$$

Applying Cramer's rule, $v_2^{(k)}, \cdots, v_{n-1}^{(k)}$, can be expressed by $v_1^{(k)}$ and $v_n^{(k)}$; i.e., for each $j \in \{2, \cdots, n-1\}$,

$$v_j^{(k)} = \frac{\mathcal{R}[j]}{\mathcal{E}[2:n-1]}, \tag{33}$$

where $\mathcal{E}[i:j]$ is defined in (17) and $\mathcal{R}[j]$ in the following:

$$\left| \lambda_k \mathbf{I} - \begin{bmatrix} b & c_2 & & & -a_1 v_1^{(k)} & & & & \\ a_2 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & c_{j-2} & & & & & \\ & & a_{j-2} & b & & & & & \\ & & & a_{j-1} & \lambda_k & c_j & & & \\ & & & & & b & c_{j+1} & & \\ & & & & & a_{j+1} & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & c_{n-2} \\ & & & & -c_{n-1} v_n^{(k)} & & & a_{n-2} & b \end{bmatrix} \right|. \tag{34}$$

Note that (33) can be written as

$$\begin{aligned} v_j^{(k)} &= \frac{\left(\prod_{i=1}^{j-1} a_i\right)\mathcal{E}[j+1:n-1]}{\mathcal{E}[2:n-1]} v_1^{(k)} \\ &\quad + \frac{\left(\prod_{i=j}^{n-1} c_i\right)\mathcal{E}[2:j-1]}{\mathcal{E}[2:n-1]} v_n^{(k)}. \end{aligned} \tag{35}$$

Since λ_k is single, the components $v_1^{(k)}$ and $v_n^{(k)}$ must not be independent. Recall that $v_2^{(k)}$ and $v_n^{(k)}$ satisfy

$$(\lambda_k - b)v_1^{(k)} - c_1 v_2^{(k)} + \Delta v_n^{(k)} = 0, \tag{36}$$

$$-\Gamma v_1^{(k)} - a_{n-1} v_{n-1}^{(k)} + (\lambda_k - b)v_n^{(k)} = 0. \tag{37}$$

Lemma 3 and (37) imply that $v_1^{(k)} = 0$ for odd k. Lemma 3 and (36) imply that $v_n^{(k)} = 0$ for even k. Thus we obtain the explicit form of $\boldsymbol{v}^{(k)}$ for $k \notin \{\frac{n}{2}, n\}$. The case $k \in \{\frac{n}{2}, n\}$ can be proved in a similar way and is skipped.

Lemma 6. Follow the conditions in Lemma 5 and let $\boldsymbol{v}^{(k)}$ be the left-eigenvector of $\mathcal{L}_2$ corresponding to λ_k. The components $v_1^{(k)}, \cdots, v_n^{(k)}$ of $\boldsymbol{v}^{(k)}$ are given as follows:

(1) When $k \in \{1, 3, 5, \cdots, n-1\} \setminus \{\frac{n}{2}\}$,
(a) if j is odd,

$$\begin{aligned} v_j^{(k)} &= \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \sin(j-1)\theta_k \\ &\quad + \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \frac{d_1}{d_2} \sin(j+1)\theta_k, \end{aligned} \tag{38}$$

(b) if j is even,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j}{2}} (\lambda_k - b) \sin j\theta_k. \tag{39}$$

(2) When $k \in \{2, 4, 6, \cdots, n-2\} \setminus \{\frac{n}{2}\}$,
(a) if j is odd,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j+1}{2}} (\lambda_k - b) \sin(j-1)\theta_k, \tag{40}$$

(b) if j is even,

$$\begin{aligned} v_j^{(k)} &= \sigma_j (d_1 d_2)^{-\frac{j}{2}} \sin(j-2)\theta_k \\ &\quad + \sigma_j (d_1 d_2)^{-\frac{j}{2}} \frac{d_2}{d_1} \sin j\theta_k. \end{aligned} \tag{41}$$

