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Approximate Feedback Linearisation and Stabilisation of Nonlinear Stochastic Systems

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Abstract: This paper addresses the design of a practically sound control architecture to solve the problem of feedback linearisation and stabilisation of single-input single-output nonlinear stochastic systems. We first present a causal method to obtain, from measurements of the state, *a-posteriori* estimates of the variations of the Brownian motion which affected the system. Then we employ these estimates to design a control law that approximately compensates for the diffusive dynamics of the system. We address the local stabilisation problem and we prove that the control law which performs the proposed stochastic compensations stabilises a broader class of systems with respect to feedback laws without compensation. We finally validate the theory through a numerical example.

1 INTRODUCTION

Control of stochastic dynamical systems is a fundamental research field in control theory. Modelling in this framework allows the designer to use the mathematical tools offered by the theory of probability and stochastic processes to assess and enforce the dynamical properties of uncertain systems. In addition to providing an advantage from a theoretical point of view, applications of stochastic theory are broad and various. Some examples include the optimal stopping problem, the production planning problem, the study of technology diffusion and of distribution of research funding, see Øksendal (2003) and Yong and Zhou (1999).

In this paper we study a class of stochastic systems described by nonlinear stochastic differential equations. In particular, we develop a framework which extends standard results for deterministic systems to the stochastic case. To this end, we recall that a popular approach for the control of nonlinear deterministic systems employs the notion of relative degree and a coordinate transformation to represent the system in a somehow "simpler" form, which is called "normal form". This allows for feedback linearisation and for the introduction of the zero dynamics, concepts which are fundamental for making classical control problems, such as stabilisation, tracking and observer design, more easily tractable, see *e.g.* Isidori (1995). The notion of normal form was firstly introduced in Isidori et al. (1981) as part of the solution of the static state-feedback non-interacting control problem, see also Zeitz (1983), Bestle and Zeitz (1983) and Krener (1987) for additional results in this direction. Before then, the solution to the problem of linearisation via feedback had already been addressed and solved for single-input single-output systems by Brockett (1978). Byrnes and Isidori (1984) introduced the concept of zero dynamics and in the later work of Byrnes and Isidori (1988) this was used to solve the local stabilisation problem.

In the stochastic framework, a special form, also called "normal form", was proposed in, *e.g.*, Arnold (2003). Therein Stratonovich calculus is used to obtain a coordinate transformation that, by anticipating the noise over a short period, yields a special form for purely diffusive systems. Other examples of coordinate transformations for stochastic systems are, *e.g.*, Gaeta and Rodríguez Quintero (1999) where symmetries for differential equations are identified, and Roberts (2008) where a normal form is used to separate fast and slow dynamics. Note that these notions of normal form are not related to the concept of relative degree.

In Mellone and Scarciotti (2019a) a normal form was introduced to address the problem of "ideal" feedback linearisation for a general class of stochastic single-input single-output systems described by nonlinear differential equations. By making a parallel with the deterministic theory presented in Isidori (1995), Mellone and Scarciotti (2019a) first define the stochastic relative degree and then obtain a normal form via a coordinate transformation which allows for linearisation via state feedback in the ideal but unrealistic case in which the Brownian motion is assumed to be known. In the later work by Mellone and Scarciotti (2020b), the concept of zero dynamics of stochastic systems is introduced and the problems of stabilisation and tracking are addressed and solved in the same ideal but unrealistic scenario. Both Mellone and Scarciotti (2019a) and Mellone and Scarciotti (2020b) propose control laws that are explicitly dependent on the noise affecting the system, *i.e.* the value of the noise must be known at all times in order to synthesise the linearising, stabilising or regulating control law. Although this assumption is not sound from a practical point of view, these works constitute a necessary preliminary step towards the design of a control architecture which can be implemented in real scenarios.

In this paper we aim at overcoming this fundamental limitation and present a theoretical framework which allows solving the problem of feedback linearisation for stochastic systems in practice. Our approach is inspired by the work of Mellone and Scarciotti (2019b), where the problem of output regulation for linear stochastic systems is addressed

and solved. Therein, a procedure to causally approximate the Brownian motion is used to design a regulator which solves the problem in a practical way. In the present paper we first adapt that procedure to nonlinear systems. We show that at periodic sampling times it is possible to obtain estimates of the increments of the Brownian motion occurred since the previous sampling time. We then use these estimates to design a hybrid controller in which a deterministic continuous-time control law is supplemented with jump corrections performed at the sampling times. These corrections are computed employing the estimates of the Brownian motion and are designed to cancel, in an approximate way, the contribution of the noise to the dynamics of the feedback-linearised system. We show that, under suitable assumptions, performing the compensations is indeed beneficial in reducing the impact of noise onto the system. Moreover, compared to the mere deterministic continuous-time control law, the stochastic jump compensations allow for solving the local stabilisation problem for a broader class of systems.
The rest of the paper is organised as follows. In Section 2 we recall some preliminary notions related to stochastic systems. In Section 3 we introduce a procedure to partially estimate the Brownian motion and to use these estimates to compensate for the stochastic dynamics. In Section 4 we give results of practical local asymptotic stabilisation. In Section 5 we show a numerical example that illustrates the theoretical results. Finally, Section 6 contains some concluding remarks.

