Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

The Zero Dynamics of Nonlinear Stochastic Systems: Stabilisation and Output Tracking in the Ideal Case

Alberto Mellone * **Giordano Scarciotti** *

* *Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK (e-mail: {a.mellone18,g.scarciotti}@imperial.ac.uk).*

Abstract: This paper introduces the notion of zero dynamics and presents results of local stabilisation and output tracking for single-input single-output nonlinear stochastic systems described by stochastic differential equations. For this class of systems we define the zero dynamics when the stochastic relative degree is strictly smaller than the order of the system. We show that, under suitable conditions on the zero dynamics, the equilibrium at the origin can be stabilised via a coordinate change and a nonlinear state feedback. In an analogous way, we show that it is possible to achieve local asymptotic output tracking of a reference signal. We validate the theory through a numerical example.

1 INTRODUCTION

Nonlinear feedback control of dynamical systems is a vast research field and major results in the past decades have contributed to its advance. In this context, particular relevance is given to systems that can be transformed into normal forms. The normal form of a nonlinear control system consists in a representation of the dynamics in a coordinate frame in which the differential equations are simpler to handle for analysis and control purposes. This facilitates the design of nonlinear control laws for stabilisation, tracking and estimation, see *e.g.* Isidori (1995). In Isidori et al. (1981) the normal form for deterministic systems was firstly introduced as a way to address the problem of static state-feedback non-interacting control, whereas further results can be found, *e.g.*, in Zeitz (1983), Bestle and Zeitz (1983) and Krener (1987). The problem of linearisation via feedback was proposed and solved for single-input single-output systems in Brockett (1978) and for multi-input multi-output systems in Jakubczyk and Respondek (1980). The concept of zero dynamics was introduced in Byrnes and Isidori (1984), and it was used to address the problem of stabilisation in Byrnes and Isidori (1988).

This paper is intended to extend the analysis of the zero dynamics to a broad class of nonlinear stochastic systems, specifically systems described by stochastic differential equations. The use of stochastic differential equations to model uncertain systems allows embedding model uncertainties in the dynamics in the form of noisy coefficients, see Øksendal (2003). Applications of stochastic systems theory are broad and vary from optimal stopping to production planning, finance, technology diffusion and research funding Øksendal (2003), Yong and Zhou (1999).

Although some notions of normal form for stochastic systems are present in the literature, they were not introduced for the purpose of controlling the systems. For example, a normal form was proposed in Arnold and Imkeller (1998) and, subsequently, in Arnold (2003). Therein a theory of normal forms is introduced by employing Stratonovich calculus on systems described by purely diffusive terms. The normal form is obtained via a coordinate transformation which requires anticipating the noise over a short time scale. Coordinate transformations for stochastic systems were also dealt with in, *e.g.*, Gaeta and Rodríguez Quintero (1999), where the authors introduce symmetries for stochastic differential equations, and Roberts (2008), where fast and slow dynamics can be separated using a normal form.

In this paper we employ a notion of normal form with the purpose of controlling a general class of nonlinear stochastic single-input single-output systems. The notion of normal form that we adopt in this work was firstly introduced in Mellone and Scarciotti (2019a). Therein the authors first define the concept of relative degree for stochastic systems, which parallels the deterministic version presented, *e.g.* in Isidori (1995), and then use this concept to derive a coordinate transformation that brings the system to a normal form which is particularly meaningful for control design. Under the assumption of full relative degree the authors also propose a linearising state feedback in the unrealistic hypothesis that the noise affecting the system is measurable.

Extending the results of Mellone and Scarciotti (2019a), in this paper we investigate the case of systems with relative degree smaller than their order. In particular, we define the concept of zero dynamics for nonlinear stochastic systems. Moreover, we present results concerning the local stabilisation and output tracking of nonlinear systems which extensively rely on the concept of the zero dynamics. We hereby point out that the control laws that we introduce depend explicitly on the white noise process affecting the system. Clearly, this hypothesis is not practically reasonable as white noise cannot be physically measured. However, this treatise is intended as a fundamental preliminary step for the solution of a practically implementable version of these results. In fact, following the ideas presented, *e.g.*,

in Mellone and Scarciotti (2019b) to solve practically the problem of output regulation for linear stochastic systems, we propose in the accompanying paper Mellone and Scarciotti (2020) an approximate, yet practical, control architecture which does not depend on measurements of the noise.
The rest of the paper is organised as follows. In Section 2 we recall some preliminary notions related to stochastic systems. In Section 3 we introduce the concept of zero dynamics and present the main results of the section, namely the local stabilisation of an equilibrium point and the local asymptotic tracking of reference inputs. In Section 4 we show a numerical example that illustrates the theoretical results of Section 3. Finally, Section 5 contains some concluding remarks.

