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Iterative Learning Control for Switched Systems in the Presence of Input Saturation ⋆

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Abstract: The paper considers iterative learning control for differential and discrete switched linear systems with control input saturation. A new design is developed based on the use of common vector Lyapunov functions. An example demonstrating the features and advantages of the new design is given.

Keywords: iterative learning control, switched systems, input saturation, repetitive processes, stability, vector Lyapunov function

1. INTRODUCTION

Iterative learning control (ILC) has been developed for applications where the mode of operation is to complete the same finite duration task over and over again. A particular example is a robot operating in 'pick and place' mode, i.e., collect a payload from a location, transport it over a finite duration, deposit it and then repeat these steps as many times as required. In the literature, each execution of the task is known as a pass (or trial) and its finite duration is known as the pass (or trial) length.

The notation for variables in this paper is, for differential dynamics, of the form $y_k(t), 0 \leq t \leq T < \infty, k \geq 0$, where y is the vector or scalar-valued variable under consideration, T is the pass length and k is the pass number. This paper considers the single-input single-output case and given a supplied reference trajectory the error on each pass can be formed and the basis of ILC design is to enforce convergence of the sequence generated in k together with regulating the dynamics along the passes. Once a pass is completed all information generated over the pass length is available for use in updating the control input for the next pass. Effective use of this information is at the core of ILC design, with the overall aim of improving performance from pass-to-pass.

The survey papers (Bristow et al., 2006; Ahn et al., 2007) are possible starting points for the literature. Design based on linear models of the dynamics have been developed for differential and discrete systems by a variety of methods and a number have been followed through to experimental validation for engineering systems, see, e.g., (Lim et al., 2017; Sammons et al., 2019) and also in healthcare, see, e.g., (Freeman et al., 2012; Ketelhut et al., 2019). The requirements of some physical systems require particular consideration of problems such as actuator saturation.

Previous research on actuator saturation in ILC includes (Xu et al., 2004; Lješnjanin et al., 2017). In (Sebastian et al., 2019), a feedback-based ILC design was developed for a class of linear-time-invariant system in the presence of input constraints. The control schemes in these last two references consist of two parts: one (feedback) deals with the performance in time domain and the other (ILC) ensures acceptable pass-to-pass tracking performance. Also in (Wei et al., 2017) a method has been developed to decompose the output feedback ILC problem into two more tractable subproblems, namely an output feedback ILC problem for a linear time-invariant system and a state-feedback stabilization problem for a nonlinear system with input saturation.

In this paper, differential and discrete linear systems are considered. A new method of ILC law design in the presence of input saturation and mode switching in the pass direction is developed, which is based on the method of vector Lyapunov functions for repetitive processes (Pakshin et al., 2016). An example demonstrating the effectiveness and advantages of the new design is also given together with a brief comparison with previously reported laws.

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In this paper, the notation $\succ 0$, respectively, $\prec 0$, denotes a symmetric positive definite, respectively, negative definite matrix. Moreover, $\succeq 0$, respectively, $\preceq 0$, denotes a symmetric positive semi-definite, respectively, negative semi-definite, matrix. Also the identity and zero matrices of compatible dimensions, respectively, are denoted by I and 0.

2. ILC DESIGN IN THE DIFFERENTIAL REPETITIVE PROCESS SETTING

Consider a linear differential linear system operating repetitive mode described by the state-space model

$$\begin{aligned} \dot{x}_k(t) &= A(k)x_k(t) + B(k)\psi_k(t),\ (A(k), B(k)) \in \mathcal{F}, \\ y_k(t) &= Cx_k(t),\ t \in [0, T],\ \ k = 0, 1, \dots, \end{aligned} \quad (1)$$

where on pass k, $x_k(t) \in \mathbb{R}^{n_x}$ is the state vector, $u_k(t) \in \mathbb{R}^{n_u}$ and $y_k(t) \in \mathbb{R}^{n_y}$ are input and output vectors, ψ represents the input saturation and $\mathcal{F} = \{(A_1, B_1), (A_2, B_2), \dots, (A_N, B_N)\}$, is a set of matrix pairs of compatible dimensions. Following the notation in (Liberzon, 2003), consider a piecewise constant mapping from the nonnegative integer numbers, $\mathbb{Z}^+ \to \mathcal{F}$. For each such mapping, there is a corresponding piecewise constant function σ from $\mathbb{Z}^+ \to \mathcal{N} = \{1, ..., N\}$ such that $A(k) = A_{\sigma(k)}$ and $B(k) = B_{\sigma(k)}$ for $k = 0, 1, 2 \dots$.