Notation. The symbol $\mathbb{Z}$ denotes the set of integer numbers, while $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively; by adding the subscript "< 0" ("≥ 0", "0") to any symbol indicating a set of numbers, we denote that subset of numbers with negative (non-negative, zero) real part. If a function g experiences a jump variation at time t, the symbols $g(t)$ and $g(t^+)$ denote the values of g immediately before and after the jump, respectively. The symbol ∂_x^n is used as a shorthand for the operator $\partial^n/\partial x^n$, while $\alpha^{(n)}$ indicates the n-th time derivative of α. $(\nabla, \mathcal{F}, \mathcal{P})$ is a probability space given by the set ∇, the σ-algebra $\mathcal{F}$ defined on ∇ and the probability measure $\mathcal{P}$ on the measurable space $(\nabla, \mathcal{F})$. A *stochastic process* with state space $\mathbb{R}^n$ is a family $\{x_t,\, t \in \mathbb{R}\}$ of $\mathbb{R}^n$-valued random variables, *i.e.* for every fixed $t \in \mathbb{R}$, $x_t(\cdot)$ is an $\mathbb{R}^n$-valued random variable and, for every fixed $w \in \nabla$, $x_.(w)$ is an $\mathbb{R}^n$-valued function of time (Arnold, 1974, Section 1.8). For ease of notation, we often indicate a stochastic process $\{x_t,\, t \in \mathbb{R}\}$ simply with x_t (this is common in the literature, see *e.g.* Arnold (1974)). With a slight abuse of notation, any subscript different from the symbol "t" indicates the corresponding component of the vector x_t, *e.g.* x_i is the i-th component of the vector x_t. Let $C_0^\infty(\mathbb{R})$ denote the space of all infinitely differentiable functions on $\mathbb{R}$ with compact support. A generalised stochastic process is a random generalised function in the sense that a random variable $\psi(\varphi)$ is assigned to every $\varphi \in C_0^\infty$, where ψ is, with probability 1, a generalised function (Arnold, 1974, Section 3.2). The symbol $\mathcal{W}_t$ indicates a standard Wiener process, also referred to as Brownian motion, whereas $\xi_t = \dot{\mathcal{W}}_t$ indicates the generalised white noise obtained by differentiating $\mathcal{W}_t$. $\mathcal{W}_t$ and ξ_t are defined on the probability space $(\nabla, \mathcal{F}, \mathcal{P})$.

2 PRELIMINARIES

In this section we recall some preliminary notions related to stochastic differential equations and the concept of stochastic relative degree. Consider the nonlinear single-input, single-output stochastic system expressed in the shorthand integral notation

$$\begin{aligned} dx_t &= (f(x_t) + g(x_t)u)dt + (l(x_t) + m(x_t)u)d\mathcal{W}_t, \\ y_t &= h(x_t), \end{aligned} \tag{1}$$

with $x_t \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y_t \in \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $l : \mathbb{R}^n \to \mathbb{R}^n$, $m : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ *smooth* functions, *i.e.* they admit continuous partial derivatives of any order. We assume that, for a fixed initial condition $x_{t=0}$, the solution of (1) is unique. For the reasons reported in (Arnold, 1974, Section 10.3), we can rewrite equation (1) in the differential notation

$$\dot{x}_t = f(x_t)+g(x_t)u+(l(x_t)+m(x_t)u)\xi_t, \quad y_t = h(x_t), \tag{2}$$

as long as ξ_t is understood as a *generalised white noise*, (Arnold, 1974, Section 10.3). Given the equivalence of the two representations in the framework of generalised stochastic processes, in the remainder of the paper equations (1) and (2) are used interchangeably, as convenient, to refer to the same underlying nonlinear stochastic system. Recall that the derivative of h along the vector field f, which is called Lie derivative and is indicated with the symbol $\mathcal{L}_f\, h$, is defined as

$$\mathcal{L}_f\, h(x) = \partial_x[h]\; f(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x).$$

We indicate the derivative of h first along the vector field f and then along the vector field g as $\mathcal{L}_g\, \mathcal{L}_f\, h(x) = \partial_x[\mathcal{L}_f\, h]\; g(x)$. We use the recursive relation $\mathcal{L}_f^k\, h(x) = \partial_x[\mathcal{L}_f^{k-1}\, h]\; f(x)$, with $\mathcal{L}_f^0\, h(x) = h(x)$, to indicate the k-th differentiation of h along f. We now recall three operators, firstly introduced in (Mellone and Scarciotti, 2019a, Section III). The first one, which indicates the second derivative of h along the vector fields f and g, is defined as

$${}^g\mathcal{G}_f h(x) = g(x)^\top \partial_x^2[h]\; f(x) = \sum_{j=1}^{n} g_j(x) \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_j \partial x_i} f_i(x).$$

Similarly to the Lie derivative, we use the notation ${}^m\mathcal{G}_l\, {}^g\mathcal{G}_f h(x) = m(x)^\top \partial_x^2[{}^g\mathcal{G}_f h]\; l(x)$, and ${}^g\mathcal{G}_f^k h(x) = g(x)^\top \partial_x^2[{}^g\mathcal{G}_f^{k-1} h]\; f(x)$, to indicate the reiterated operations. The second operator is ${}^l\mathcal{S}_f h$, which is employed to define the *stochastic* Lie derivative of h along the drift vector field f and the diffusion vector field l, namely

$${}^l\mathcal{S}_f h(\xi_t, x) = \mathcal{L}_f\, h(x) + \mathcal{L}_l\, h(x)\xi_t + \frac{1}{2}\, {}^l\mathcal{G}_l h(x).$$