Notation. The symbol $\mathbb{Z}$ denotes the set of integer numbers, while $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively; by adding the subscript "< 0" ("≥ 0", "0") to any symbol indicating a set of numbers, we denote that subset of numbers with negative (non-negative, zero) real part. The symbol ∂_x^n is used as a shorthand for the operator $\partial^n/\partial x^n$, while $\alpha^{(n)}$ indicates the n-th time derivative of α. $(\nabla, \mathcal{F}, \mathcal{P})$ is a probability space given by the set ∇, the σ-algebra $\mathcal{F}$ defined on ∇ and the probability measure $\mathcal{P}$ on the measurable space $(\nabla, \mathcal{F})$. A *stochastic process* with state space $\mathbb{R}^n$ is a family $\{x_t,\, t \in \mathbb{R}\}$ of $\mathbb{R}^n$-valued random variables, *i.e.* for every fixed $t \in \mathbb{R}$, $x_t(\cdot)$ is an $\mathbb{R}^n$-valued random variable and, for every fixed $w \in \nabla$, $x_\cdot(w)$ is an $\mathbb{R}^n$-valued function of time (Arnold, 1974, Section 1.8). For ease of notation, we often indicate a stochastic process $\{x_t,\, t \in \mathbb{R}\}$ simply with x_t (this is common in the literature, see *e.g.* Arnold (1974)). With a slight abuse of notation, any subscript different from the symbol "t" indicates the corresponding component of the vector x_t, *e.g.* x_i is the i-th component of the vector x_t. Let $C_0^\infty(\mathbb{R})$ denote the space of all infinitely differentiable functions on $\mathbb{R}$ with compact support. A generalised stochastic process is a random generalised function in the sense that a random variable $\psi(\varphi)$ is assigned to every $\varphi \in C_0^\infty$, where ψ is, with probability 1, a generalised function (Arnold, 1974, Section 3.2). The symbol $\mathcal{W}_t$ indicates a standard Wiener process, also referred to as Brownian motion, whereas $\xi_t = \dot{\mathcal{W}}_t$ indicates the generalised white noise obtained by differentiating $\mathcal{W}_t$. $\mathcal{W}_t$ and ξ_t are defined on the probability space $(\nabla, \mathcal{F}, \mathcal{P})$.

2 PRELIMINARIES

In this section we recall some preliminary notions related to stochastic differential equations and the concept of stochastic relative degree. Consider the nonlinear single-input, single-output stochastic system expressed in the shorthand integral notation

$$\begin{aligned} dx_t &= (f(x_t) + g(x_t)u)dt + (l(x_t) + m(x_t)u)d\mathcal{W}_t, \\ y_t &= h(x_t), \end{aligned} \tag{1}$$

with $x_t \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y_t \in \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $l : \mathbb{R}^n \to \mathbb{R}^n$, $m : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ *smooth* functions, *i.e.* they admit continuous partial derivatives of any order. We assume that, for a fixed initial condition $x_{t=0}$, the solution of (1) is unique. For the reasons reported in (Arnold, 1974, Section 10.3), we can rewrite equation (1) in the differential notation

$$\dot{x}_t = f(x_t) + g(x_t)u + (l(x_t) + m(x_t)u)\xi_t, \quad y_t = h(x_t), \tag{2}$$

as long as ξ_t is understood as a *generalised white noise*, (Arnold, 1974, Section 10.3). Given the equivalence of the two representations in the framework of generalised stochastic processes, in the remainder of the paper equations (1) and (2) are used interchangeably, as convenient, to refer to the same underlying nonlinear stochastic system. Recall that the derivative of h along the vector field f, which is called Lie derivative and is indicated with the symbol $\mathcal{L}_f\, h$, is defined as

$$\mathcal{L}_f\, h(x) = \partial_x[h]\ f(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x).$$

We indicate the derivative of h first along the vector field f and then along the vector field g as $\mathcal{L}_g\, \mathcal{L}_f\, h(x) = \partial_x[\mathcal{L}_f\, h]\ g(x)$. We use the recursive relation $\mathcal{L}_f^k\, h(x) = \partial_x[\mathcal{L}_f^{k-1}\, h]\ f(x)$, with $\mathcal{L}_f^0\, h(x) = h(x)$, to indicate the k-th differentiation of h along f. We now recall three operators, firstly introduced in (Mellone and Scarciotti, 2019a, Section III). The first one, which indicates the second derivative of h along the vector fields f and g, is defined as

$${}^g\mathcal{G}_f h(x) = g(x)^\top \partial_x^2[h]\ f(x) = \sum_{j=1}^{n} g_j(x) \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_j \partial x_i} f_i(x).$$

Similarly to the Lie derivative, we use the notation ${}^m\mathcal{G}_l\, {}^g\mathcal{G}_f h(x) = m(x)^\top \partial_x^2[{}^g\mathcal{G}_f h]\ l(x)$, and ${}^g\mathcal{G}_f^k h(x) = g(x)^\top \partial_x^2[{}^g\mathcal{G}_f^{k-1} h]\ f(x)$, to indicate the reiterated operations. The second operator is ${}^l\mathcal{S}_f h$, which is employed to define the *stochastic* Lie derivative of h along the drift vector field f and the diffusion vector field l, namely

$${}^l\mathcal{S}_f h(\xi_t, x) = \mathcal{L}_f\, h(x) + \mathcal{L}_l\, h(x)\xi_t + \frac{1}{2}\, {}^l\mathcal{G}_l h(x).$$