The mapping σ is the switching signal in the pass-to-pass (k) direction. Assume that mode switching occurs at the beginning of the corresponding pass and define the switching instances $k_1, k_2, \dots$ as the pass numbers on which (1) changes mode. This switching signal specifies, on each pass k, the index $\sigma(k) \in \mathcal{N}$ of the active subsystem, i.e. if $\sigma(k) = i$ then the subsystem (mode) i is active at the instant k and and model (1) can also be represented as a family of subsystems with given switching rule between them:

$$\begin{aligned} \dot{x}_k(t) &= A_i x_k(t) + B_i\psi_k(t),\ i \in \mathcal{N}, \\ y_k(t) &= Cx_k(t),\ t \in [0, T],\ \ k = 0, 1, \dots. \end{aligned} \quad (2)$$

In applications, it can be the case that the input computed for the next pass exceeds the safe operating range of the actuators used and it is necessary to limit the input to within safe limits. Also in operation, the actuator may start to degrade and hence the actual control input is not applied. This paper considers the case of a saturating actuator described by

$$\begin{aligned} \psi_k(t)_i &= \mathrm{sat}(u_k(t))_i \\ &= \begin{cases} U_i & \text{if } u_{k,i}(t) > U_i, \\ u_{k,i}(t) & \text{if } -U_i \le u_{k,i}(t) \le U_i, \\ -U_i & \text{if } u_{k,i} < -U_i, \end{cases}, \\ & \qquad i = 1, 2, \dots, n_u,\ k = 0, 1, \dots, \end{aligned} \quad (3)$$

where U_i are specified positive constants. Also let $y_{ref}(t)$, $0 \le t \le T$, denote the supplied reference trajectory. Then $e_k(t) = y_{ref}(t) - y_k(t)$ is the error on pass k and the design problem is to construct a sequence of pass inputs $\{u_k\}$ such that

$$\begin{aligned} ||e_{k+1}(t)|| \le ||e_k(t)||,\ \lim_{k\to\infty} ||e_k(t)|| = ||e_\infty(t)||, \\ \lim_{k\to\infty} ||u_k(t) - u_\infty(t)|| = 0, \end{aligned} \quad (4)$$

where $||\cdot||$ denotes the norm on the underlying function space $e_\infty(t)$ and $u_\infty(t)$ are bounded variables and $u_\infty(t)$ is termed the learned control. Ideally, the design should enforce $e_\infty(t) = 0$ uniformly in t but, since (3) must hold on each pass, this requirement may not always be achievable.

In ILC design, the input for next pass is most often formed as that used on the previous pass plus a correction, i.e.,

$$u_{k+1}(t) = u_k(t) + \Delta u_{k+1}(t), \quad (5)$$

where $\Delta u_{k+1}(t)$ is the correction term to be designed. The design of the last term in this control law can make use of the complete previous pass error. In particular, at time instance t on pass $k+1$ information from the complete previous pass error is available instead of just the same time instant. This feature is sometimes termed 'non-causal' and if such action is not used then the ILC law can be replaced by a standard feedback control loop and no added benefit arises.

To write model of the system in the form of a differential repetitive process, assume that each subsystem from the family (2) is active for at least two consecutive passes and introduce the auxiliary vectors $\eta_{k+1}(t) = \int_0^t (x_{k+1}(\tau) - x_k(\tau))d\tau$. Assume also that $y_{ref}(t)$ is differentiable, $\dot{y}_k(t)$ is available to the controller and $CB_i \neq 0$. Then the dynamics of the considered system can be written in terms of $\eta_{k+1}(t)$ and $e_k(t)$ as

$$\begin{aligned} \dot{\eta}_{k+1}(t) &= A_i\eta_{k+1}(t) + B_i\phi_{k+1}(t), \\ e_{k+1}(t) &= -CA_i\eta_{k+1}(t) + e_k(t) - CB_i\phi_{k+1}(t),\ i \in \mathcal{N}, \end{aligned} \quad (6)$$

where

$$\phi_{k+1}(t) = \int_0^t [\mathrm{sat}(u_{k+1}(\tau)) - \mathrm{sat}(u_k(\tau))]d\tau.$$

Given (6), the stability theory for differential repetitive processes can now be applied to ILC design. In this paper, the theory developed in (Pakshin et al., 2016) is used.

Choose

$$\Delta u_{k+1}(t) = K_1 \frac{d\eta_{k+1}(t)}{dt} + K_2 \frac{de_k(t)}{dt} \quad (7)$$

and rewrite (6) and (7) in the following equivalent form

$$\begin{aligned} \dot{\eta}_{k+1}(t) &= (A_i + BK_1)\eta_{k+1}(t) + B_iK_2e_k(t) + B_i\varphi_k(t), \\ e_{k+1}(t) &= -C(A_i + B_iK_1)\eta_{k+1}(t) \\ &\quad + (I + B_iK_2)e_k(t) - CB_i\varphi_k(t), \end{aligned} \quad (8)$$

where

$$\varphi_k(t) = \phi_{k+1}(t) - \int_0^t \Delta u_{k+1}(\tau)d\tau, \quad (9)$$

and $\varphi_k(t) = 0$ if $\Delta u_{k+1}(t) = 0$.

