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Stability conditions of TS fuzzy systems with switched polynomial Lyapunov functions ⋆

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Abstract: Exploring properties of membership functions, sufficient conditions based on linear matrix inequalities (LMIs) for a existence of a switched polynomial Lyapunov function are proposed. To obtain the LMIs, the time derivative of membership functions are described as a finite polytopic representation, leading to less conservative conditions than other published results. A numerical example illustrates the efficiency of the stabilizing conditions.

Keywords: TS fuzzy system; Polynomial Lyapunov functions; Stability analysis; Linear matrix inequalities.

1. INTRODUCTION

Takagi-Sugeno (TS) fuzzy models represent an important tool for the analysis of smooth nonlinear systems. The main attractiveness of this model consists in describing the nonlinear system as a weighted sum of linear systems. The matrices of each local models are known and the weights of the sum are called membership functions (MFs) which hold convex sum properties (Takagi and Sugeno, 1985; Taniguchi et al., 2001; Tanaka and Wang, 2001). TS models can be applied to solve academic or industrial problems, such as, temperature control of chemical reactors (Chiou et al., 2010), stabilization of power system in co-generation plants (Arrifano et al., 2007) and trajectory tracking of quadrotors (Araujo et al., 2019).

Stability analysis of TS fuzzy systems are usually investigated via the direct Lyapunov method. This approach allows to analyze the system without knowing its solutions. On the other hand, there is no systematic approach to obtain Lyapunov functions for nonlinear systems. As a consequence, the most used strategy in the literature consists in selecting a function as a candidate and checking if it satisfies the Lyapunov conditions. In general, this step is conservative and in some cases it is necessary to test several types of functions to find the desired result. The first results of the literature used a common quadratic Lyapunov function (CQLF) to guarantee stability (Tanaka and Wang, 2001; Tuan et al., 2001; Mansouri et al., 2009). However, to find a common matrix satisfying all Lyapunov inequalities for a TS fuzzy system may be too conservative (Johansson et al., 1999). Then, different types of Lyapunov functions have been employed, among which, it can be cited the piecewise Lyapunov functions (Johansson et al., 1999; Tognetti and Oliveira, 2010), fuzzy Lyapunov functions (FLFs) (Guerra and Bernal, 2009; Mozelli et al., 2009; Faria et al., 2013; de Souza et al., 2014; Campos et al., 2017; Valentino et al., 2019; Faria et al., 2019) and polynomial Lyapunov functions (PLFs) (Bernal and Guerra, 2010; Lee et al., 2012; Kim et al., 2016; Meng et al., 2018).

This paper proposes less conservative LMI-based conditions for the existence of a polynomial fuzzy Lyapunov function. Exploring the structure of membership functions imposed by the sector nonlinearity approach (Tanaka and Wang, 2001) an alternative method is proposed to describe the polytopic representation of the time derivative of the membership functions presented in Geromel and Colaneri (2006); Mozelli and Adriano (2019). The strategy decreases the number of LMIs to be solved, reducing the conservativeness in the numerical process. This process was previously explored in the work by Elias et al. (2020) to obtain less conservative stability conditions using FLFs. In this work we extend the result for a switched polynomial Lyapunov function. A numerical example illustrates the efficiency of the proposed results.

2. PRELIMINARIES

Consider a nonlinear model

$$\dot{\mathbf{x}}(t) = f(\mathbf{z}(t))\mathbf{x}(t) \tag{1}$$

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where $f(\mathbf{z}(t))$ is a smooth nonlinear function, $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector and $\mathbf{z}(t) = \mathbf{L}\mathbf{x}(t)$ is the premise vector, with $\mathbf{L} \in \mathbb{R}^{p \times n_x}$, bounded in the compact set

$$\mathscr{C} = \{\mathbf{x}(t) \in \mathbb{R}^{n_x} : |x_i(t)| \leq \tilde{x}_i\}$$

where $i \in \{1, 2, \ldots, p\}$ is the set of indexes for state variables $x_i(t)$ which compose the premise vector and $\tilde{x}_i$ are parameters defined by designer. In this paper, we consider that $p \leq n_x$.

