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Inventory Control based on Dynamic Programming with State Probability Distribution

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1. INTRODUCTION

Besides the randomness from demands, which has been considered in a lot of research, inventory inaccuracy is another kind of uncertainty in inventory systems, which have significantly negative impact on the performance of inventory control (Raman and DeHoratius, 2008). Kang and Gershwin (2005) found that a small rate of invisible stock loss could lead to severe stock-out. A lot of research has been conducted to hedge against inventory inaccuracy and the results indicate that elimination of inventory inaccuracy could reduce supply chain costs and out-of-stock level (DeHoratius et al., 2008; Fleisch and Tellkamp, 2005; Agrawal and Sharda, 2012; Sahin and Yves, 2009).

In this abstract, we describe the demand and the inventory loss rate by Poison distribution and the uniform distribution respectively, based on which the probability distribution of the inventory level can be calculated. After obtaining this probability distribution, we develop a dynamic programming with state probability distribution to obtain the optimal ordering quantity so as to minimize the cost.

2. PROBLEM DESCRIPTION AND MODELING

We consider a finite horizon and single-product inventory system, which has a random stock loss rate and random demand rate. In addition, the true inventory level cannot be observed exactly. The objective of the optimal inventory control problem is to minimize the overall cost over a finite horizon.

The notifications of the mathematical model of this optimal inventory control problem are as follows:

x_k: inventory level at the beginning of period k (which cannot be observed accurately);

x_1^0: initial inventory level;

q_k: inventory loss rate (a random variable with known probability distribution, or an uncertain variable with known bounds);

u_k: replenishment quantity in period k;

w_k: random disturbance in period k;

r_k: demand (a random variable with known probability distribution);

c^+: unit inventory holding cost;

c^-: unit demand backlog cost;

α: proportional ordering cost;

β: fixed ordering cost.

At the beginning of period k, we decide the replenishment quantity u_t according to the inaccurate inventory level x_k which can be described as a probability distribution function F_k^X (x_k)=Prob$\{X_k \le x_k\}$. Suppose that the lead time is zero. Then, the ordered products arrive, which raises the inventory level to x_k+u_k. We assume that the invisible stock loss happens immediately after the ordered products arrive, with the stock loss rate q_k, which makes the inventory level decrease to $(1-q_k)(x_k+u_k)$. After that, the demand r_k occurs. All the events are completed by the end of period k, which determines the inventory level x_{k+1} at the beginning of period k+1. This process can be captured by

$$x_{k+1} = h_k\left(x_k, u_k, w_k\right) = \left(1-q_k\right)\left(x_k + u_k\right) - r_k \tag{1}$$

The objective of the inventory control problem is to minimize the expectation of the overall cost over all the periods,

$$E_{x_k} \min \sum_{k=1}^{K} g_k\left(x_k, u_k\right) \tag{2}$$

where
$g_k\left(x_k,u_k\right) = c^+ \cdot \max\left(x_{k+1},0\right) + c^- \cdot \max\left(-x_{k+1},0\right) + \alpha \cdot u_k + \beta \cdot \delta\left(u_k\right)$
is the reward based on the decision in period k, and $\boxed{\delta(x)}$ is the indicating function that takes the value of one is x is positive and zero otherwise.

3. APPRIXIMATED ISDP ALGORITHM

We rewrite the objective Eq. (2) as a dynamic programming equation:

$$J_k\left(x_k\right) = \min_{u_k \in U\left(x_k\right)} E_{r_k, q_k}\left\{g_k\left(x_k, u_k\right) + J_{k+1}\left(x_{k+1}\right)\right\} \tag{3}$$

where $J_k(x_k)$ is the value function based on state x_k in period k and $x_{k+1} = h_k\left(x_k, u_k, w_k\right)$ for $k = 1, \ldots, K$.

Because the inaccurate inventory level x_k can be captured by F_k^X (x_k). An dynamic programming equation with state probability distribution can be write as follows from Eq. (3),

$$J_k\left(F_k^X\right)=\min_{u_k\in U_k\left(F_k^X\right)}\left\{G_k\left(F_k^X,u_k\right)+J_{k+1}\left(F_{k+1}^X\right)\right\} \tag{4}$$

with $F_{k+1}^X=H_k\left(F_k^X,u_k,w_k\right)$

and $G_k\left(F_k^X,u_k\right)=\int g_k\left(x_k,u_k\right)\mathrm{d}F_k^X\left(x_k\right)$.