Consider a vector Lyapunov function of the form

$$V(\eta_k(t), e_k(t)) = \begin{bmatrix} V_1(\eta_k(t)) \\ V_2(e_k(t)) \end{bmatrix}, \quad (10)$$

where $V_1(\eta) > 0,\ \eta \neq 0,\ \ V_2(e) > 0,\ e \neq 0,\ \ V_1(0) = 0,\ V_2(0) = 0$, with associated divergence operator

$$\mathcal{D}_cV(\eta_k(t), e_k(t)) = \frac{dV_1(\eta_k(t))}{dt} + \Delta_kV_2(e_k(t)), \quad (11)$$

where $\Delta_kV_2(e_k(t)) = V_2(e_{k+1}(t)) - V_2(e_k(t))$.

The following result gives sufficient conditions for convergence in the sense that the conditions of (4) hold.

Theorem 1. If there exists a vector Lyapunov function (10) and positive scalars $c_1, \dots, c_4$ and γ such that

$$c_1||\eta_k(t)||^2 \le V_1(\eta_k(t)) \le c_2||\eta_k(t)||^2, \quad (12)$$
$$c_1||e_k(t)||^2 \le V_2(e_k(t)) \le c_2||e_k(t)||^2, \quad (13)$$
$$\mathcal{D}_cV(\eta_k(t), e_k(t)) \le \gamma - c_3(||\eta_k(t)||^2 + ||e_k(t)||^2), \quad (14)$$
$$\frac{\partial V_1(\eta)}{\partial \eta} \le c_4||\eta||. \quad (15)$$

Then, for dynamics described by (8), $||e_k(t)||$ is monotonically decreasing as $k \to \infty$ uniformly in t and the conditions of (4) hold.

The proof follows directly from results in (Pakshin et al., 2016). A function (10) satisfying (12) – (15) is a common vector Lyapunov function for the family (2) and hence could give conservative convergence conditions for some examples.

Introduce the notation

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ -CA_i & I \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} B_i \\ -CB_i \end{bmatrix},$$
$$\bar{A}_{ci} = \bar{A}_i + \bar{B}_iK, \ I^{(1,0)} = \mathrm{diag}[I_{n_x} \ 0_{n_y}],$$
$$I^{(0,1)} = \mathrm{diag}[0_{n_x} \ I_{n_y}], \ \zeta_k(t) = [\eta_{k+1}^T(t) \ e_k(t)]^T, \quad (16)$$

then

$$\Delta u_{k+1}(t) = K\zeta_k(t),$$

where $K = [K_1 \ \ K_2]$ is a matrix to be designed. By construction, the entries of $\varphi_k(t)$, see (9), satisfy the constraints

$$-2TU_i - (K\zeta_k)_i \le (\varphi_k(t))_i \le 2TU_i - (K\zeta_k)_i, \quad i = 1, 2, \ldots, n_u. \quad (17)$$

Also it follows from (17) that $\varphi_k(t)$ satisfies the following quadratic constraints

$$-\zeta_k^T(t)K^TGK\zeta_k(t) - 2\zeta_k^T(t)K^TG\varphi_k(t) - \varphi_k^T(t)G\varphi_k(t) + 4T^2U^TGU \ge 0, \quad (18)$$

where G is diagonal matrix with positive entries and $U = \mathrm{diag}[U_1 \ldots U_{n_u}]^T$.

Theorem 2. Suppose that there exist matrices $W = \mathrm{diag}[W_1 \ \ W_2] \succ 0$, $S = \mathrm{diag}[S_1 \ldots S_{n_u}] \succ 0$, where $W_1 \in \mathbb{R}^{n_x \times n_x}$, $W_2 \in \mathbb{R}^{n_y \times n_y}$ and a sufficiently small positive scalar ϵ satisfying the linear matrix inequalities (LMIs)

$$\begin{bmatrix} -I^{(0,1)}W + W\bar{A}_{ci}^TI^{(1,0)} + I^{(1,0)}\bar{A}_{ci}W + \epsilon I & -WK^T & W\bar{A}_{ci}^TI^{(0,1)} \\ S\bar{B}_i^TI^{(1,0)} - KW & -S & S\bar{B}_i^TI^{(0,1)} \\ I^{(0,1)}\bar{A}_{ci} & I^{(0,1)}\bar{B}_iS & -WI^{(0,1)} \end{bmatrix} \preceq 0, \ i \in \mathcal{N}, \quad (19)$$

then $||e_k(t)||$ is monotonically decreasing as $k \to \infty$ uniformly in t and the conditions of (4) hold.

Proof. Choose the entries of (10) as the quadratic forms

$$V_1(\eta_k(t)) = \eta_k^{\mathrm{T}}(t)P_1\eta_k(t), \ V_2(e_k(t)) = e_k^T(t)P_2e_k(t),$$

where $P_1 \succ 0$ and $P_2 \succ 0$ and set $P = \mathrm{diag}[P_1 \ \ P_2]$. Computing divergence of (10) with these entries along the trajectories of (6) gives

$$\mathcal{D}_cV(\eta, e) = [(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi]^TPI^{(1,0)}\zeta + \zeta^TPI^{(1,0)}[(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi] + [(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi]^TPI^{(0,1)}[(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi] - \zeta^TPI^{(0,1)}\zeta. \quad (20)$$

Moreover, since $V_1(\eta)$ and $V_2(e)$ are positive definite quadratic forms (12),(13) and (15) of Theorem 1 hold.