If ${}^l\mathcal{S}_f h(\xi_t, x) = {}^l\mathcal{S}_f h(x)$ is a deterministic expression, *i.e.* the white noise does not appear explicitly or equivalently $\mathcal{L}_l\, h \equiv 0$, then, similarly to the deterministic Lie derivative, we use the notation ${}^l\mathcal{S}_f^2 h(\xi_t, x) = {}^l\mathcal{S}_f\, {}^l\mathcal{S}_f h(\xi_t, x)$. Iteratively, if ${}^l\mathcal{S}_f^{k-1} h(\xi_t, x) = {}^l\mathcal{S}_f^{k-1} h(x)$ is deterministic, ${}^l\mathcal{S}_f^k h(\xi_t, x) = {}^l\mathcal{S}_f\, {}^l\mathcal{S}_f^{k-1} h(\xi_t, x)$, with ${}^l\mathcal{S}_f^0 h(x) = h(x)$ by definition. Finally, we define a third stochastic differential operator

$${}^m_g\mathcal{A}_l h(\xi_t, x) = \mathcal{L}_g\, h(x) + \mathcal{L}_m\, h(x)\xi_t + {}^m\mathcal{G}_l h(x).$$

By using Itô's formula, it is easy to see that the first derivative of the output of system (2) is given by

$$y_t^{(1)} = {}^l\mathcal{S}_f h(\xi_t, x_t) + {}^m_g\mathcal{A}_l h(\xi_t, x_t)u + \frac{1}{2}\,{}^m\mathcal{G}_m h(x_t)u^2.$$

Using the previous definitions, in the remainder we employ the definition of *stochastic relative degree* as given in Mellone and Scarciotti (2019a).

Assumption 1. Let r be the stochastic relative degree of system (2) at $\bar{x}$. Assume that the row vectors $\partial_x[h(\bar{x})]$, $\partial_x[{}^l\mathcal{S}_f h(\bar{x})]$, ..., $\partial_x[{}^l\mathcal{S}_f^{r-1} h(\bar{x})]$ are linearly independent.

Suppose that the relative degree of system (2) at $\bar{x} = 0$ is $r \leq n$. Since Assumption 1 holds, by the definition of stochastic relative degree there exist functions $\phi_i(x)$, $i = r+1, ..., n$ such that Φ, given by

$$\Phi(x) = \begin{bmatrix} h(x) & {}^l\mathcal{S}_f h(x) & \dots & {}^l\mathcal{S}_f^{r-1}h(x) & \phi_{r+1}(x) & \dots & \phi_n(x) \end{bmatrix}^\top,$$

is a local diffeomorphism in a neighbourhood U of $\bar{x}$ such that the dynamics of system (2), written in the new coordinates $z_t = \Phi(x_t)$, is expressed by (Mellone and Scarciotti (2019a))

$$\begin{aligned} \dot{z}_i &= z_{i+1}, & i &= 1, ..., r-1, \\ \dot{z}_r &= c(\xi_t, z_t) + b(\xi_t, z_t)u + a(z_t)u^2, \\ \dot{z}_j &= p_j(\xi_t, z_t) + q_j(\xi_t, z_t)u + s_j(z_t)u^2, & j &= r+1, ..., n, \\ y_t &= z_1, \end{aligned} \tag{3}$$

with $c(\xi_t, z_t) = {}^l\mathcal{S}_f^r h(\xi_t, \Phi^{-1}(z_t))$, $b(\xi_t, z_t) = {}^m_g\mathcal{A}_l\,{}^l\mathcal{S}_f^{r-1} h(\xi_t, \Phi^{-1}(z_t))$, $a(z_t) = \frac{1}{2}\,{}^m\mathcal{G}_m\,{}^l\mathcal{S}_f^{r-1} h(\Phi^{-1}(z_t))$, $p_j(\xi_t, z_t) = {}^l\mathcal{S}_f \phi_j(\xi_t, \Phi^{-1}(z_t))$, $q_j(\xi_t, z_t) = {}^m_g\mathcal{A}_l \phi_j(\xi_t, \Phi^{-1}(z_t))$, $s_j(z_t) = \frac{1}{2}\,{}^m\mathcal{G}_m \phi_j(\Phi^{-1}(z_t))$. In particular, observe that the dependence of the coefficients c, b, p_j and q_j on the white noise ξ_t is affine, *i.e.* they can be decomposed as

$$\begin{aligned} c(\xi_t, z_t) &= c_d(z_t) + c_s(z_t)\xi_t, \\ b(\xi_t, z_t) &= b_d(z_t) + b_s(z_t)\xi_t, \\ p_j(\xi_t, z_t) &= p_{d,j}(z_t) + p_{s,j}(z_t)\xi_t, \quad j = r+1, ..., n, \\ q_j(\xi_t, z_t) &= q_{d,j}(z_t) + q_{s,j}(z_t)\xi_t, \quad j = r+1, ..., n, \end{aligned}$$

where all the quantities are uniquely defined as a consequence of the previous observation. For compactness, we define $p = [p_{r+1} \dots p_n]^\top$, $q = [q_{r+1} \dots q_n]^\top$ and $s = [s_{r+1} \dots s_n]^\top$. When designing the control input of system (3) we need to distinguish between two cases.

(A) $a(z_t) \equiv 0$ in a neighbourhood U_1 of $\bar{x}$. Then, by definition of relative degree, there exists a neighbourhood U_2 of $\bar{x}$ such that $b(\xi_t, z_t) \neq 0$ in $\Phi(U_2)$. Let $U = U_1 \cap U_2$.