To deal with the "curse of dimensionality", an approximate algorithm is designed for calculating the states captured by probability distribution functions with 3 steps as follows:

(1) *State updating*: $F_{k+1}^X=H_k\left(F_k^X,u_k,w_k\right)$

$$\begin{aligned}F_{k+1}^X\left(x_{k+1}\right)&=\mathrm{Prob}\left\{\left(1-q_k\right)\left(x_k+u_k\right)-r_k\le x_{k+1}\right\}\\&=\mathrm{Prob}\left\{x_k\le\frac{x_{k+1}+r_k}{1-q_k}-u_k\right\}\\&=\frac{1}{\bar{q}}\int_0^{\bar{q}}\left\{\sum_{n=0}^{\infty}F_k^X\left(\frac{x_{k+1}+r_k}{1-q_k}-u_k\right)P\left(r_k=n\right)\right\}\mathrm{d}q_k\end{aligned} \tag{5}$$

where $F_k^W(w_k)$ is the known PDF of the random variable w_k.

The initial state $F_1^X(x_1)$ is supposed to be known, e.g.,

$$F_1^X\left(x_1\right)=\begin{cases}0, & \text{if } x_1<x_1^0\\1, & \text{if } x_1\ge x_1^0\end{cases} \tag{6}$$

(2) *Grid construction for state space*

We construction the grid for state space as

Period 1: 1 state;

Period k (k = 2, …, K−1): $N(k-1)$ states.

$u_{k,i}=\dfrac{i}{N\left(k-1\right)-1}u_{\max}$ for i = 0, 1, …, $N(k-1)-1$, where N is the grid construction parameters.

$$F_{k+1,i}^X=H_k\left[F_{k,j(i)}^X,u_{k,i},w_k\right]$$

Period K: $N(K-1)$ states

$$u_{k,i}^*=\arg\min G_k\left(F_{K+1,i}^X,u_{k,i}\right)$$

$$F_{K+1,i}^X=H_K\left[F_{K,i}^X,u_{k,i}^*,w_k\right]$$

In figure 1, we give a grid of state space for 5 periods and N=3.

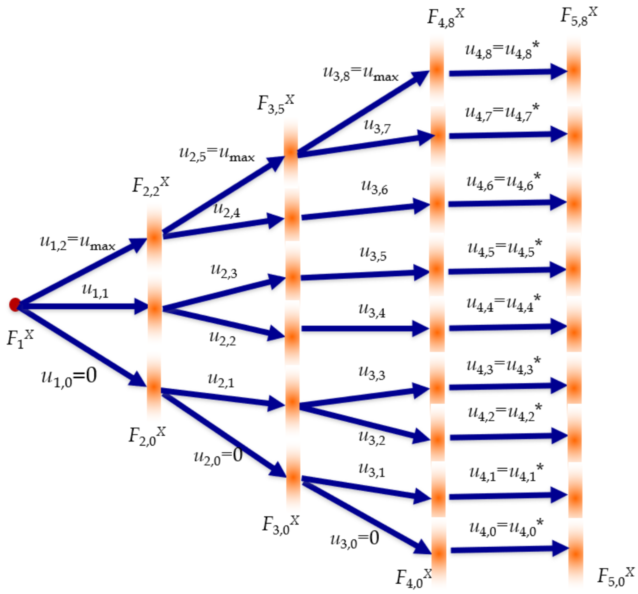

Fig. 1. Grid construction for state space for 5 periods when N=3

(3) *Backward optimization with interpolation*

For k=K, solve the single stage optimization problem:

$$J_K\left(F_{K,i}^X\right)=\min_{u_{K,i}}G_K\left(F_{K+1,i}^X,u_{K,i}\right)\quad\text{for } i=0,1,\ldots,N(K-1)-1. \tag{7}$$