(3) When $k \in \{\frac{n}{2}, n\}$,
(a) if j is odd,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} (\lambda_k - b) \sin\left(\frac{n}{2} - j + 1\right)\theta_k, \tag{42}$$

(b) if j is even,

$$v_j^{(k)} = \sigma_j (d_1 d_2)^{-\frac{j-2}{2}} \cos\left(\frac{n}{2} - j\right)\theta_k \sin 2\theta_k. \tag{43}$$

Proof. The results above can be derived in a similar way to that for Lemma 5 and thus is skipped.

If ${d_1}^2 = {d_2}^2$, both $\mathcal{L}_1$ and $\mathcal{L}_2$ have a repeated eigenvalue b. We discuss this case in the following.

Lemma 7. Suppose in Lemma 5, $d_1^2 = d_2^2$. If $k \notin \{\frac{n}{2}, n\}$, $\boldsymbol{v}^{(k)}$ is the same as in Lemma 5, otherwise, the components $v_1^{(k)}, \cdots, v_n^{(k)}$ of $\boldsymbol{v}^{(k)}$ are described by

$$v_j^{(k)} = \begin{cases} \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \sin \dfrac{j\pi}{2}, & j \text{ is odd}, \\ 0, & j \text{ is even}, \end{cases} \tag{44}$$

when $n \equiv 0 \pmod 4$ and

$$v_j^{(k)} = \begin{cases} 0, & j \text{ is odd}, \\ \sigma_j (d_1 d_2)^{-\frac{j}{2}} \cos \dfrac{j\pi}{2}, & j \text{ is even}, \end{cases} \tag{45}$$

when $n \equiv 2 \pmod 4$.

Proof. The result can be obtained by solving the system of linear difference equations:

$$a_i v_i + c_{i+1} v_{i+2} = 0, \quad i \in \{1, \cdots, n-2\},$$

and the boundary conditions:

$$c_1 v_2 - \Delta v_n = 0,$$
$$\Gamma v_1 + a_{n-1} v_{n-1} = 0.$$

Similarly, we have the following result.

Lemma 8. Suppose in Lemma 6, ${d_1}^2 = {d_2}^2$. If $k \notin \{\frac{n}{2}, n\}$, then $\boldsymbol{v}^{(k)}$ is in Lemma 6, otherwise, the components $v_1^{(k)}, \cdots, v_n^{(k)}$ of $\boldsymbol{v}^{(k)}$ are described by

$$v_j^{(k)} = \begin{cases} 0, & j \text{ is odd}, \\ \sigma_j (d_1 d_2)^{-\frac{j}{2}} \cos \dfrac{j\pi}{2}, & j \text{ is even}, \end{cases} \tag{46}$$

when $n \equiv 0 \pmod 4$ and

$$v_j^{(k)} = \begin{cases} \sigma_j (d_1 d_2)^{-\frac{j-1}{2}} \sin \dfrac{j\pi}{2}, & j \text{ is odd}, \\ 0, & j \text{ is even}, \end{cases} \tag{47}$$

when $n \equiv 0 \pmod 4$.

The lemmas above suggest that the as $d_1^2 = d_2^2$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are both non-diagonalizable. This makes it possible to control the ring graph by only one input, even though there exists the symmetry in some algebraic sense, which leads to a repeated eigenvalue. In the sequel we present a sufficient condition for the controllability of the family of ring graphs satisfying Assumption 1.

Theorem 9. Suppose $n \equiv 0 \pmod 4$. The system $(\mathcal{L}, \boldsymbol{b})$ is controllable if

(1) $\mathcal{L} = \mathcal{L}_1$ and $\boldsymbol{b} \in \{\boldsymbol{e}_{\frac{n}{2}-1}, \boldsymbol{e}_{\frac{n}{2}+1}\} \setminus \{\boldsymbol{e}_1\}$;
(2) $\mathcal{L} = \mathcal{L}_2$ and $\boldsymbol{b} \in \{\boldsymbol{e}_{\frac{n}{2}}, \boldsymbol{e}_{\frac{n}{2}+2}\} \setminus \{\boldsymbol{e}_n\}$.