A sufficient condition for (14) to hold under the constraints (18) can (Tarbouriech et al., 2011) be written as

$$\mathcal{D}_cV(\eta, e) - \zeta^TK^TGK\zeta - 2\zeta^TK^TG\varphi - \varphi^TG\varphi + 4T^2U^TGU \le \gamma - \zeta^T(\epsilon PP)\zeta, \quad (21)$$

where $\epsilon > 0$ is small enough. Choose $\gamma = 4T^2U^TGU$ and then condition (14) of Theorem 1 holds if

$$[(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi]^TPI^{(1,0)}\zeta + \zeta^TPI^{(1,0)}[(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi] + [(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi]^TPI^{(0,1)}[(\bar{A}_i + \bar{B}_iK)\zeta + \bar{B}_i\varphi] - \zeta^TPI^{(0,1)}\zeta + \zeta^T(\epsilon PP)\zeta - 2\zeta^TK^TG\varphi - \varphi^TG\varphi \preceq 0,$$

or

$$[\zeta^T \ \varphi^T]M(i)[\zeta^T \ \varphi^T]^T, \preceq 0,$$

where $M(i) = \begin{bmatrix} M_{11}(i) & M_{12}(i) \\ M_{12}^T(i) & M_{22}(i) \end{bmatrix}$,

$$M_{11}(i) = \bar{A}_{ci}^TPI^{(1,0)} + PI^{(1,0)}\bar{A}_{ci} + \bar{A}_{ci}^TPI^{(0,1)}\bar{A} - PI^{(0,1)} + \epsilon PP,$$
$$M_{12}(i) = \bar{B}_i^TI^{(1,0)}P + \bar{A}_{ci}^TPI^{(0,1)}\bar{B}_i - K^TG,$$
$$M_{22}(i) = \bar{B}_i^TPI^{(0,1)}\bar{B}_i - G.$$

The matrix $M(i)$ can be rewritten as

$$M(i) = \begin{bmatrix} -PI^{(0,1)} + \bar{A}_{ci}^TPI^{(1,0)} + PI^{(1,0)}\bar{A}_{ci} + \epsilon PP & PI^{(1,0)}\bar{B}_i - K^TG \\ \bar{B}_i^TI^{(1,0)}P - GK & -G \end{bmatrix} + \begin{bmatrix} \bar{A}_{ci}^T \\ \bar{B}_i^T \end{bmatrix} I^{(0,1)}PI^{(0,1)}[\bar{A}_{ci} \ \bar{B}_i]$$

and by the Schur's complement formula $M(i) \preceq 0$ if and only if

$$\begin{bmatrix} -PI^{(0,1)} + \bar{A}_{ci}^TI^{(1,0)}P + PI^{(1,0)}\bar{A}_{ci} + \epsilon PP & PI^{(1,0)}\bar{B}_i - K^TG & \bar{A}_{ci}I^{(0,1)} \\ \bar{B}_i^TI^{(1,0)}P - GK & -G & \bar{B}_i^TI^{(0,1)} \\ I^{(0,1)}\bar{A}_{ci} & I^{(0,1)}\bar{B}_i & -P^{-1} \end{bmatrix} \preceq 0. \quad (22)$$

Pre and post multiplying (22) by $\mathrm{diag}[P^{-1} \ G^{-1} \ I]$ and setting $P^{-1} = W$, $G^{-1} = S$ gives that (19) holds. All conditions of Theorem 1 therefore hold and the proof is complete.

To find gain matrix K it is possible to consider (8) without saturation and use a linear design. If the gain obtained in this way satisfies LMI's (19) then by Theorem 2 the corresponding ILC law with saturation guarantees that $||e_k(t)||$ is monotonically decreasing as $k \to \infty$ uniformly in t and the conditions of (4) hold.

Introduce the notation

$$A_{11}(i) = A(i),\ A_{12} = 0,$$
$$A_{21}(i) = -CA(i), A_{22} = I,\ B_1(i) = B_i,\ B_2(i) = -CB_i.$$

Then the next theorem gives a design method for the ILC law with input saturation.