Using the sector nonlinearity approach (Tanaka and Wang, 2001; Taniguchi et al., 2001) system (1) can be exactly represented by the following TS fuzzy system:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{A}_i\mathbf{x}(t) \tag{2}$$

where $\mathbf{A}_i \in \mathbb{R}^{n_x \times n_x}$ are known constant matrices, r is the number of local models of the TS fuzzy system. The membership functions $h_i(\mathbf{z}(t))$ are built as

$$h_i(\mathbf{z}(t)) = \prod_{j=1}^{p} w^j_{k_{ij}}(\mathbf{z}_j) \tag{3}$$

with $k_{ij} \in \{0,\ 1\}$, $j \in \{1, \ldots, p\}$

$$w^j_0(\mathbf{z}(t)) = \frac{\overline{nl_j} - nl_j(\mathbf{z}(t))}{\overline{nl_j} - \underline{nl_j}} \quad w^j_1(\mathbf{z}(t)) = 1 - w^j_0(\mathbf{z}(t)) \tag{4}$$

and $nl_j(\mathbf{z}(t)) \in [\underline{nl_j}\ ,\ \overline{nl_j}]$ are the nonlinearities (Tanaka and Wang, 2001). By (3) and (4), $h_i(\mathbf{z}(t))$ satisfy the convex properties

$$\forall\, i \in \mathcal{R}, \quad h_i(\mathbf{z}(t)) \geq 0 \ \text{ and } \ \sum_{i=1}^{r} h_i(\mathbf{z}(t)) = 1 \tag{5}$$

with $\mathcal{R} := \{1, 2, \ldots, r\}$. By (5) it follows that

$$\sum_{i=1}^{r} \dot{h}_i(\mathbf{z}(t)) = 0. \tag{6}$$

When convenient, the argument of the functions will be omitted.

3. STABILITY ANALYSIS WITH POLYNOMIAL FUNCTIONS

This work uses a polynomial function

$$V(\mathbf{x}(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \mathbf{x}(t)'\mathbf{P}_{ij}\mathbf{x}(t) \tag{7}$$

as a Lyapunov candidate to study the dynamic behaviour of system (2). Function (7) is known in the literature as a polynomial Lyapunov function (PLF) and its time derivative is given by

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}(t)' \left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{i=k}^{r} h_i h_j h_k \left(\mathbf{A}'_k\mathbf{P}_{ij} + \mathbf{P}_{ij}\mathbf{A}_k\right) \right.$$
$$\left. + \sum_{i=1}^{r}\sum_{\rho=1}^{r} h_i \dot{h}_\rho \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i}\Big) \right] \mathbf{x}(t). \tag{8}$$

Due to property (6), function $\dot{V}(\mathbf{x}(t))$ is nonconvex. To analyze (2) using LMIs, it is necessary to obtain an alternative form of (8). Thus, for chosen parameters ϕ_ρ we define the set

$$\mathscr{F} = \{\mathbf{x}(t) \in \mathscr{C} : |\dot{h}_\rho| \leq \phi_\rho, \quad \rho \in \mathcal{R}\}. \tag{9}$$

Let $\mathscr{D}$ be

$$\mathscr{D} = \{\mathbf{x}(0) \in \mathscr{C} : \lim_{t \longrightarrow \infty} \mathbf{x}(t) = \mathbf{0}\}$$

the set of initial conditions for which the solutions of system (2) converges to the origin. An estimative of the domain attraction $\mathscr{D}$ can be obtained by the level set

$$\mathcal{B} = \{\mathbf{x}(\mathbf{t}) \in \mathscr{C} :\ V(\mathbf{x}(\mathbf{t})) < \mathbf{C}\} \tag{10}$$

where $\mathbf{C} > 0$ is the maximum value such that all the vectors $\mathbf{x}(t)$ belong to $\mathscr{F}$.

Taking into account (7) and (9), Lee et al. (2012) presented the following result.