If ${}^l\mathcal{S}_f h(\xi_t, x) = {}^l\mathcal{S}_f h(x)$ is a deterministic expression, *i.e.* the white noise does not appear explicitly or equivalently $\mathcal{L}_l\, h \equiv 0$, then, similarly to the deterministic Lie derivative, we use the notation ${}^l\mathcal{S}_f^2 h(\xi_t, x) = {}^l\mathcal{S}_f\, {}^l\mathcal{S}_f h(\xi_t, x)$. Iteratively, if ${}^l\mathcal{S}_f^{k-1} h(\xi_t, x) = {}^l\mathcal{S}_f^{k-1} h(x)$ is deterministic, ${}^l\mathcal{S}_f^k h(\xi_t, x) = {}^l\mathcal{S}_f\, {}^l\mathcal{S}_f^{k-1} h(\xi_t, x)$, with ${}^l\mathcal{S}_f^0 h(x) = h(x)$ by definition. Finally, we define a third stochastic differential operator

$${}^m_g\mathcal{A}_l h(\xi_t, x) = \mathcal{L}_g\, h(x) + \mathcal{L}_m\, h(x)\xi_t + {}^m\mathcal{G}_l h(x).$$

By using Itô's formula, it is easy to see that the first derivative of the output of system (2) is given by

$$y_t^{(1)} = {}^l\mathcal{S}_f h(\xi_t, x_t) + {}^m_g\mathcal{A}_l h(\xi_t, x_t)u + \frac{1}{2}\, {}^m\mathcal{G}_m h(x_t)u^2.$$

Using the previous definitions, we recall the concept of stochastic relative degree.

Definition 1. (*Stochastic Relative Degree*, Mellone and Scarciotti (2019a)) Assume that there exists $\bar{r}$ such that

$$\mathcal{L}_l\, {}^l\mathcal{S}_f^k h(\xi_t, x) = 0, \tag{3}$$

for all k in $\{0, 1, ..., \bar{r} - 2\}$ and for all x in a neighbourhood of $\bar{x}$. System (2) is said to have *stochastic relative degree* r at a point $\bar{x}$ if $\bar{r} = r$ and

(a) all the following conditions are satisfied

$$\begin{aligned} 0 &= \mathcal{L}_g\, {}^l\mathcal{S}_f^k h(\xi_t, x) + {}^m\mathcal{G}_l\, {}^l\mathcal{S}_f^k h(\xi_t, x), \\ 0 &= \mathcal{L}_m\, {}^l\mathcal{S}_f^k h(\xi_t, x), \\ 0 &= {}^m\mathcal{G}_m\, {}^l\mathcal{S}_f^k h(\xi_t, x), \end{aligned}$$

for all x in a neighborhood of $\bar{x}$ and all $k \in \{0, 1, ..., r-2\}$.

(b) one of the following conditions is satisfied

$$0 \neq \mathcal{L}_g \, {}^l\mathcal{S}_f^{r-1} h(\xi_t, \bar{x}) + {}^m\mathcal{G}_l \, {}^l\mathcal{S}_f^{r-1} h(\xi_t, \bar{x}),$$
$$0 \neq \mathcal{L}_m \, {}^l\mathcal{S}_f^{r-1} h(\xi_t, \bar{x}),$$
$$0 \neq {}^m\mathcal{G}_m \, {}^l\mathcal{S}_f^{r-1} h(\xi_t, \bar{x}).$$

Assumption 1. Let r be the stochastic relative degree of system (2) at $\bar{x}$. Assume that the row vectors $\partial_x[h(\bar{x})]$, $\partial_x[\,{}^l\mathcal{S}_f h(\bar{x})]$, ..., $\partial_x[\,{}^l\mathcal{S}_f^{r-1} h(\bar{x})]$ are linearly independent.

Suppose that the relative degree of system (2) at $\bar{x} = 0$ is $r \leq n$. Since Assumption 1 and the standing assumption (3) hold [1], then there exist functions $\phi_i(x)$, $i = r+1, ..., n$ such that Φ, given by

$$\Phi(x) = \Big[h(x) \;\; {}^l\mathcal{S}_f h(x) \; \ldots \; {}^l\mathcal{S}_f^{r-1} h(x) \;\; \phi_{r+1}(x) \; \ldots \; \phi_n(x)\Big]^\top,$$

is a local diffeomorphism in a neighbourhood U of $\bar{x}$ such that the dynamics of system (2), written in the new coordinates $z_t = \Phi(x_t)$, is expressed by (Mellone and Scarciotti (2019a))

$$\begin{aligned} \dot{z}_i &= z_{i+1}, & i = 1, ..., r-1,\\ \dot{z}_r &= c(\xi_t, z_t) + b(\xi_t, z_t)u + a(z_t)u^2, &\\ \dot{z}_j &= p_j(\xi_t, z_t) + q_j(\xi_t, z_t)u + s_j(z_t)u^2, & j = r+1, ..., n,\\ y_t &= z_1, & \end{aligned} \tag{4}$$