Theorem 3. Let $K_1 = Y_1X_1^{-1}$ and $K_2 = Y_2X_2^{-1}$, where X_1, X_2, Y_1 and Y_2 solve the LMIs

$$\left[\begin{array}{ccccccc} \begin{array}{c}(A_{11}(i)X_1 + B_1(i)Y_1)\\ +(A_{11}(i)X_1 + B_1(i)Y_1)^T\end{array} & (A_{12}X_2 + B_1(i)Y_2) & 0 & (A_{21}(i)X_1 + B_2(i)Y_1)^T & X_1 & 0 & Y_1^T \\ (A_{12}X_2 + B_1(i)Y_2)^T & -X_2 & 0 & (A_{22}X_2 + B_2(i)Y_2)^T & 0 & X_2 & Y_2^T \\ 0 & 0 & -X_1 & 0 & 0 & 0 & 0 \\ (A_{21}(i)X_1 + B_2(i)Y_1) & (A_{22}X_2 + B_2(i)Y_2) & 0 & -X_2 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & -Q_1^{-1} & 0 & 0 \\ 0 & X_2(i) & 0 & 0 & 0 & -Q_2^{-1} & 0 \\ Y_1 & Y_2(i) & 0 & 0 & 0 & 0 & -R^{-1} \end{array}\right] \preceq 0,$$

$$X_1 \succ 0,\ X_2 \succ 0.\ i \in \mathcal{N}. \quad (23)$$

and suppose that the LMIs (19) with $K = [K_1\ K_2]$ are feasible. Then for (8) without saturation ($\varphi_k(t) = 0$) $||e_k(t)||$ exponentially tends to zero as $k \to \infty$ and ILC law

$$\psi_{k+1}(t) = \text{sat}(\psi_k(t) + \Delta u_{k+1}(t)) \quad (24)$$

with update given by (7) ensures that $||e_k(t)||$ is monotonically decreasing as $k \to \infty$ uniformly in t and the conditions (4) hold.

Proof. Choose the entries of (10) as the quadratic forms

$$V_1(\eta_k(t)) = \eta_k^{\mathrm{T}}(t)P_1\eta_k(t),\ V_2(e_k(t)) = e_k^T(t)P_2e_k(t),$$

where $P_1 = X_1^{-1} \succ 0$ and $P_2 = X_2^{-1} \succ 0$, where X_1 and X_2 are solution to (23). Computing the divergence of (10), along the trajectories of (8) with $\varphi_k(t) = 0$, gives

$$\begin{aligned}\mathcal{D}_cV(\eta_{k+1}(t), e_k(t)) = \zeta_k^T(t)(\bar{A}_{ci}^T I^{(1,0)}P \\ + PI^{(1,0)}\bar{A}_{ci} + \bar{A}_{ci}^T I^{(0,1)}P\bar{A}_{ci} \\ - I^{(0,1)}P)\zeta_k(t),\ i \in \mathcal{N} \quad (25)\end{aligned}$$

and using (23)

$$\mathcal{D}_cV(\eta_{k+1}(t), e_k(t)) \le -\zeta_k^T(t)(Q + K^TRK)\zeta_k^T(t),$$

where $Q = \text{diag}[Q_1\ Q_2]$. By Theorem 2 from (Pakshin et al., 2016), it follows from this inequality that (7) ensures exponential stability of (8), hence $||e_k(t)||$ exponentially tends to zero as $k \to \infty$. Finally, since (19) holds, Theorem 2 gives that the ILC law (24) ensures that $||e_k(t)||$ is monotonically decreasing as $k \to \infty$ uniformly in t and the conditions of (4) hold.

3. ILC DESIGN IN THE DISCRETE REPETITIVE PROCESS SETTING

Consider a discrete switched system operating in repetitive mode described by the family of the state space models

$$x_k(p+1) = A_ix_k(p) + B_i\psi_k(p),\ y_k(p) = Cx_k(p), \quad (26)$$
$$p = 0, 1, \dots T-1,\ i \in \mathcal{N},\ k = 0, 1, \dots,$$

where on pass k, $x_k(p) \in \mathbb{R}^{n_x}$ is the state vector, $\psi_k(p)$ is the input and $y_k(p)$ is the output. In this case T denotes the number of samples along the pass length (T times the sampling period gives the pass length.) The problem statement is the same as in previous case, with t replaced by the sample number p. Also the notation $\bar{A}_i$, $\bar{A}_{ci}$ and $\bar{B}_i$ used below is defined as in (16) with the matrices involved replaced by those from (26).

Introduce $\eta_{k+1}(p+1) = x_{k+1}(p) - x_k(p)$. Then the counterpart of (6) for the controlled dynamics in this case is

$$\begin{aligned}\eta_{k+1}(p+1) &= A_i\eta_{k+1}(p) + B_i\phi_{k+1}(p), \\ e_{k+1}(p) &= -CA_i\eta_{k+1}(p) + e_k(p) \quad (27) \\ &\quad - CB_i\phi_{k+1}(p),\ i \in \mathcal{N},\ k = 0, 1, \dots\end{aligned}$$

where

$$\phi_{k+1}(p) = \text{sat}[u_{k+1}(p-1)] - \text{sat}[u_k(p-1)].$$

Choose

$$\Delta u_{k+1}(p-1) = K_1\eta_{k+1}(p) + K_2e_k(p), \quad (28)$$

and rewrite (27), (28) in the following equivalent form

$$\begin{aligned}\eta_{k+1}(p+1) &= (A_i + BK_1)\eta_{k+1}(p) \\ &\quad + B_iK_2e_k(p) + B_i\varphi_k(p), \\ e_{k+1}(p) &= -C(A_i + B_iK_1)\eta_{k+1}(p) \quad (29) \\ &\quad + (I + B_iK_2)e_k(p) - CB_i\varphi_k(p),\end{aligned}$$