(B) There exists a neighbourhood U of $\bar{x}$ such that $a(z_t) \neq 0$ in $\Phi(U)$.

Note that, by the definition of relative degree, no other cases are possible. If a system is such that case **(A)** is satisfied, then the control input does not appear quadratically in the r-th derivative of the output for x_t in U, whereas if **(B)** is satisfied, then the square of the input does appear in the r-th derivative of the output. Due to space limitation, in this paper we consider case **(A)**. Note that case **(B)** is analogous.

3 APPROXIMATE STOCHASTIC COMPENSATION

In this section we illustrate the practical limitations of the control laws introduced in Mellone and Scarciotti (2019a) and Mellone and Scarciotti (2020b) and we introduce a method to overcome them.

Let the stochastic relative degree of system (2) be $1 \leq r \leq n$ and consider the problem of partial feedback linearisation. Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a generic input yet to be selected. System (2) can be partially feedback-linearised in $\Phi(U)$ by

$$u_t^{lin} = \frac{1}{b(\xi_t, z_t)}(-c(\xi_t, z_t) + v), \quad \text{if } \mathbf{(A)}.$$

Remark 1. In (Mellone and Scarciotti, 2020b, Section 3) it is pointed out that a control law u_t that explicitly depends on ξ_t may compromise the equivalence of (1) and (2). Hence, therein a definition of *admissible* control laws, *i.e.* laws such that this equivalence is preserved, is given. However, the framework of the present paper is that of practical scenarios where ξ_t is not available for feedback. Therefore the control laws which we design in Sections 3 and 4 are merely functions of measurements of the state. Consequently, this implies that such control laws are always admissible.

For compactness, set $\zeta_t = [z_1 \dots z_r]^\top$ and $\eta_t = [z_{r+1} \dots z_n]^\top$. From the theory of normal forms of deterministic systems, recall that necessarily $\zeta_t = 0$ at $\bar{x}$ and that the value of η_t at $\bar{x}$ can be arbitrarily chosen. It is straightforward to observe that these two facts hold also for stochastic systems. Therefore let $\Phi(x_t)$ be such that $\eta_t = 0$ as well at $\bar{x}$, which makes zero an equilibrium point of system (3). The partially feedback linearised system by the control law u_t^{lin} has the form

$$\begin{aligned} \dot{\zeta}_t &= A\zeta_t + Bv, \\ \dot{\eta}_t &= p(\xi_t, \zeta_t, \eta_t) + q(\xi_t, \zeta_t, \eta_t)u_t^{lin} + s(\zeta_t, \eta_t)(u_t^{lin})^2, \end{aligned} \tag{4}$$

with $B = [0 \dots 0 \; 1]^\top$. We also recall that if there exists a control law $u_t = u_{z,t}(\eta_t)$ such that $\dot{\zeta}_t \equiv \zeta_t \equiv 0$, then the autonomous stochastic differential equation

$$\dot{\eta}_t = p(\xi_t, 0, \eta_t) + q(\xi_t, 0, \eta_t)u_{z,t} + s(\zeta_t, \eta_t)u_{z,t}^2$$

is called the *zero dynamics* of the stochastic system (2). The limitation highlighted in Mellone and Scarciotti (2019a) and Mellone and Scarciotti (2020b) is that the control u_t^{lin} requires the knowledge of the stochastic process ξ_t and therefore cannot be applied in practice. We address how to overcome this issue in the next sections.

3.1 Estimation of the Brownian motion

By sampling the state, we now present a method to obtain a causal partial estimate of the Brownian motion affecting the system between sampling times. Let $\{t_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be a sequence of equally-spaced sampling times, with $t_k - t_{k-1} = \varepsilon$ for all $k \in \mathbb{Z}_{>0}$. Define the differences $\Delta W_\varepsilon(k) = \mathcal{W}_{t_k} - \mathcal{W}_{t_{k-1}}$ and $\Delta x(k) = x_{t_k} - x_{t_{k-1}}$. Our aim is to show that it is possible to compute a causal estimate $\Delta\widehat{W}_\varepsilon(k)$ of the quantity $\Delta W_\varepsilon(k)$ by comparing the samples of the state of the system at times t_{k-1} and t_k. In particular, we want this estimate to "converge", in a sense to be defined, to the stochastic differential $d\mathcal{W}_t$ as the sampling period ε converges to zero. Let $\mathfrak{L}_I$ be the space of functions that are integrable in Itô's sense. Then with the notation $\Delta\widehat{W}_\varepsilon \xrightarrow{\varepsilon} d\mathcal{W}_t$ we mean that for all $\alpha \in \mathfrak{L}_I$ $\lim_{\varepsilon\to 0}\sum_k \alpha(t_{k-1}, w)\Delta\widehat{W}_\varepsilon(k) = \int_0^t \alpha(\tau, w)d\mathcal{W}_\tau$. For ease of notation, define $F_{t_k} = f(x_{t_k}) + g(x_{t_k})u_{t_k}$

and $L_{t_k} = l(x_{t_k}) + m(x_{t_k})u_{t_k}$, which are the drift and diffusion terms, respectively, of system (1) evaluated at time t_k. It is reasonable to assume that the vector L_{t_k} is non-zero for almost all $k \in \mathbb{Z}_{\geq 0}$, as otherwise the system would display a deterministic behaviour and $\Delta\widehat{W}_\varepsilon$ could be selected as zero. Consequently, the Moore-Penrose left pseudo-inverse of L_{t_k}, *i.e.* $L^+_{t_k} = (L^\top_{t_k} L_{t_k})^{-1} L^\top_{t_k}$, is well-defined almost surely. A procedure to estimate *a posteriori* the Brownian motion affecting *linear* stochastic system has been presented by Mellone and Scarciotti (2020a). The following Lemma extends those results to systems with *nonlinear* drift and diffusion terms.