Theorem 1. Let ϕ_ρ, $\rho \in \mathcal{R}$, be positive known numbers. Considering $|\dot{h}_\rho| \leq \phi_\rho, \forall \rho \in \mathcal{R}$, TS fuzzy system (2) is asymptotically stable, if there exist symmetric matrices $\mathbf{P}_{ij} \in \mathbb{R}^{n \times n}$, $\mathbf{M}_{ij} \in \mathbb{R}^{n \times n}$ and matrices $\mathbf{L}_i \in \mathbb{R}^{n \times n}$, $\mathbf{R}_i \in \mathbb{R}^{n \times n}$, satisfying the following LMIs

$$\mathbf{P}_{ij} + \mathbf{P}_{ji} \succ \mathbf{0}, \quad i \leq j, \quad i, j \in \mathcal{R} \tag{11}$$
$$\Upsilon_{ij} + \Upsilon_{ji} \prec \mathbf{0}, \quad i \leq j, \quad i, j \in \mathcal{R} \tag{12}$$
$$\tilde{\Upsilon}^\rho_{ij} + \tilde{\Upsilon}^\rho_{ji} \succ \mathbf{0}, \quad i \leq j, \quad i, j, \rho \in \mathcal{R} \tag{13}$$

where $\tilde{\Upsilon}^\rho_{ij} = \mathbf{P}_{\rho j} + \mathbf{P}_{i\rho} + \mathbf{M}_{ij}$ and

$$\Upsilon_{ij} = \begin{bmatrix} \mathbf{A}'_i\mathbf{L}'_j + \mathbf{L}_j\mathbf{A}_i + \sum_{\rho=1}^{r}\phi_\rho\tilde{\Upsilon}^\rho_{ij} & \star \\ \mathbf{P}_{ij} - \mathbf{L}'_i + \mathbf{R}'_j\mathbf{A}_i & -\mathbf{R}_i - \mathbf{R}'_i \end{bmatrix}.$$

Proof. See Theorem 2 of Lee et al. (2012).

To ensure the negativity of (8) using LMIs, Theorem 1 uses the relationship

$$\sum_{i=1}^{r}\sum_{\rho=1}^{r} h_i\dot{h}_\rho\Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i}\Big) <$$
$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{\rho=1}^{r} h_i h_j \phi_\rho \Big(\mathbf{P}_{\rho j} + \mathbf{P}_{i\rho} + \mathbf{M}_{ij}\Big), \tag{14}$$

which is conservative, to study the time derivative of the membership functions. To overcome this drawback, we use the representation given in Mozelli and Adriano (2019) and rewrite the left hand side of (14)

$$\sum_{i=1}^{r}\sum_{\rho=1}^{r} h_i\dot{h}_\rho\Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i}\Big) =$$
$$\sum_{i=1}^{r}\sum_{\ell=1}^{\eta}\sum_{\rho=1}^{r} h_i\alpha_\ell(t)\tilde{g}^\ell_\rho\Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i}\Big) \tag{15}$$

where $\alpha_\ell(t)$ are unknown functions satisfying

$$\alpha_\ell(t) \geq 0, \qquad \sum_{\ell=1}^{\eta} \alpha_\ell(t) = 1 \tag{16}$$

and $\tilde{g}_\rho^\ell$ is the element of the ρ^{th} row and ℓ^{th} column of matrix

$$\tilde{\mathbf{G}} := \left[\tilde{\mathbf{g}}^1, \tilde{\mathbf{g}}^2, \ldots, \begin{pmatrix} \tilde{g}_1^\ell \\ \tilde{g}_2^\ell \\ \vdots \\ \tilde{g}_r^\ell \end{pmatrix}, \ldots, \tilde{\mathbf{g}}^{\tilde{\eta}} \right] \tag{17}$$

that describe the region

$$\Omega := \{\tilde{g}^\ell \in \mathbb{R}^r; -\phi_\rho \leq \tilde{g}_\rho^\ell \leq \phi_\rho,\ c^T \tilde{g}^\ell = 0\} \tag{18}$$

with $c^T = [1, 1, \ldots, 1] \in \mathbb{R}^r$ and $\tilde{\eta} = \dfrac{r!}{(\frac{r}{2})!\ (\frac{r}{2})!}$.