with $c(\xi_t, z_t) = {}^l\mathcal{S}_f^r h(\xi_t, \Phi^{-1}(z_t))$, $b(\xi_t, z_t) = {}^m_g\mathcal{A}_l \, {}^l\mathcal{S}_f^{r-1} h(\xi_t, \Phi^{-1}(z_t))$, $a(z_t) = \frac{1}{2} \, {}^m\mathcal{G}_m \, {}^l\mathcal{S}_f^{r-1} h(\Phi^{-1}(z_t))$, $p_j(\xi_t, z_t) = {}^l\mathcal{S}_f \phi_j(\xi_t, \Phi^{-1}(z_t))$, $q_j(\xi_t, z_t) = {}^m_g\mathcal{A}_l \phi_j(\xi_t, \Phi^{-1}(z_t))$, $s_j(z_t) = \frac{1}{2} \, {}^m\mathcal{G}_m \phi_j(\Phi^{-1}(z_t))$. In particular, observe that the dependence of the coefficients c, b, p_j and q_j on the white noise ξ_t is affine, *i.e.* they can be decomposed as

$$\begin{aligned} c(\xi_t, z_t) &= c_d(z_t) + c_s(z_t)\xi_t,\\ b(\xi_t, z_t) &= b_d(z_t) + b_s(z_t)\xi_t,\\ p_j(\xi_t, z_t) &= p_{d,j}(z_t) + p_{s,j}(z_t)\xi_t, \quad j = r+1, ..., n,\\ q_j(\xi_t, z_t) &= q_{d,j}(z_t) + q_{s,j}(z_t)\xi_t, \quad j = r+1, ..., n, \end{aligned}$$

where all the quantities are uniquely defined as a consequence of the previous observation. For compactness, we define $p = [p_{r+1} \ldots p_n]^\top$, $q = [q_{r+1} \ldots q_n]^\top$ and $s = [s_{r+1} \ldots s_n]^\top$. When designing the control input of system (4) we need to distinguish between two cases.

(A) $a(z_t) \equiv 0$ in a neighbourhood U_1 of $\bar{x}$. Then, by definition of relative degree, there exists a neighbourhood U_2 of $\bar{x}$ such that $b(\xi_t, z_t) \neq 0$ in $\Phi(U_2)$. Let $U = U_1 \cap U_2$.

(B) There exists a neighbourhood U of $\bar{x}$ such that $a(z_t) \neq 0$ in $\Phi(U)$.

Note that, by the definition of relative degree, no other cases are possible. If a system is such that case **(A)** is satisfied, then the control input does not appear quadratically in the r-th derivative of the output for x_t in U, whereas if **(B)** is satisfied, then the square of the input does appear in the r-th derivative of the output. The distinction between these two cases will be used in the remainder to define two separate control strategies, depending on which case is satisfied for the system under consideration.

[1] Condition (3) assures that ${}^l\mathcal{S}_f^k h(\xi_t, x) = {}^l\mathcal{S}_f^k h(x)$ is deterministic for all k in $\{0, 1, ..., r-1\}$. This implies that the noise does not appear in the first $r-1$ equations of the normal form.

3 THE ZERO DYNAMICS OF A STOCHASTIC SYSTEM

Suppose that the stochastic relative degree r of system (2) is strictly less than n at $\bar{x}$. For compactness, set $\zeta_t = [z_1 \ldots z_r]^\top$ and $\eta_t = [z_{r+1} \ldots z_n]^\top$. From the theory of normal forms of deterministic systems, recall that necessarily $\zeta_t = 0$ at $\bar{x}$, whereas it is straightforward to observe that the value of η_t at $\bar{x}$ can be arbitrarily chosen for stochastic systems as well. Therefore let $\Phi(x_t)$ be such that $\eta_t = 0$ as well at $\bar{x}$, which makes zero an equilibrium of system (4). We hereby extend the definition of zero dynamics to nonlinear stochastic systems, *i.e.* the internal dynamics of the system when the input and the initial conditions constrain the output to be identically zero.

Definition 2. (*Zero Dynamics*) The stochastic differential equation

$$\dot{\eta}_t = p(\xi_t, 0, \eta_t) + q(\xi_t, 0, \eta_t)u_{z,t} + s(0, \eta_t)u_{z,t}^2,$$

with η_0 in a neighbourhood of zero and $u_{z,t}$, if it exists, given by either **(A)**

$$u_{z,t} = -\frac{c(\xi_t, 0, \eta_t)}{b(\xi_t, 0, \eta_t)},$$

or **(B)**

$$u_{z,t} = \frac{-b(\xi_t, 0, \eta_t) \pm \sqrt{b(\xi_t, 0, \eta_t)^2 - 4a(0, \eta_t)c(\xi_t, 0, \eta_t)}}{2a(0, \eta_t)},$$

as long as $b(\xi_t, 0, \eta_t)^2 - 4a(0, \eta_t)c(\xi_t, 0, \eta_t) \geq 0$, is called the *zero dynamics* of system (2).