Proof. Let $\boldsymbol{v}$ be a left-eigenvector of $\mathcal{L}_1$, with components $v_1, \cdots, v_n$. From Lemma 5 and Lemma 7, $\boldsymbol{v}\boldsymbol{e}_j$, $j \in \{\frac{n}{2} - 1, \frac{n}{2} + 1\} \setminus \{1\}$, is nonzero. Thus, from Theorem 1, system $(\mathcal{L}_1, \boldsymbol{e}_j)$, $j \in \{\frac{n}{2} - 1, \frac{n}{2} + 1\} \setminus \{1\}$, is controllable. Other cases can be shown in a similar way and are skipped.

Example 10. Consider the ring graph in Fig 1, where $\gamma = -1$ and $\delta = 1$. The Laplacian matrix of the graph is

$$\mathcal{L}_1 = \begin{bmatrix} 5 & -4 & & & & & & 1 \\ -4 & 5 & -1 & & & & & \\ & -1 & 5 & 4 & & & & \\ & & 4 & 5 & -1 & & & \\ & & & -1 & 5 & -4 & & \\ & & & & -4 & 5 & 1 & \\ & & & & & 1 & 5 & -4 \\ -1 & & & & & & -4 & 5 \end{bmatrix}^T. \tag{48}$$

Eigenvalues of $\mathcal{L}_1$ are given by

$$\lambda_1 = 5 + \sqrt{17 + 4\sqrt{2}}, \qquad \lambda_2 = 5 + \sqrt{17}, \tag{49}$$
$$\lambda_3 = 5 + \sqrt{17 - 4\sqrt{2}}, \qquad \lambda_4 = 5 + \sqrt{15}, \tag{50}$$
$$\lambda_5 = 5 - \sqrt{17 - 4\sqrt{2}}, \qquad \lambda_6 = 5 - \sqrt{17}, \tag{51}$$
$$\lambda_7 = 5 - \sqrt{17 + 4\sqrt{2}}, \qquad \lambda_8 = 5 - \sqrt{15}. \tag{52}$$

The eigenvectors corresponding to λ_k, $k \notin \{4, 8\}$, are

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ \sqrt{17+4\sqrt{2}} \\ 4+\sqrt{2} \\ -\sqrt{34+8\sqrt{2}} \\ 4\sqrt{2}+1 \\ \sqrt{17+4\sqrt{2}} \\ -4 \end{bmatrix}^T, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} 4 \\ -\sqrt{17} \\ 1 \\ 0 \\ 4 \\ -\sqrt{17} \\ -1 \\ 0 \end{bmatrix}^T, \tag{53}$$

$$\boldsymbol{v}^{(3)} = \begin{bmatrix} 0 \\ -1 \\ \sqrt{17-4\sqrt{2}} \\ 4-\sqrt{2} \\ \sqrt{34-8\sqrt{2}} \\ -4\sqrt{2}+1 \\ \sqrt{17-4\sqrt{2}} \\ -4 \end{bmatrix}^T, \quad \boldsymbol{v}^{(5)} = \begin{bmatrix} 0 \\ -1 \\ -\sqrt{17-4\sqrt{2}} \\ 4-\sqrt{2} \\ -\sqrt{34-8\sqrt{2}} \\ -4\sqrt{2}+1 \\ -\sqrt{17-4\sqrt{2}} \\ -4 \end{bmatrix}^T, \tag{54}$$