Defining the set

$$\Sigma = \bigcap_{j,u} \left\{ \mathbf{x}(t) : \left| \frac{\partial w_0^j}{\partial z_j} x_u \right| \leq \lambda_{ju} \right\} \tag{19}$$

with $j \in \{1, 2, \ldots, p\}$, $u \in \{1, 2, \ldots, n_x\}$ and using (15) the following result is proposed.

Theorem 2. If there exist symmetric matrices $\mathbf{P}_{ij} \in \mathbb{R}^{n \times n}$, and matrices $\mathbf{L}_i \in \mathbb{R}^{n \times n}$, $\mathbf{R}_i \in \mathbb{R}^{n \times n}$, satisfying LMIs (20)-(22), then every solution of TS fuzzy system (2) contained in set (19), with $\tilde{\phi} = \max_{\forall j,u}\{\lambda_{ju}\}$, is attracted to the origin.

$$\mathbf{P}_{ij} + \mathbf{P}_{ji} \succ 0, \quad i \leq j \tag{20}$$

$$\Upsilon_{ii}^\ell \prec 0 \tag{21}$$

$$\frac{2}{r-1} \Upsilon_{ii}^\ell + \Upsilon_{ij}^\ell + \Upsilon_{ji}^\ell \prec 0, \quad i \neq j \tag{22}$$

where $i, j \in \mathcal{R}$, $\ell \in \{1, \ldots, \tilde{\eta}\}$

$$\Upsilon_{ij}^\ell = \begin{bmatrix} \mathbf{A}_i' \mathbf{L}_j' + \mathbf{L}_j \mathbf{A}_i + \tilde{\mathbf{P}}_{ij}^\ell & \star \\ \mathbf{P}_{ij} - \mathbf{L}_i' + \mathbf{R}_j' \mathbf{A}_i & -\mathbf{R}_i - \mathbf{R}_i' \end{bmatrix}$$

$$\tilde{\mathbf{P}}_{ij}^\ell = \tilde{\phi} \sum_{\rho=1}^{r} \tilde{g}_\rho^\ell (\mathbf{P}_{i\rho} + \mathbf{P}_{\rho j}).$$

Proof. Let us consider a PLF candidate (7). If (20) holds then $V(\mathbf{x}(t)) > 0$. Moreover, if (21) and (22) hold, then by Tuan et al. (2001) it follows that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Upsilon_{ij}^\ell \prec 0.$$

Multiplying inequality above by $\alpha_\ell(t)$ and adding the terms we obtain

$$\begin{bmatrix} \mathbf{A}(h)' \mathbf{L}(h)' + \mathbf{L}(h) \mathbf{A}(h) + \tilde{\mathbf{P}} & \star \\ \mathbf{P}(h) - \mathbf{L}(h) + \mathbf{R}(h) \mathbf{A}(h) & -\mathbf{R}(h) - \mathbf{R}'(h) \end{bmatrix} \prec 0 \tag{23}$$

where

$$\mathbf{A}(h) := \sum_{i=1}^{r} h_i \mathbf{A}_i, \quad \mathbf{L}(h) := \sum_{i=1}^{r} h_i \mathbf{L}_i$$

$$\mathbf{R}(h) := \sum_{i=1}^{r} h_i \mathbf{R}_i, \quad \mathbf{P}(h) := \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \mathbf{P}_{ij}$$

$$\tilde{\mathbf{P}} := \tilde{\phi} \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\ell=1}^{\eta} h_i h_j \alpha_\ell(t) \sum_{\rho=1}^{r} \tilde{g}_\rho^\ell \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho j} \Big)$$