Remark 1. By selecting u as in Definition 2, we are enlarging the class of systems (2) to allow control laws which depend explicitly on ξ_t. In general, this yields a closed-loop system that is not equivalent to (1). To avoid this, we hereby define a control law to be *admissible* if it preserves the equivalence of the closed-loop systems (1) and (2), *i.e.* the dependence of the dynamics in (2) is affine in ξ_t. To address the most general case, we refrain from making additional assumptions which would be sufficient, although restrictive, to guarantee the admissibility of u and we leave this condition to be checked *a posteriori*. See Section 4.1 for an example.

Remark 2. Since $u_{z,t}$ is uniquely dependent on the state η_t and the noise ξ_t, the zero dynamics is autonomous and can be denoted by

$$\dot{\eta}_t = \hat{p}(\xi_t, 0, \eta_t) = p(\xi_t, 0, \eta_t) + q(\xi_t, 0, \eta_t)u_{z,t} + s(0, \eta_t)u_{z,t}^2.$$

Remark 3. It might be possible to choose $\phi_{r+1}, ..., \phi_n$ such that $q \equiv 0$ and $s \equiv 0$ in a neighbourhood of zero, which reduces the zero dynamics to

$$\dot{\eta}_t = \hat{p}(\xi_t, 0, \eta_t) = p(\xi_t, 0, \eta_t).$$

While for nonlinear deterministic systems it is always possible to find functions ϕ_j such that the dynamics of η_t does not explicitly depend on the control input (Isidori, 1995, Proposition 4.1.3), for stochastic systems this property is currently under investigation.

3.1 Local asymptotic stabilisation

In this section we show that under the assumption that the zero dynamics is almost surely asymptotically stable it is possible to stabilise the equilibrium at the origin via nonlinear feedback. Consider the transformed system (4). Let $v(\zeta_t) = -(d_0 z_1 + d_1 z_2 + ... + d_{r-1} z_r)$ and the feedback control law be given by $u_t = u_t^{stab}$, where u_t^{stab} is either **(A)**

$$u_t^{stab} = -\frac{c(\xi_t, \zeta_t, \eta_t) - v(\zeta_t)}{b(\xi_t, \zeta_t, \eta_t)},$$

or **(B)**

$$u_t^{stab} = (2a(\zeta_t, \eta_t))^{-1}\Big[-b(\xi_t, \zeta_t, \eta_t) \pm \sqrt{b(\xi_t, \zeta_t, \eta_t)^2 - 4a(\zeta_t, \eta_t)[c(\xi_t, \zeta_t, \eta_t) - v(\zeta_t)]}\Big],$$

where d_i, with $i = 0, ..., r-1$, are, if possible, such that $b^2 - 4a(c - v) \geq 0$.
Then the transformed system has the form

$$\dot{\zeta}_t = A\zeta_t, \quad \dot{\eta}_t = \hat{p}(\xi_t, \zeta_t, \eta_t), \tag{5}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -d_0 & -d_1 & -d_2 & \dots & -d_{r-1} \end{bmatrix}.$$

Therefore, the eigenvalues of the matrix A are the roots of the polynomial

$$\Lambda(s) = s^r + d_{r-1}s^{r-1} + ... + d_1 s + d_0 \tag{6}$$

and the following result holds.

Theorem 1. Suppose that the equilibrium at $\eta = 0$ of the zero dynamics of system (2) is locally asymptotically stable almost surely and the roots of the polynomial $\Lambda(s)$ in (6) have negative real part. Then the control law $u_t = u_t^{stab}$ renders the equilibrium $(\zeta, \eta) = (0, 0)$ locally asymptotically stable almost surely.

3.2 Local asymptotic output tracking

We now show that under further hypotheses on the zero dynamics, it is possible to control the system so that its output tracks reference trajectories while its internal variables remain bounded.

Assumption 2. The dependence of the zero dynamics of system (2) on the white noise is affine, *i.e.*

$$\dot{\eta}_t = \hat{p}(\xi_t, 0, \eta_t) = \hat{p}_d(0, \eta_t) + \hat{p}_s(0, \eta_t)\xi_t.$$

Note that Assumption 2 is trivially satisfied if it is possible to find ϕ_j such that $\hat{p}(\xi_t, 0, \eta_t) = p(\xi_t, 0, \eta_t)$.