where

$$\varphi_k(p) = \phi_{k+1}(p) - \Delta u_{k+1}(p-1), \quad (30)$$

and $\varphi_k(p) = 0$ if $\Delta u_{k+1}(p-1) = 0$. It is easy to show as in previous section that $\varphi_k(p)$ satisfies constraints

$$\begin{aligned}&-\zeta_k^T(p)K^TGK\zeta_k(p) - 2\zeta_k^T(p)K^TG\varphi_k(p) \\ &\quad - \varphi_k^T(p)G\varphi_k(p) + 4U^TGU \ge 0. \quad (31)\end{aligned}$$

Define the operator $\mathcal{D}_d$ as follows

$$\begin{aligned}\mathcal{D}_dV(\eta_{k+1}(p), e_k(p)) = V_1(\eta_k(p+1)) \\ - V_1(\eta_k(p)) + V_2(e_{k+1}(p)) - V_2(e_k(p)). \quad (32)\end{aligned}$$

Sufficient conditions for convergence such that the conditions of (4) hold for this case are given by the following counterpart of Theorem 1.

Theorem 4. If there exists a vector Lyapunov function (10) (with t replaced by p) and positive scalars $c_1 - c_4$ and γ such that

$$c_1||\eta_k(p)||^2 \le V_1(\eta_k(p)) \le c_2||\eta_k(p)||^2, \quad (33)$$
$$c_1||e_k(p)||^2 \le V_2(e_k(p)) \le c_2||e_k(p)||^2, \quad (34)$$
$$\mathcal{D}_dV(\eta_k(p), e_k(p)) \le \gamma - c_3(||\eta_k(p)||^2 + ||e_k(p)||^2). (35)$$

then $||e_k(p)||$ is monotonically decreasing as $k \to \infty$ for all $p = 0, 1, \dots T-1$ and the conditions (4) hold.

The proof, as in the differential dynamics case, is a direct consequence of the results by (Pakshin et al., 2016). In this case, (10) (with t replaced by p) satisfying (33)–(35) is a common vector Lyapunov function for the system (29) and can give conservative stability conditions.

Assume that X is a solution of the LMIs

$$\begin{bmatrix} X & (\bar{A}_iX+\bar{B}_iY)^T & X & Y^T \\ (\bar{A}_iX+\bar{B}_iY) & X & 0 & 0 \\ X & 0 & Q^{-1} & 0 \\ Y & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0,$$
$$X = \mathrm{diag}[X_1\ X_2] \succ 0,\ i \in \mathcal{N}, \quad (36)$$

where $Q \succ 0$ and $R \succ 0$ are weighting matrices. Then the next theorem gives a design method for the ILC law with input saturation.

Theorem 5. Let $K = YX^{-1}$, where X is solution to (36) and suppose that there exist matrices $W = \mathrm{diag}[W_1\ W_2] \succ 0$, $S = \mathrm{diag}[S_1 \dots S_{n_u}] \succ 0$, where $W_1 \in \mathbb{R}^{n_x \times n_x}$, $W_2 \in \mathbb{R}^{n_y \times n_y}$ and a sufficiently small positive scalar ϵ such that

$$\begin{bmatrix} -W+\epsilon I & -WK^T & W\bar{A}_{ci}^T \\ -KW & -S & S\bar{B}_i^T \\ \bar{A}_{ci}W & \bar{B}_iS & -W \end{bmatrix} \preceq 0,\ i \in \mathcal{N}. \quad (37)$$

Then for (29) without saturation $||e_k(t)||$ exponentially tends to zero as $k \to \infty$ and ILC law

$$\psi_{k+1}(p) = \mathrm{sat}(\psi_k(p) + \Delta u_{k+1}(p)) \quad (38)$$

with update given by (28) ensures that $||e_k(p)||$ is monotonically decreasing as $k \to \infty$ for all $p = 0, 1, \dots T-1$ and the conditions of (4) hold.

Proof. If the inequalities (36) are solvable, then a vector Lyapunov function of the form (10) with entries $\eta^T(p)X_1^{-1}\eta(p)$ and $e^T(p)X_2^{-1}e(p)$ satisfies the conditions of Theorem 1 in (Pakshin et al., 2016) and guarantees exponential stability of (29) in the absence of saturation.