Lemma 1. Consider system (2). Let $\{\Delta\widehat{W}_\varepsilon(k)\}_{k>0}$ be a sequence of scalars defined as $\Delta\widehat{W}_\varepsilon(k) = L^+_{t_{k-1}}[\Delta x(k) - F_{t_{k-1}}\varepsilon]$. Then $\Delta\widehat{W}_\varepsilon(k) \xrightarrow{\varepsilon} d\mathcal{W}_t$ almost surely.

3.2 Compensating control

In this section we discuss how to implement a feedback law that, by exploiting the estimates $\{\Delta\widehat{W}_\varepsilon(k)\}_k$, partially linearises the system in an approximate way. The advantage of this control law with respect to u^{lin}_t is that it is practically implementable while the accuracy of the approximation can be tuned by reducing the sampling period ε. We begin with defining a control u^d_t which corresponds to u^{lin}_t when $\xi_t \equiv 0$, namely

$$u^d_t = \frac{1}{b_d(z_t)}(-c_d(z_t) + v), \quad \text{if } \mathbf{(A)}. \tag{5}$$

Note that to ensure that u^d_t is well-defined (*i.e.* $b_d(z_t) \neq 0$) in a neighbourhood of zero we need the following assumption.

Assumption 2. $\mathcal{L}_g\, {}^l\mathcal{S}^{r-1}_f h(\bar{x}) + {}^m\mathcal{G}_l\, {}^l\mathcal{S}^{r-1}_f h(\bar{x}) \neq 0$.

Observe that the term u^d_t is the deterministic approximation of u^{lin}_t when no estimation of the white noise is performed. In fact, since the white noise cannot be measured, the most reasonable approximation is obtained by replacing ξ_t with its mean value, hence computing the feedback control using $c_d(z_t) = c(0, z_t)$ and $b_d(z_t) = b(0, z_t)$ instead of $c(\xi_t, z_t)$ and $b(\xi_t, z_t)$, respectively.
We now define the control $u^{app}_t = u^d_t + u^s_t$, with u^s_t to be specified. By construction, when the control u^{app}_t is applied, the dynamics of the transformed system is

$$\begin{aligned}
\dot{z}_i &= z_{i+1}, \qquad\qquad\qquad\qquad i = 1, \ldots, r-1,\\
\dot{z}_r &= v + \left[c_s(z_t) + b_s(z_t)u^d_t\right]\xi_t + \left[b_d(z_t) + b_s(z_t)\xi_t\right]u^s_t,\\
\dot{\eta}_t &= p(\xi_t, \zeta_t, \eta_t) + q(\xi_t, \zeta_t, \eta_t)u^{app}_t + s(\zeta_t, \eta_t)(u^{app}_t)^2,\\
y_t &= z_1.
\end{aligned}$$

Specifically, observe that the term $\left[c_s(z_t) + b_s(z_t)u^d_t\right]\xi_t$ arises because the approximation u^d_t of the feedback linearising control u^{lin}_t has been adopted.

Using the estimates $\{\Delta\widehat{W}_\varepsilon(k)\}_k$ introduced in Section 3.1, we wonder whether it is possible to design the control u^s_t to reduce the noisy contribution to the dynamics of the transformed system. To this end, we look at the evolution of the state z_r between two consecutive sampling times, namely

$$z_{r,t_{k+1}} = z_{r,t_k} + \int_{t_k}^{t_{k+1}} v d\tau + \beta_d(k+1) + \int_{t_k}^{t_{k+1}} b_d(z_\tau)u^s_\tau d\tau + \int_{t_k}^{t_{k+1}} b_s(z_\tau)u^s_\tau d\mathcal{W}_\tau,$$

where $\beta_d(k+1) = \int_{t_k}^{t_{k+1}} \left[c_s(z_\tau) + b_s(z_\tau)u^d_\tau\right] d\mathcal{W}_\tau$. Our goal is to minimise the contribution of the noise to the dynamics using u^s_t and the estimate $\Delta\widehat{W}_\varepsilon(k+1)$ obtained at time t_{k+1}. The fact that $\Delta\widehat{W}_\varepsilon(k+1)$ is available *a posteriori* at the sampling time t_{k+1} suggests that u^s_t should compensate for $\beta_d(k+1)$ in the form of an impulse at time t_{k+1}. Iterating over k, this yields a control u^s_t of the form

$$u^s_t = \sum_{i=0}^{k} u^*(i+1)\delta(t - t_{i+1}), \qquad t \leq t_{k+1}, \tag{6}$$

where $\delta(t)$ is a Dirac delta and $\{u^*(k)\}_k$ is a sequence of scalars depending on $\{\Delta\widehat{W}_\varepsilon(k)\}_k$ which needs to be defined. Since we introduced an impulsive control, it is necessary to adopt the jump notation in the expression of z_{r,t_k}. In particular, the sampling property of the Dirac delta yields

$$z_{r,t^+_{k+1}} = z_{r,t^+_k} + \int_{t_k}^{t_{k+1}} v d\tau + \beta_d(k+1) + b_d(z_{t_{k+1}})u^*(k+1) + b_s(z_{t_{k+1}})u^*(k+1)\xi_{t_{k+1}}, \tag{7}$$

where $z_{t_{k+1}}$ is the value of z_t immediately before the jump happening at time t_{k+1}. Thus, we have reduced the problem of approximate partial feedback linearisation to the problem of finding the sequence $\{u^*(k+1)\}_k$ such that the contributions of $\beta_d(k+1)$, $b_d(z_{t_{k+1}})u^*(k+1)$ and $b_s(z_{t_{k+1}})u^*(k+1)\xi_{t_{k+1}}$ are compensated. Unfortunately, this problem is impossible to solve as detailed in the next remark.