Premultiplying and posmultiplying LMI (23) by matrix $[\mathbf{x}(t)'\ \ \mathbf{x}(t)'\mathbf{A}(h)']$ and its transpose, respectively, it yields

$$\mathbf{x}(t)' \left[\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{i=k}^{r} h_i h_j h_k \left(\mathbf{A}_k' \mathbf{P}_{ij} + \mathbf{P}_{ij} \mathbf{A}_k \right) + \tilde{\mathbf{P}} \right] \mathbf{x}(t) < 0. \tag{24}$$

For all $\mathbf{x}(t) \in \Sigma$ we have $|\dot{h}_\rho| \leq \tilde{\phi}$, $\forall \rho \in \mathcal{R}$. From convex property (5) and using (15), the following holds

$$\begin{aligned} \tilde{\mathbf{P}} &= \tilde{\phi} \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\ell=1}^{\eta} h_i h_j \alpha_\ell(t) \sum_{\rho=1}^{r} \tilde{g}_\rho^\ell \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho j} \Big) \\ &= \sum_{i=1}^{r} \sum_{\ell=1}^{\eta} h_i \alpha_\ell(t) \tilde{\phi} \sum_{\rho=1}^{r} \tilde{g}_\rho^\ell \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i} \Big) \\ &\geq \sum_{i=1}^{r} \sum_{\rho=1}^{r} h_i \dot{h}_\rho \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i} \Big). \end{aligned} \tag{25}$$

Then, substituting (25) into (24), it yields

$$\mathbf{x}(t)' \Bigg[\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{i=k}^{r} h_i h_j h_k \left(\mathbf{A}_k' \mathbf{P}_{ij} + \mathbf{P}_{ij} \mathbf{A}_k \right) + \sum_{i=1}^{r} \sum_{\rho=1}^{r} h_i \dot{h}_\rho \Big(\mathbf{P}_{i\rho} + \mathbf{P}_{\rho i} \Big) \Bigg] \mathbf{x}(t) < 0 \tag{26}$$

Then, by (15) and (26), (8) is negative which ensures that the solution of system (1) is attracted to the origin.

Altough Theorem 2 gives less conservative stability conditions than Theorem 1, the number of LMIs increases factorially with the number of local models, increasing the computational burden. Thus, in the next section, less conservative conditions are proposed. The result explores the use of a switching Lyapunov function and the properties of membership functions imposed by the sector nonlinearity approach (Tanaka and Wang, 2001; Taniguchi et al., 2001) to decrease the number of columns in matrix (17).

3.1 Relaxing the stability conditions

In this section, less conservative results are proposed using a polynomial Lyapunov function candidate of the form

$$V_{\sigma(\mathbf{x})}(\mathbf{x}(t)) = \mathbf{x}(t)' \mathbf{P}_{\sigma(\mathbf{x})}(h) \mathbf{x}(t) \tag{27}$$

where $\sigma(\mathbf{x}) : \mathbb{R}^{n_x} \to \mathcal{P}$, with $\mathcal{P} = \{1, 2, \ldots, p\}$, is a piecewise constant function of the state called switching law and

$$\mathbf{P}_{\sigma(\mathbf{x})}(h) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \mathbf{P}_{\sigma(\mathbf{x})ij}.$$

If system (2) is continuous and function $V_\sigma(\mathbf{x}(t))$ is decreasing, then it is possible to obtain a Lyapunov function for system (2) using the following switching law:

$$\sigma(\mathbf{x}) = \arg\left\{\min_{1\le\rho\le p} \mathbf{x}(t)'\left(\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \mathbf{P}_{\rho ij}\right)\mathbf{x}(t)\right\}. \tag{28}$$

In what follows, we consider a particular case of region Ω in (18) such that $\tilde{\phi} = \max_{\forall j,u}\{\lambda_{ju}\}$. Properties of the membership functions are investigated to reduce the conservatism of the LMI conditions given by Theorem 2.