Choose the entries of (10) as the quadratic forms

$$V_1(\eta_{k+1}(p)) = \eta_{k+1}^{\mathrm{T}}(p)P_1\eta_{k+1}(p),$$
$$V_2(e_k(p)) = e_k^T(p)P_2e_k(p),$$

where $P_1 \succ 0$ and $P_2 \succ 0$ and set $P = \mathrm{diag}[P_1\ P_2]$. Computing divergence of the function (10) with these entries along the trajectories of (29) gives

$$\mathcal{D}_dV(\eta,e) = [\bar{A}_{ci}\zeta + \bar{B}_i\varphi]^TP[\bar{A}_{ci}\zeta + \bar{B}\varphi] - \zeta^TP\zeta. \quad (39)$$

Since $V_1(\eta)$ and $V_2(e)$ are positive definite quadratic forms (33) and (34) of Theorem 4 hold. Also (Tarbouriech et al., 2011) a sufficient condition for (35) to hold under the constraints (31) can be written as

$$\mathcal{D}_dV(\eta,e) - \zeta^T(K\bar{C})^TGK\bar{C}\zeta$$
$$- 2\zeta^T(K\bar{C})^TG\varphi - \varphi^TG\varphi + 4U^TGU$$
$$\le \gamma - \epsilon\zeta^TPP\zeta, \quad (40)$$

Choose $\gamma = 4U^TGU$ and then (35)from Theorem 4 holds if

$$[\bar{A}_{ci}\zeta + \bar{B}_i\varphi]^TP[\bar{A}_{ci}\zeta + \bar{B}_i\varphi] - \zeta^TP\zeta$$
$$+ \epsilon\zeta^TPP\zeta - 2\zeta^TK^TG\varphi - \varphi^T(t)G\varphi \preceq 0. \quad (41)$$

The proof is completed by following, with routine changes, identical steps to those in the proof of Theorem 2.

4. EXAMPLE

As an example consider a discrete linear systems example where

$$A_1 = \begin{bmatrix} 1 & 0.0308 & 0.0099 & 0.0001 \\ 0 & 0.9522 & 0.0001 & 0.0098 \\ 0 & 6.0908 & 0.9812 & 0.0308 \\ 0 & -9.4572 & 0.0187 & 0.9522 \end{bmatrix}$$

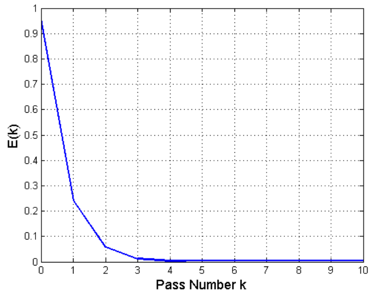

Fig. 1. Progression of $E(k)$ with plant switching on the passes 4 and 7 passes without saturation

$$A_2 = \begin{bmatrix} 1 & 0.0308 & 0.0099 & 0.0001 \\ 0 & 0.9580 & 0.0001 & 0.0099 \\ 0 & 6.1029 & 0.9812 & 0.0308 \\ 0 & -8.3128 & 0.0188 & 0.9580 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.0238 \\ -0.0237 \\ 4.7124 \\ -4.6852 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.0237 \\ 4.7124 \\ -4.6946 \\ -4.6946 \end{bmatrix}$$

It is assumed that all the state variables are available for measurement and the reference trajectory is of 3 sec duration and is formed by sampling a continuous time signal at and given by

$$y_{ref}(p) = \frac{10^{-4}\pi}{6}p^2 - \frac{10^{-6}\pi}{27}p^3,\ p = 0, 1, \dots, 300 \quad (42)$$

The linear ILC law without saturation is

$$u_{k+1}(p) = u_k(p) + K_1(x_{k+1}(p) - x_k(p))$$
$$+ K_2e_k(p+1), \quad (43)$$

where the control law matrices K_1 and K_2, together with P_i, are given by the solution of the minimization problem $-(\mathrm{tr}X_1 + \mathrm{tr}X_2)$ under the LMI constraints (36). Solving this problem for

$$Q = \mathrm{diag}[10^{-3}\ 10^{-3}\ 10^{-3}\ 10^{-3}\ 10^{3}], \quad R = 0.01,$$

gives

$$K_1 = [\,-31.9444\ 0.2018\ -0.5174\ -0.149\,],$$
$$K_2 = 23.8705. \quad (44)$$

To measure the performance of this ILC law, introduce

$$E(k) = \sqrt{\frac{1}{T}\sum_{p=0}^{T}||e_k(p)||^2} \quad (45)$$

i.e., the mean squared value of the error for each pass over the pass length, which is plotted against the pass number in Fig. 1 for the case of switching from mode 1 to mode 2 on the 4th pass and back to mode 1 on the 7th pass without saturation. In this case monotonic pass-to-pass error convergence is achieved, where after 4 passes the learning error is zero. Fig. 2 shows the control progression for this case.