Remark 2. The noise term $\xi_{t_{k+1}}$ appearing in (7) is inevitably introduced by the impulsive control $u^*(k+1)\delta(t - t_{k+1})$. In fact, if $b_s(z_{t_{k+1}}) \neq 0$, a control impulse at time t_{k+1} causes a jump variation of the state z_r, the amplitude of which is proportional to the white noise at time t_{k+1}. The effect of this term cannot be compensated for because the white noise process evaluated at time t_{k+1} is independent of its values ξ_t for $t < t_{k+1}$.

Thus, we revise our goal and look for a sequence $\{u^*(k)\}_k$ and a set of assumptions under which the control u^s_t generates a model which is "closer", in a sense to be defined, to (4) than the selection $u^s_t \equiv 0$. To this end, we introduce two assumptions which are instrumental to the statement of the main result of this section.

Assumption 3. There exists a neighbourhood U of $\bar{x} = 0$ and $\delta_0 \in [0, 1)$ such that $\left|\frac{b_s(z)}{b_d(z)}\right| \leq \delta_0$ for all $z \in \Phi(U)$.

Remark 3. The rationale of this assumption is that in a neighbourhood of zero the noise introduced by the control action is "dominated" by the deterministic contribution.

Assumption 4. The control v is such that, if $z_{t_k} \in \Phi(U)$, then $\mathcal{P}\left(\lim_{\varepsilon\to 0} z_{t_{k+1}} \in \Phi(U)\right) = 1$.

Remark 4. The previous assumption requires that the input v, by means of which we would like to control the linearised system, is such that with probability one it does not drive the state z outside the neighbourhood $\Phi(U)$ after a time ε which tends to zero. Intuitively, to attain this, on the one hand the control v has to stabilise the ideally linearised system around a trajectory contained in $\Phi(U)$ with a sufficiently high gain. On the other hand the noise introduced by the term $\left[c_s(z_t)+b_s(z_t)u_t^d\right]\xi_t$ in the period ε has to be sufficiently small not to compromise the action of v.

We are now ready to give the main result of this section.

Proposition 1. Consider system (2) and suppose that Assumptions 1, 2, 3 and 4 hold. Let $z_{t_0}\in\Phi(U)$ and

$$\beta_s(k+1)=\beta_d(k+1)+\int_{t_k}^{t_{k+1}} b_d(z_\tau)u_\tau^s d\tau+\int_{t_k}^{t_{k+1}} b_s(z_\tau)u_\tau^s d\mathcal{W}_\tau\,.$$

If $u_t=u_t^{app}$ with u_t^s given by (6) where

$$u^*(k+1)=-\frac{\widehat{\beta}_d^E(k+1)}{b_d(z_{t_{k+1}})},\tag{8}$$

then $\mathcal{P}\left(\lim_{\varepsilon\to 0}|\beta_s(k+1)|\le\delta_0\left|\beta_d(k+1)\xi_{t_{k+1}}\right|\right)=1$ for all $k\in\mathbb{Z}$.

Note that when u_t^{app} is applied with u^* given by (8), the noisy terms in the dynamics of z_r reduce to

$$-\frac{b_s(z_{t_{k+1}})}{b_d(z_{t_{k+1}})}\xi_{t_{k+1}}\widehat{\beta}_d^E(k+1)+o(\varepsilon^2),\tag{9}$$

whereas the noisy term when only u_t^d is applied is the stochastic integral $\beta_d(k+1)$. The previous proposition states that, under Assumptions 3 and 4, the noisy term (9) in the dynamics of ζ_t introduced by u_t^{app} is smaller in norm than the noisy term introduced by only u_t^d, with a probability closer to 1 as the sampling period is made smaller.

4 LOCAL ASYMPTOTIC STABILISATION UNDER STOCHASTIC COMPENSATION

In Section 3 we have illustrated how estimates of the Brownian motion obtained causally can be employed to design a controller that compensates, in an approximate way, the stochastic dynamics of the state ζ_t. We now address specifically the problem of local asymptotic stabilisation and show that such stochastic compensations have indeed a crucial role in the solution of the problem.

Let u_t^d be given as in (5), with $v(\zeta_t)=-(d_0z_1+d_1z_2+...+d_{r-1}z_r)$, with d_i, $i=0,1,...,r-1$, coefficients to be chosen. Suppose that the input u_t^{app} is applied to the system, with u_t^s given by (6) and (8). Recall that $B=[0\ \dots\ 0\ 1]^\top$. Then the dynamics of the state ζ_t is given by

$$\dot\zeta_t=A\zeta_t+B\left[\left(c_s(z_t)+b_s(z_t)u_t^d\right)\xi_t+\left(b_d(z_t)+b_s(z_t)\xi_t\right)u_t^s\right]$$

where the matrix A can be made negative definite by a proper selection of the coefficients d_i. We now present two results. The first is a sufficient condition on the term $c(\xi_t,\zeta_t,\eta_t)$ for the stabilisation of the system by using either the control $u_t=u_t^d$ or the control $u_t=u_t^{app}$. The second is a sufficient condition on the term $b_s(z_t)$ for the stabilisation of the system by using $u_t=u_t^{app}$.