Since $w_1^j = 1 - w_0^j$, it follows that

$$\dot{w}_1^j = -\dot{w}_0^j,\ \forall j \in \{1,\ldots,p\}. \tag{29}$$

Using properties (6) and (29), it is possible to establish relationships between functions $\dot{h}_i, i = 1,\cdots,r$, at the vertices of polytope Ω. That is, for any $j \in \{1,\ldots,p\}$, there are integer numbers ν and ρ such that

$$\dot{h}_{\nu+(\rho-1)2^j} = -\dot{h}_{\nu+(2\rho-1)2^{j-1}} \tag{30}$$

where $\rho \in \{1,\ldots,2^{p-j}\}$ and $\nu \in \{1,\ldots,2^{j-1}\}$. Property (30) is illustrated in Table 1 for r membership functions. By the relationships obtained in (30), a simplified convex description of matrix $\tilde{\mathbf{G}}$ in (17) can be obtained.

To exemplify the use of Table 1, consider $p = 2$. The columns 1 and 2 of Table 1 hold the following relationships respectively

$$\begin{bmatrix}\dot{h}_1\\ \dot{h}_2\\ \dot{h}_3\\ \dot{h}_4\end{bmatrix} = \begin{bmatrix}-\dot{h}_2\\ \dot{h}_2\\ -\dot{h}_4\\ \dot{h}_4\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\dot{h}_1\\ \dot{h}_2\\ \dot{h}_3\\ \dot{h}_4\end{bmatrix} = \begin{bmatrix}-\dot{h}_3\\ -\dot{h}_4\\ \dot{h}_3\\ \dot{h}_4\end{bmatrix}. \tag{31}$$

Using the left equality of (31) to describe the vertices of the polytope Ω we obtained the matrix $\mathbf{G_1}$. Following the same steps with the right side of (31) we obtained the matrix $\mathbf{G_2}$. Both matrices are showed below

$$\mathbf{G}_1 = \tilde{\phi}\begin{bmatrix}-1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\\ -1 & -1 & 1 & 1\\ 1 & 1 & -1 & -1\end{bmatrix},\ \mathbf{G}_2 = \tilde{\phi}\begin{bmatrix}-1 & 1 & -1 & 1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{bmatrix}. \tag{32}$$

Note that the addition of relationships of Table 1 decrease the vectors number of Ω by $\eta = 2^{r/2}$.

For comparison purposes, note that matrix $\tilde{\mathbf{G}}$ in (17), again for the particular case $\tilde{\phi} = \max_{\forall j,u}\{\lambda_{ju}\}$, is given by

$$\tilde{\mathbf{G}} = \tilde{\phi}\begin{bmatrix}-1 & -1 & 1 & -1 & 1 & 1\\ -1 & 1 & -1 & 1 & -1 & 1\\ 1 & -1 & -1 & 1 & 1 & -1\\ 1 & 1 & 1 & -1 & -1 & -1\end{bmatrix}. \tag{33}$$

As a matter of fact, matrix $\tilde{\mathbf{G}}$ in (33) has $\tilde{\eta} = 6$ while matrices $\mathbf{G}_j$, $j = 1,2$, in (32) have four columns. Taking into account Theorem 2 and Table 1, the following theorem is established.

Theorem 3. If there exist symmetric matrices $\mathbf{P}_{\beta ij} \in \mathbb{R}^{n\times n}$, and matrices $\mathbf{L}_{\beta i} \in \mathbb{R}^{n\times n}$, $\mathbf{R}_{\beta i} \in \mathbb{R}^{n\times n}$, satisfying LMIs (34)-(34), then every solution of TS fuzzy system (2) contained in set (19), with $\tilde{\phi} = \max_{\forall j,u}\{\lambda_{ju}\}$, is attracted to the origin.