$$F_{K+1,i}^X=H_K\left(F_{K,i}^X,u_{K,i},w_K\right)$$

For k<K, solve the following recursive optimization problem:

$$J_k\left(F_{k,i}^X\right)=\min_{u_{k,i}}\left\{G_k\left(F_k^X,u_{k,i}\right)+\tilde{J}_{k+1}\left[H_k\left(F_{k,i}^X,u_{k,i},w_k\right)\right]\right\}$$

for i=0, 1, …, $N(k-1)-1$ (8)

where $\tilde{J}_{k+1}\left[H_k\left(F_{k,i}^X,u_{k,i},w_k\right)\right]$ is the approximated value function. If the state distribution $F_{K+1,i}^X=H_K\left[F_{K,i}^X,u_{K,i},w_K\right]$ is in the constructed grid, we calculate $\tilde{J}_{k+1}\left[H_k\left(F_{k,i}^X,u_{k,i},w_k\right)\right]$ based on $F_{K+1,i}^X$ directly. Otherwise, $\tilde{J}_{k+1}\left[H_k\left(F_{k,i}^X,u_{k,i},w_k\right)\right]$ can be obtained by linear interpolation as follows:

$$\begin{aligned}&\tilde{J}_{k+1}\left[H_k\left(F_{k,i}^X,u_{k,i},w_k\right)\right]\\&=\tilde{J}_{k+1}\left(F_{k+1,\alpha}^X\right)\\&=\frac{1}{100}\sum_{\theta=0.01}^{1}\left\{\begin{aligned}&\left[\frac{\left(F_{k+1,\alpha}^X\right)^{-1}(\theta)-\left(F_{k+1,j}^X\right)^{-1}(\theta)}{\left(F_{k+1,j+1}^X\right)^{-1}(\theta)-\left(F_{k+1,j}^X\right)^{-1}(\theta)}\right]J_{k+1}\left(F_{k+1,j+1}^X\right)\\&+\left[\frac{\left(F_{k+1,j+1}^X\right)^{-1}(\theta)-\left(F_{k+1,\alpha}^X\right)^{-1}(\theta)}{\left(F_{k+1,j+1}^X\right)^{-1}(\theta)-\left(F_{k+1,j}^X\right)^{-1}(\theta)}\right]J_{k+1}\left(F_{k+1,j}^X\right)\end{aligned}\right\}\end{aligned}$$

4. NUMERICAL EXPERIMENTS

In this section, we design a two period inventory optimization experiment, in which c^{+}=1, c^{-}=25, α=10, β=300 and x_1^0=50, to test the effect of the approximate algorithm for the dynamic programming with state probability distribution. The demand is subject to a Poison distribution which is truncated in the probability distribution range [0.001,0.999]. Besides, the inventory loss rate is uniformly distributed in [0, 0.1] and the maximum order quantity is 200.

In the first period, we discretize the range of the order quantity to M nodes and construct the grid for the state space. Second, we optimize the cost of the second period for every state distribution and find the best order quantity in the first period by backward optimization and liner interpolation. The result is shown in Table 1.

Table 1. Accuracy of Interpolation

M	Optimal order quantity obtained by interpolation	$\frac{\tilde{u}_1 - u^*}{u^*} \times 100\%$
3	74	-56.21%
5	150	-11.24%
6	160	-5.33%
11	160	-5.33%
21	170	0.59%
26	168	-0.59%
41	170	0.59%
51	168	-0.59%
The best order quantity under the integer discretization: u^*=169		

From Table 1, we can see that the optimal order quantity is 169, which can be regarded as the actual optimal order quantity u^*, when the order quantity is discretized into integer. Compared to the u^*, we can find that the error becomes smaller as the number discretization nodes increases. When $M \geq 6$, the error is less than 6%; and when $M \geq 21$, the error is less than 1%.

5. CONCLUSION

In this abstract, we model the inventory control problem under inaccurate inventory record over a finite horizon by a dynamic programming with state probability distribution. To solve the problem, an approximated algorithm by considering the backward optimization and the interpolation of state probability distribution is developed. Numerical experiments verify the effectiveness of the proposed dynamic programming model and the approximation algorithm.

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