Let y_R be a reference signal which is continuously differentiable r times with values in a neighbourhood of zero. We assume that the initial state of the system is arbitrary while in a neighbourhood of zero and we seek a feedback control u_t that makes the output y_t of the system asymptotically converge to y_R.
Consider the transformed system (4). Let

$$v(\zeta_t, y_R(t)) = y_R^{(r)} - \sum_{i=1}^{r} d_{i-1}(z_i - y_R^{(i-1)})$$

and the feedback control law be given by $u_t = u_t^{track}$, with u_t^{track}, if it exists, given by either **(A)**

$$u_t^{track} = -\frac{c(\xi_t, \zeta_t, \eta_t) - v(\zeta_t, y^R(t))}{b(\xi_t, \zeta_t, \eta_t)},$$

or **(B)**

$$u_t^{track} = (2a(\zeta_t, \eta_t))^{-1}\Big[-b(\xi_t, \zeta_t, \eta_t) \pm \sqrt{b(\xi_t, \zeta_t, \eta_t)^2 - 4a(\zeta_t, \eta_t)[c(\xi_t, \zeta_t, \eta_t) - v(\zeta_t, y_R(t))]}\Big],$$

where d_i, with $i = 0, ..., r-1$, are, if possible, such that $b^2 - 4a(c - v) \geq 0$.
Define the tracking error $e_t := y_t - y_R(t)$. Then it is easy to see that, under the control u_t^{track}, the tracking error has dynamics $e_t^{(r)} + d_{r-1}e_t^{(r-1)} + ... + d_1 e_t^{(1)} + d_0 e_t$, which can be forced exponentially to zero by suitably selecting the coefficients d_i. We are also interested in analysing the boundedness of the states z_i and of the internal variable η_t under the control u_t^{track}, when the reference output and its first $r-1$ time derivatives are bounded. Define $\zeta_R(t) = [y_R(t) \ \dots \ y_R^{(r-1)}(t)]^\top$ and $\theta_t = [e_t \ \dots \ e_t^{(r-1)}]^\top$. Then the following result holds.

Theorem 2. Consider system (2) and let Assumptions 1 and 2 hold. Suppose $y_R(t)$, $y_R^{(1)}(t), \dots, y_R^{(r-1)}(t)$ are bounded. Let $\eta_{R,t}$ be the solution of

$$\dot{\eta}_{R,t} = \hat{p}(\xi_t, \zeta_R(t), \eta_{R,t}), \quad \eta_{R,0} = 0 \tag{7}$$

and let $\hat{p}_d$ and $\hat{p}_s$ be Lipschitz continuous. Moreover, assume that there exists a strict Lyapunov function $V(x, t)$ for (7) such that $\frac{\partial V}{\partial x_i}(x, t)$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}(x, t)$ are bounded for all $x \in U$ and $t \geq 0$. Suppose that the roots of the polynomial $\Lambda(s)$ in (6) have negative real part. Then for sufficiently small $\epsilon_R > 0$, if

$$|z_i(\bar{t}) - y_R^{(i-1)}(\bar{t})| < \epsilon_R, \ 1 \leq i \leq r, \qquad \|\eta_{\bar{t}} - \eta_{R,\bar{t}}\| < \epsilon_R,$$

then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z_i(\bar{t}) - y_R^{(i-1)}(\bar{t})| < \delta \rightarrow |z_i(t) - y_R^{(i-1)}(t)| < \epsilon, \quad 1 \leq i \leq r, \text{ for all } t \geq \bar{t} \geq 0,$$

$$\|\eta_{\bar{t}} - \eta_{R,\bar{t}}\| < \delta \rightarrow \|\eta_t - \eta_{R,t}\| < \epsilon \ \text{ for all } t \geq \bar{t} \geq 0,$$

almost surely, *i.e.* the response z_i and η_t, $t \geq \bar{t} \geq 0$, of system (2) under the control law u_t^{track} is bounded almost surely.

The previous theorem solves the local asymptotic output tracking problem, *i.e.* the output $y_t = z_1$ asymptotically converges to y_R whilst the state z_t remains bounded almost surely.

Remark 4. The white noise ξ_t explicitly appears in the control laws u_t^{stab} and u_t^{track}, implying that in general its knowledge is needed in order to compute the control input. This is impossible in practice. However, the results presented in this paper lay the theoretical grounds to approach the stabilisation and tracking problem in a practically viable way. Specifically, in Mellone and Scarciotti (2020) we show how to obtain a causal estimation of the increments of the Brownian motion and, subsequently, how to use them to approximate the locally stabilising control law u_t^{stab}. The practical counterpart of the control law u_t^{track} will be the topic of forthcoming papers.

4 EXAMPLE

Consider the nonlinear stochastic system

$$\dot{x}_t = \begin{bmatrix} s_2(1 + x_1) \\ -2\tan x_2 \\ 2x_3 + x_1 s_2 - \frac{2x_1^2 s_2}{c_2^2} \end{bmatrix} + \begin{bmatrix} e^{x_3} \\ 0 \\ e^{x_3} \end{bmatrix} u + \begin{bmatrix} x_1 \\ -\frac{2x_1}{c_2} \\ -x_1 \end{bmatrix} \xi_t + \begin{bmatrix} x_1^2 \\ 0 \\ x_1^2 \end{bmatrix} u\xi_t, \tag{8}$$

with the output $y_t = x_1 + s_2 - x_3$, where $s_i = \sin(x_i)$ and $c_i = \cos x_i$. We study the system in a neighbourhood $U = (-\pi/2, \pi/2)^3$ of zero. We look for a change of coordinates Φ to bring the system into the normal form. By setting $z_1 = y_t$ simple computations yield $z_2 = \dot{z}_1 = -s_2 - 2x_3$. As neither the control input nor the white noise appear in the expression of z_2, the standing assumption (3) is satisfied and the relative degree of the system is larger than 1. Therefore, we compute the second derivative of the output, which results in