$$\mathbf{P}_{\beta ij} + \mathbf{P}_{\beta ji} \succ 0, \quad i \le j \tag{34}$$

$$\Upsilon^{\ell}_{\beta ii} \prec 0 \tag{35}$$

$$\frac{2}{r-1}\Upsilon^{\ell}_{\beta ii} + \Upsilon^{\ell}_{\beta ij} + \Upsilon^{\ell}_{\beta ji} \prec 0, \quad i \ne j \tag{36}$$

where $i,j \in \mathcal{R}$, $\ell \in \{1,\ldots,\eta\}$, $\beta \in \{1,\ldots,p\}$

$$\Upsilon^{\ell}_{\beta ij} = \begin{bmatrix}\mathbf{A}_i'\mathbf{L}_{\beta j}' + \mathbf{L}_{\beta j}\mathbf{A}_i + \tilde{\mathbf{P}}^{\ell}_{\beta ij} & \star\\ \mathbf{P}_{\beta ij} - \mathbf{L}_{\beta i}' + \mathbf{R}_{\beta j}'\mathbf{A}_i & -\mathbf{R}_{\beta i} - \mathbf{R}_{\beta i}'\end{bmatrix}$$

$$\tilde{\mathbf{P}}^{\ell}_{\beta ij} = \tilde{\phi}\sum_{\rho=1}^{r} g^{\ell}_{\beta\rho}(\mathbf{P}_{\beta i\rho} + \mathbf{P}_{\beta\rho j})$$

and $g^{\ell}_{\beta\rho}$ is the element of the ρ^{th} row and ℓ^{th} column of matrix $\mathbf{G}_\beta$ given by relations of Table 1.

Proof. Let us consider a PLF candidate (27). If (34) holds then $V_{\sigma(\mathbf{x})}(\mathbf{x}(t)) > 0$. Moreover, if (35) and (36) hold, by Tuan et al. (2001) it follows that

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Upsilon^{\ell}_{\beta ij} \prec 0.$$

Following the same steps of the proof of Theorem 2 we obtain

$$\mathbf{x}(t)'\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{i=k}^{r} h_i h_j h_k\left(\mathbf{A}_k'\mathbf{P}_{\beta ij} + \mathbf{P}_{\beta ij}\mathbf{A}_k\right) + \tilde{\mathbf{P}}_\beta\right]\mathbf{x}(t) < 0. \tag{37}$$

For all $\mathbf{x}(t) \in \Sigma$ we have $|\dot{h}_\rho| \le \tilde{\phi}$, $\forall\rho \in \mathcal{R}$. By Table 1 and property (15), for any t there is at least one $\beta \in \{1,\ldots,p\}$, such that

$$\begin{aligned}\tilde{\mathbf{P}}_\beta &= \tilde{\phi}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{\ell=1}^{\eta} h_i h_j \alpha_\ell(t)\sum_{\rho=1}^{r} g^{\ell}_{\beta\rho}(\mathbf{P}_{\beta i\rho} + \mathbf{P}_{\beta\rho j})\\ &= \sum_{i=1}^{r}\sum_{\ell=1}^{\eta} h_i\alpha_\ell(t)\tilde{\phi}\sum_{\rho=1}^{r} g^{\ell}_{\beta\rho}\Big(\mathbf{P}_{\beta i\rho} + \mathbf{P}_{\beta\rho i}\Big)\\ &\ge \sum_{i=1}^{r}\sum_{\rho=1}^{r} h_i\dot{h}_\rho\Big(\mathbf{P}_{\beta i\rho} + \mathbf{P}_{\beta\rho i}\Big).\end{aligned} \tag{38}$$

Then, replacing (38) in (37) it yields

$$\begin{aligned}\mathbf{x}(t)'\Bigg[&\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{i=k}^{r} h_i h_j h_k\left(\mathbf{A}_k'\mathbf{P}_{\beta ij} + \mathbf{P}_{\beta ij}\mathbf{A}_k\right)\\ &+ \sum_{i=1}^{r}\sum_{\rho=1}^{r} h_i\dot{h}_\rho\Big(\mathbf{P}_{\beta i\rho} + \mathbf{P}_{\beta\rho i}\Big)\Bigg]\mathbf{x}(t) < 0.\end{aligned} \tag{39}$$

Thus, by (39) there is at least one β such that $\dot{V}_{\sigma(\mathbf{x})}(\mathbf{x}(t))$ is negative for any t, which ensures that the solution of system (2), with switching law (28), is attracted to the origin.