For K_1 and K_2 given by (44), the LMI's (37) hold and hence the ILC law (43) and (44) and (as a numerical example) saturation level $U = 0.0075$ results in monotonic

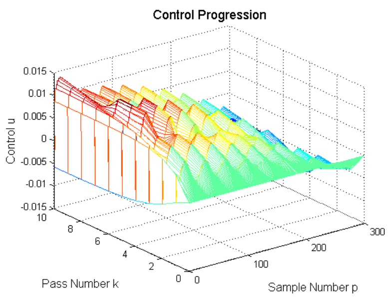

Fig. 2. Control progression with plant switching on the passes 4 and 7 without saturation

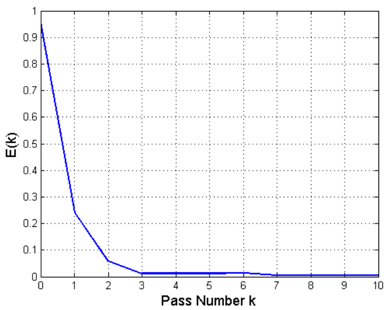

Fig. 3. Progression of $E(k)$ with plant switching on the passes 4 and 7 with saturation level $U = 0.0075$

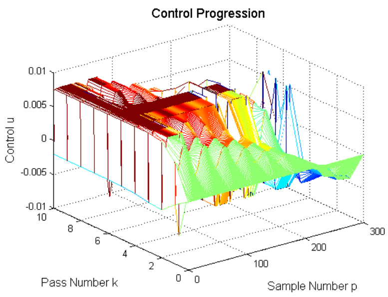

Fig. 4. Control progression with plant switching on the passes 4 and 7 with saturation level $U = 0.0075$

pass-to-pass error convergence. This is confirmed by Fig. 3, which plots the function (45) for the case of switching from mode 1 to mode 2 on the 4th pass and back on the 7th pass with saturation level $U = 0.0075$. Fig. 4 shows the corresponding control progression.

This example demonstrates that, with the same switching mode, the presence of saturation reduces the convergence rate by almost half compared to the case without saturation. In addition, with a further decrease in saturation level the tolerance begins to decrease significantly and the choice of this level is application specific.

5. CONCLUSIONS AND FUTURE RESEARCH

This paper has used the stability theory of repetitive processes to develop a new ILC law for differential and discrete dynamics in the presence of saturating action on the control input. The saturation operator is applied to the control input on the current pass, in contrast to other approaches, see, e.g., (Xu et al., 2004; Lješnjanin et al., 2017; Sebastian et al., 2019), where the control law is of the form

$$v_k(t) = \text{sat}(v_{k-1}(t)) + f(e_k(t), e_{k-1}(t)),$$
$$u_k(t) = \text{sat}(v_k(t)).$$

There is a need to further investigate the relative merits of the new design against alternatives, were the laws developed in this paper apply saturation directly to the current pass input only. Future work could also be directed to obtaining less conservative results by, e.g., aiming to develop a multiple vector Lyapunov function approach.

REFERENCES

Ahn, H.S., Chen, Y.Q., and Moore, K.L. (2007). Iterative learning control: Brief survey and categorization. *IEEE Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews*, 37(6), 1099–1121.

Bristow, D., Tharayil, M., and Alleyne, A. (2006). A survey of iterative learning control. *IEEE Control Systems Magazine*, 26(3), 96–114.

Freeman, C.T., Rogers, E., Hughes, A.M., Burridge, J.H., and Meadmore, K.L. (2012). Iterativellearning control in health care: electrical stimulation and robotic-assisted upper-limb stroke rehabilitation. *IEEE Control Systems Magazine*, 47, 70–80.

Ketelhut, M., Stemmler, S., Gesenhues, J., Hein, M., and Abel, D. (2019). Iterative learning control of ventricular assist devices with variable cycle durations. *Control Engineering Practice*, 83, 33–44.

Liberzon, D. (2003). *Switching in Systems and Control*. Birkhäuser,, Boston, MA.

Lim, I., Hoelzle, D.J., and Barton, K.L. (2017). A multi-objective iterative learning control approach for additive manufacturing applications. *Control Engineering Practice*, 64, 74–87.

Lješnjanin, M., Tan, Y., Oetomo, D., and Freeman, C.T. (2017). Spatial iterative learning control: Systems with input saturation. In *2017 American Control Conference*, 5121–5126. Seattle, USA.

Pakshin, P., Emelianova, J., Emelianov, M., Galkowski, K., and Rogers, E. (2016). Dissipativity and stabilization of nonlinear repetitive processes. *Systems & Control Letters*, 91, 14–20.

Sammons, P., Gegel, M., Bristow, D.A., and Landers, R.G. (2019). Repetitive process control of additive manufacturing with application to laser metal deposition. *IEEE Transactions on Control Systems Technology*, 27(2), 566–575.

Sebastian, G., Tan, Y., and Oetomo, D. (2019). Convergence analysis of feedback-based iterative learning control with input saturation. *Automatica*, 101, 44–52.

Tarbouriech, S., Garcia, G., da Silva Jr., J.G., and Queinnec, I. (2011). *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer-Verlag, London.

Wei, Z.B., Quan, Q., and Cai, K.Y. (2017). Output feedback ILC for a class of nonminimum phase nonlinear systems with input saturation: An additive-state-decomposition-based method. *IEEE Trans. Autom. Control*, 62(1), 502–508.

Xu, J.X., Tan, Y., and Lee, T.H. (2004). Iterative learning control design based on composite energy function with input saturation. *Automatica*, 40, 1371–1377.