$$\begin{aligned}\dot{z}_2 &= 2s_2 - 4x_3 - 2x_1 s_2 + 6\frac{x_1^2 s_2}{c_2^2} + 4x_1\xi_t - 2e^{x_3}u - 2x_1^2 u\xi_t \\ &= \tilde{c}(\xi_t, x_t) + \tilde{b}(\xi_t, x_t)u,\end{aligned}$$

with $\tilde{c}(\xi_t, x_t) = c(\xi_t, \Phi(x_t))$ and $\tilde{b}(\xi_t, x_t) = b(\xi_t, \Phi(x_t))$. As the control input appears on the right-hand side and $\tilde{b}$ is non-zero at $\bar{x} = 0$, the relative degree of the system at the origin is 2. To complete the definition of the change of coordinates Φ, we set $z_3 = x_1 - x_3$, which in turn yields

$$\dot{z}_3 = s_2 - 2x_3 + \frac{2x_1^2 s_2}{c_2^2} + 2x_1\xi_t = \tilde{p}(\xi_t, x_t) = p(\xi_t, \Phi(x_t)). \tag{9}$$

We now want to study the stability of the zero dynamics. Define $\zeta_t = [z_1 \;\, z_2]^\top$ and $\eta_t = z_3$. We set $\zeta_t = 0$ and, using the definition of z_3, we obtain the system of equations

$$0 = x_1 + s_2 - x_3, \quad 0 = -s_2 - 2x_3, \quad z_3 = x_1 - x_3,$$

which yields

$$x_1 = \frac{3}{2}z_3, \qquad s_2 = -z_3, \qquad x_3 = -\frac{1}{2}z_3.$$

Substituting these values in (9), we get

$$\dot{\eta}_t = p(\xi_t, 0, \eta_t) = -2\eta_t + \frac{9\eta_t^3}{2(\eta_t^2 - 1)} + 3\eta_t\xi_t.$$

The linear approximation of this system is

$$\dot{\eta}_t = A_\eta\eta_t + F_\eta\eta_t\xi_t = -2\eta_t + 3\eta_t\xi_t,$$

which is asymptotically stable almost surely if $2A_\eta - F_\eta^2 < 0$, see, *e.g.*, Gard (1988). Since this condition is verified, we conclude that the zero dynamics is locally asymptotically stable almost surely.

4.1 Local asymptotic stabilisation

We first show an example of stabilisation of the equilibrium of the system at the origin. We set $v(\zeta_t) = -6z_1 - 5z_2$ and we apply the control law u_t^{stab} (case **(A)**) with $c(\xi_t, \zeta_t, \eta_t) = \tilde{c}(\xi_t, \Phi^{-1}(z_t))$ and $b(\xi_t, \zeta_t, \eta_t) = \tilde{b}(\xi_t, \Phi^{-1}(z_t))$. This control law is admissible because in (8) the expression $(g + m\xi_t)u_t^{stab} = [\frac{1}{2} \; 0 \; \frac{1}{2}]^\top(\tilde{c}(\xi_t, x_t) - \tilde{v}(x_t))$, with $\tilde{v}(x_t) = v(\Phi(x_t))$, is affine in ξ_t (and so it can be rewritten in the form (1)). Moreover, the control law brings the system to the form (5), where the matrix A has eigenvalues $\{-2, -3\}$. The assumptions of Theorem 1 are satisfied and therefore the closed-loop system has an almost surely asymptotically stable equilibrium at the origin.

Figure 1 shows the time histories of the state of the system for the case of local asymptotic stabilisation. In particular, Figure 1(a) shows the time history of the state in the coordinates x_t and Figure 1(b) shows the time history of the state in the coordinates z_t. While all the three components of x_t display a noisy behaviour, the change of coordinates Φ decouples the noise from the first two components of the vector z_t and projects this onto the third component z_3. This is in line with the standing assumption (3). Moreover, Figure 1(b) confirms that the state $\zeta_t = [z_1 \;\, z_2]^\top$ has linear, deterministic and asymptotically stable dynamics, whereas $\eta_t = z_3$ has nonlinear, stochastic yet asymptotically stable dynamics.

(a) Time history of the state in the coordinates x_t.