$$\boldsymbol{v}^{(6)} = \begin{bmatrix} 4 \\ \sqrt{17} \\ 1 \\ 0 \\ 4 \\ \sqrt{17} \\ -1 \\ 0 \end{bmatrix}^T, \quad \boldsymbol{v}^{(7)} = \begin{bmatrix} 0 \\ -1 \\ -\sqrt{17+4\sqrt{2}} \\ 4+\sqrt{2} \\ \sqrt{34+8\sqrt{2}} \\ 4\sqrt{2}+1 \\ -\sqrt{17+4\sqrt{2}} \\ -4 \end{bmatrix}^T, \tag{55}$$

respectively. Obviously, the 3rd and 5th entries of eigenvectors $\boldsymbol{v}^{(k)}$, $k \notin \{4, 8\}$, are nonzero. To show that the 5th entry of $\boldsymbol{v}^{(4)}$ is nonzero, we consider the equation

$$\underbrace{\begin{bmatrix} 4 & \sqrt{15} & 1 & & & & \\ & 1 & \sqrt{15} & -4 & & & \\ & & -4 & \sqrt{15} & & & \\ & & & 1 & 4 & & \\ & & & & \sqrt{15} & -1 & \\ & & & & -1 & \sqrt{15} & 4 \\ 1 & & & & & 4 & \sqrt{15} \end{bmatrix}}_{M} \begin{bmatrix} v_1^{(4)} \\ v_2^{(4)} \\ v_3^{(4)} \\ v_4^{(4)} \\ v_6^{(4)} \\ v_7^{(4)} \\ v_8^{(4)} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ -1 \\ -\sqrt{15} \\ -4 \\ \\ \end{bmatrix} v_5^{(4)}.$$

Note that $\det(M) = 4^3\sqrt{15} \neq 0$. If $v_5^{(4)} = 0$, then $\boldsymbol{v}^{(4)}$ is a zero vector, a contradiction. Similarly, we can show that the 3rd entry of $\boldsymbol{v}^{(4)}$, the 3rd, and 5th entries of $\boldsymbol{v}^{(8)}$ are all nonzero. We conclude that this graph is controllable by either u_1 or u_2.

4. CONCLUSION

We have considered the Laplacian controllability of a class of non-simple ring graphs in this paper. It has been shown that a ring graph is in general Laplacian controllable by two controllers connected to two neighboring nodes respectively. Under the condition of periodic constant product, we have shown that one controller might be enough to render the ring graph controllable. This is made possible by the condition that results in a non-diagonalizable Laplacian matrix when it has a repeated eigenvalue. Our main results are based on the detailed exploration on the Laplacian eigenspace of the graph. Apparently, deriving the closed form of eigenpairs directly from the Laplacian matrices is challenging. A potential research topic of interest is on the possibility of relaxing our condition (for example, generalizing the period of constant product from 2 to any positive integer) while maintaining the tractability of eigenspace analysis. It is also interesting to know if our result is applicable to a wider class, such as the non-simple regular graphs.

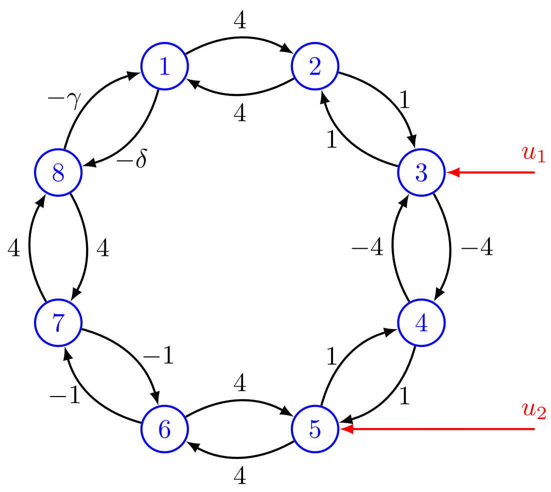

Fig. 1. Let $\gamma = -1$, $\sigma = 1$. The ring graph is single-input Laplacian controllable, by either u_1 or u_2.

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