Theorem 1. Consider system (2) and let Assumptions 1 and 2 hold. Suppose that the zero dynamics is asymptotically stable almost surely, that $c(\xi_t,0,\eta)\equiv 0$ for η in a neighbourhood of zero and that $\frac{\partial c}{\partial\zeta}(\xi_t,0,0)=0$. Then the control law $u_t=u_t^d$ makes the equilibrium at the origin of the closed-loop system asymptotically stable almost surely. Moreover, under the additional Assumptions 3 and 4, the control law $u_t=u_t^{app}$ makes the equilibrium at the origin of the closed-loop system asymptotically stable almost surely as well.

The previous theorem states that under a condition on the term $c(\xi_t,z_t)$ both the control law with and without compensation can achieve local stabilisation. We now show that under a different condition on the term $b_s(z_t)$, $u_t=u_t^{app}$ can still solve the stabilisation problem.

Theorem 2. Consider system (2) and let Assumptions 1, 2, 3 and 4 hold. Suppose that the zero dynamics is asymptotically stable almost surely, that $b_s(0,\eta)\equiv 0$ for all η in a neighbourhood of zero and that $\frac{\partial b_s}{\partial\zeta}(0,0)=0$. Then the control law $u_t=u_t^{app}$ makes the equilibrium at the origin of the closed-loop system asymptotically stable almost surely for ε going to zero.

Theorem 2 gives sufficient conditions under which the control law $u_t=u_t^{app}$ stabilises system (2). In the next section we provide a counter-example that shows that the use of just $u_t=u_t^d$ is not stabilising when these conditions are met. This proves that the proposed control law which makes use of the impulsive correction (6)-(8) and of the estimates $\{\Delta\widehat{W}_\varepsilon(k)\}_k$ is able to stabilise a class of stochastic systems that the standard deterministic continuous-time law is not able to stabilise.

Remark 5. An important subclass of stochastic systems, which is largely studied because simpler than the general case, is represented by systems with $m(x_t)\equiv 0$, therefore $b_s(z_t)\equiv 0$. For this subclass of systems a control input $u_t\ne 0$ does not introduce noise in the differential dx_t. It is trivial to observe that, for such systems, Assumption 3 and the other hypotheses on b_s in Theorem 2 hold. Moreover the noisy contribution (9) to the dynamics of z_r reduces to just $o(\varepsilon^2)$, which means that the system dynamics under the input u_t^{app} can be made arbitrarily "close" to the dynamics under u_t^{lin} by reducing the sampling time ε.

5 A COUNTER-EXAMPLE

In this section we provide a numerical example to illustrate the theory. Consider the nonlinear stochastic system

$$\dot z_t=\begin{bmatrix}e^{z_2}\\-z_2+s_1\end{bmatrix}+\begin{bmatrix}e^{z_1}+z_1s_2\\0\end{bmatrix}u+\begin{bmatrix}\mu s_2\\z_2-z_2^2\end{bmatrix}\xi_t+\begin{bmatrix}z_1^2\\0\end{bmatrix}u\xi_t,$$

where $s_i=\sin(z_i)$ and $\mu\in\mathbb{R}$. Our goal is to make the origin an asymptotically stable equilibrium almost surely. To this end let $y_t=z_1$ be a fictitious output and note that for this selection the system has relative degree one at zero. We first study the stability of the zero dynamics. Define $\zeta_t=z_1$ and $\eta_t=z_2$. Setting $\zeta_t=0$ we obtain the system $\dot\eta_t=p(\xi_t,0,\eta_t)=-\eta_t+\eta_t\xi_t-\eta_t^2\xi_t$. Note that this subsystem is asymptotically stable almost surely in the first approximation. In fact, the linear approximation of this system is $\dot\eta_t=A_\eta\eta_t+F_\eta\eta_t\xi_t=-\eta_t+\eta_t\xi_t$ which is asymptotically stable almost surely if $2A_\eta-F_\eta^2<0$ see, *e.g.*, Gard (1988). Since this condition is verified, we conclude that the zero dynamics is locally asymptotically stable almost surely.

Let $u_t=u_t^d$ be given by (5). Figure 1 shows the time histories of the state (top) of the system and of the control input (bottom). We notice that the equilibrium is not asymptotically stable. Then we set $u_t=u_t^{app}=u_t^d+u_t^s$,

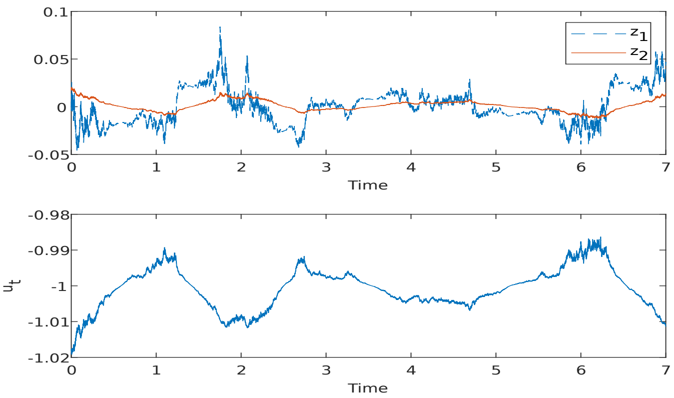

Fig. 1. Time history of the states z_t (top) and the control input u_t^d (bottom).