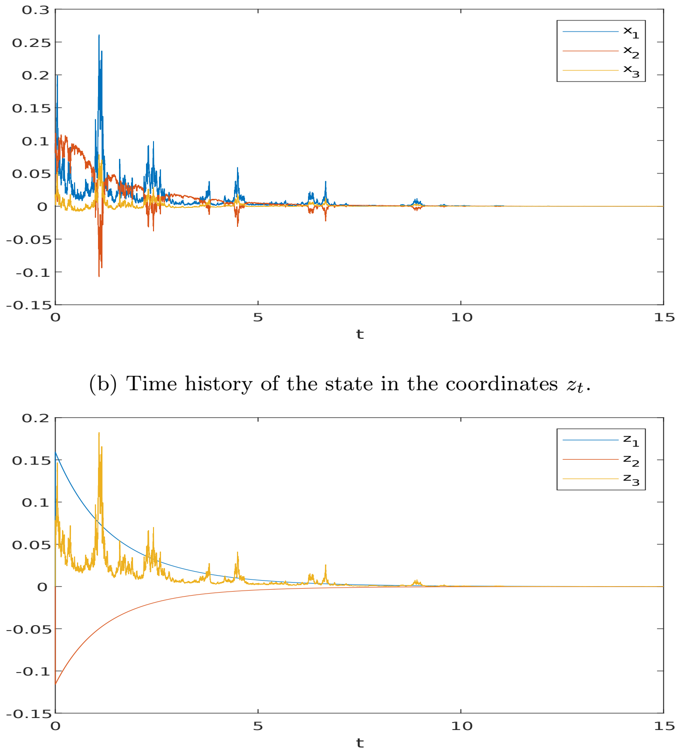


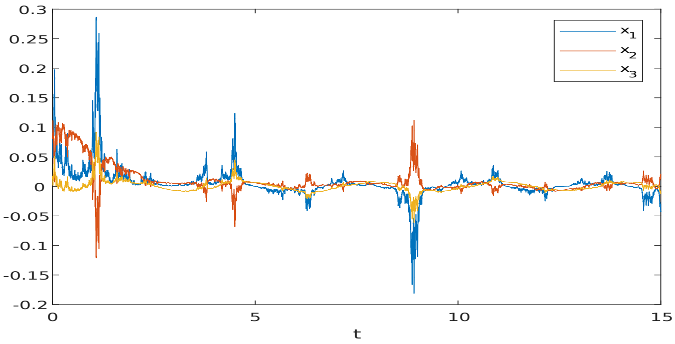
Fig. 1. Local asymptotic stabilisation of system (8).

4.2 Local asymptotic output tracking

We now consider an example of local asymptotic output tracking. We assume that the reference signal $y_R(t) = 0.01\sin(2t)$ has to be tracked by the output y_t. In this case we set $v(\zeta_t, y_R(t)) = -0.04\sin(2t) - 6(z_1 - 0.01\sin(2t)) - 5(z_2 - 0.02\cos(2t))$ and we apply the control law u_t^{track} (case **(A)**) with $c(\xi_t, \zeta_t, \eta_t) = \tilde{c}(\xi_t, \Phi^{-1}(z_t))$ and $b(\xi_t, \zeta_t, \eta_t) = \tilde{b}(\xi_t, \Phi^{-1}(z_t))$. This control law is admissible and asymptotically stabilises the tracking error e_t.

Figure 2 shows the time histories of the state of the system for the case of local asymptotic output tracking. In particular, Figure 2(a) shows the time history of the state in the coordinates x_t and Figure 2(b) shows the time history of the state in the coordinates z_t. Similarly to the case of stabilisation, the isomorphism Φ and the control u_t^{track} project the noise on the third component z_3 of the state in the coordinates z_t, while z_1 and z_2 display a linear deterministic behaviour. Moreover, the control is such that the output $y_t = z_1$ converges asymptotically to the reference y_R while the internal variable z_3 remains bounded almost surely.

(a) Time history of the state in the coordinates x_t.



(b) Time histories of the state in the coordinates z_t and of the reference output y_R.

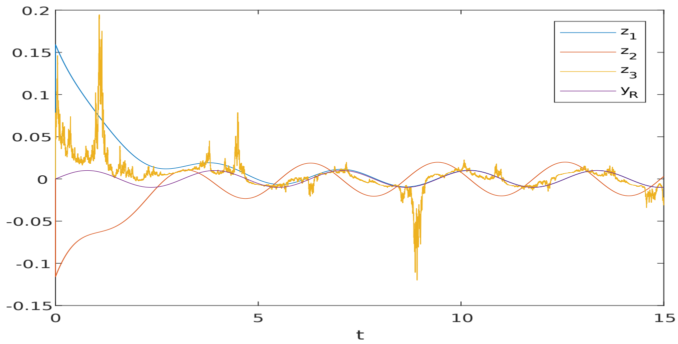


Fig. 2. Local asymptotic output tracking of system (8).

5 CONCLUSIONS

In this paper we have introduced the notion of zero dynamics of nonlinear stochastic systems described by a general class of stochastic differential equations and we have employed it to address the problems of local stabilisation and asymptotic output tracking. We have showed that, under the assumption of an almost surely asymptotically stable zero dynamics, a change of coordinates and a nonlinear state feedback solve the problems. Finally, we have demonstrated the validity of the theory with a numerical example.

In this work we have stressed that the nonlinear state feedback yielding stabilisation and tracking require the knowledge of the white noise affecting the system. Although this is impossible in practical circumstances, the treatise in this paper is a fundamental preliminary step towards a practical solution of the problems. This is the subject of the accompanying paper Mellone and Scarciotti (2020).

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