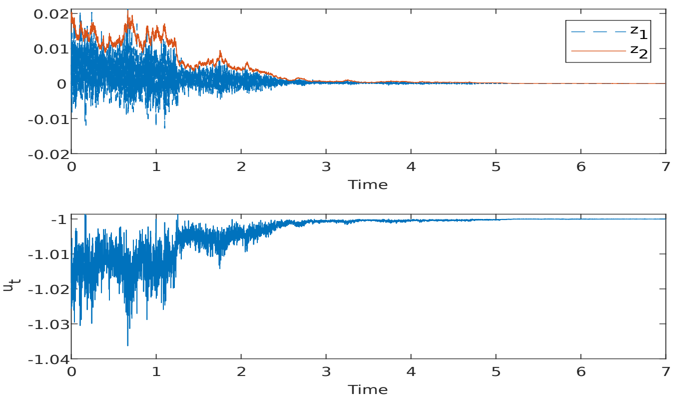

Fig. 2. Time history of the states z_t (top) and the control input u_t^{app} (bottom).

where u_t^s is given by (6) and (8) with $\varepsilon = 10^{-3}$. Figure 2 shows the new time histories of the state (top) of the system and of the control input (bottom). We observe that the equilibrium at the origin is now asymptotically stable almost surely.

A few observations are in order. Firstly we notice that, when enforcing approximate control laws, the dynamics of $\zeta_t = z_1$ remains stochastic, since the noise is not known and it cannot be compensated for perfectly. This is an unavoidable issue when tackling the control of stochastic systems in practical scenarios. Secondly, we notice that in the ideal case the control input takes values which are several order of magnitude greater than the state of the system, whereas the approximate (and implementable) controls have reasonable orders of magnitude and can therefore be practically applied in real scenarios.

6 CONCLUSIONS

In this paper we have addressed the problem of designing a control law to practically feedback-linearise and stabilise nonlinear stochastic systems in an approximate way. Specifically, we have shown that it is possible to causally obtain estimates of the increments of the Brownian motion that affected the system from measurements of the states. We have then used these estimates to synthesise a hybrid control law that compensates for the stochastic dynamics and we have shown that it solves the local stabilisation problem.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations*. A Wiley-Interscience publication. Wiley.

Arnold, L. (2003). *Random Dynamical Systems*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg.

Bestle, D. and Zeitz, M. (1983). Canonical form observer design for non-linear time-variable systems. *International Journal of Control*, 38(2), 419–431.

Brockett, R. (1978). Feedback invariants for nonlinear systems. *IFAC Proceedings Volumes*, 11(1), 1115–1120.

Byrnes, C.I. and Isidori, A. (1984). A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control. In *The 23rd IEEE Conference on Decision and Control*, 1569–1573.

Byrnes, C.I. and Isidori, A. (1988). Local stabilization of minimum-phase nonlinear systems. *Systems & Control Letters*, 11(1), 9–17.

Gaeta, G. and Rodríguez Quintero, N. (1999). Lie-point symmetries and stochastic differential equations. *Journal of Physics A: Mathematical and General*, 32(48), 8485–8505.

Gard, T. (1988). *Introduction to Stochastic Differential Equations*. Monographs and textbooks in pure and applied mathematics. M. Dekker.

Isidori, A. (1995). *Nonlinear Control Systems*. Communications and Control Engineering. Springer-Verlag London.

Isidori, A., Krener, A., Gori-Giorgi, C., and Monaco, S. (1981). Nonlinear decoupling via feedback: A differential geometric approach. *IEEE Transactions on Automatic Control*, 26(2), 331–345.

Krener, A.J. (1987). Normal forms for linear and nonlinear systems. *Contemporary Mathematics*, 68, 157–189.

Mellone, A. and Scarciotti, G. (2019a). Normal form and exact feedback linearisation of nonlinear stochastic systems: the ideal case. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 3503–3508.

Mellone, A. and Scarciotti, G. (2019b). ε-Approximate Output Regulation of Linear Stochastic Systems: a Hybrid Approach. In *2019 European Control Conference (ECC)*, 287–292.

Mellone, A. and Scarciotti, G. (2020a). Output Regulation of Linear Stochastic Systems. *IEEE Transactions on Automatic Control, under review.*

Mellone, A. and Scarciotti, G. (2020b). The Zero Dynamics of Nonlinear Stochastic Systems: Stabilisation and Output Tracking in the Ideal Case. In *21st IFAC World Congress (IFAC 2020)*. To appear.

Øksendal, B. (2003). *Stochastic Differential Equations (Sixth Edition)*. Springer-Verlag.

Roberts, A. (2008). Normal form transforms separate slow and fast modes in stochastic dynamical systems. *Physica A: Statistical Mechanics and its Applications*, 387(1), 12–38.

Yong, J. and Zhou, X.Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Stochastic Modelling and Applied Probability. Springer New York.

Zeitz, M. (1983). Controllability canonical (phase-variable) form for non-linear time-variable systems. *International Journal of Control*, 37(6), 1449–1457.