The efficiency of Theorem 3 is illustrated by two numerical examples.

Table 1. Relationships between the derivatives of the membership functions at the vertices

1	2	3	...	j	...	p
$\dot h_1 = -\dot h_2$ $\dot h_3 = -\dot h_4$ $\dot h_5 = -\dot h_6$ $\dot h_7 = -\dot h_8$ $\dot h_9 = -\dot h_{10}$ $\dot h_{11} = -\dot h_{12}$ $\dot h_{13} = -\dot h_{14}$ $\dot h_{15} = -\dot h_{16}$ $\vdots$	$\dot h_1 = -\dot h_3$ $\dot h_2 = -\dot h_4$ $\dot h_5 = -\dot h_7$ $\dot h_6 = -\dot h_8$ $\dot h_9 = -\dot h_{11}$ $\dot h_{10} = -\dot h_{12}$ $\dot h_{13} = -\dot h_{15}$ $\dot h_{14} = -\dot h_{16}$ $\vdots$	$\dot h_1 = -\dot h_5$ $\dot h_2 = -\dot h_6$ $\dot h_3 = -\dot h_7$ $\dot h_4 = -\dot h_8$ $\dot h_9 = -\dot h_{13}$ $\dot h_{10} = -\dot h_{14}$ $\dot h_{11} = -\dot h_{15}$ $\dot h_{12} = -\dot h_{16}$ $\vdots$	...	$\dot h_1 = -\dot h_{1+2^{j-1}}$ $\dot h_2 = -\dot h_{2+2^{j-1}}$ $\vdots$ $\dot h_{2^{j-1}} = -\dot h_{2\times 2^{j-1}}$ $\dot h_{1+2\times 2^{j-1}} = -\dot h_{1+3\times 2^{j-1}}$ $\dot h_{2+2\times 2^{j-1}} = -\dot h_{2+3\times 2^{j-1}}$ $\vdots$ $\dot h_{3\times 2^{j-1}} = -\dot h_{4\times 2^{j-1}}$ $\vdots$	...	$\dot h_1 = -\dot h_{1+2^{p-1}}$ $\vdots$
$\dot h_{r-2^0} = -\dot h_r$	$\dot h_{r-2^1} = -\dot h_r$	$\dot h_{r-2^2} = -\dot h_r$	...	$\dot h_{r-2^{j-1}} = -\dot h_r$	...	$\dot h_{r-2^{p-1}} = -\dot h_r$

NUMERICAL EXAMPLE

Consider the same numerical TS fuzzy system given in Faria et al. (2013)

$$\mathbf{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -4 & -4 \\ \dfrac{(3b-2)}{5} & \dfrac{3a-4}{5} \end{bmatrix}$$
$$\mathbf{A}_3 = \begin{bmatrix} -3 & -4 \\ \dfrac{2b-3}{5} & \dfrac{2a-6}{5} \end{bmatrix}, \mathbf{A}_4 = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix} \tag{40}$$

and membership functions, for $|x_i| \leq \pi/2$,

$$\begin{aligned} h_1 &= w_0^1(x_1)w_0^2(x_2), & h_2 &= w_0^1(x_1)w_1^2(x_2) \\ h_3 &= w_1^1(x_1)w_0^2(x_2), & h_4 &= w_1^1(x_1)w_1^2(x_2) \end{aligned} \tag{41}$$

with $w_0^i(x_i) = \dfrac{1-\sin(x_i)}{2}$, $w_1^i(x_i) = 1 - w_0^i(x_i)$.

Stability of system (2), with local models (40) and membership functions (41), was verified with Theorems 1, 2 and 3. The LMIs were solved using YALMIP (Löfberg, 2004) and SeDuMi (Sturm, 1999) with parameters $\phi_\rho = \tilde{\phi} = 0.9425$, $\forall \rho \in \mathcal{R}$, $a \in [-3 \quad -1]$ and $b \in [0 \quad 200]$. Figure 1 shows the stable region of each compared method.

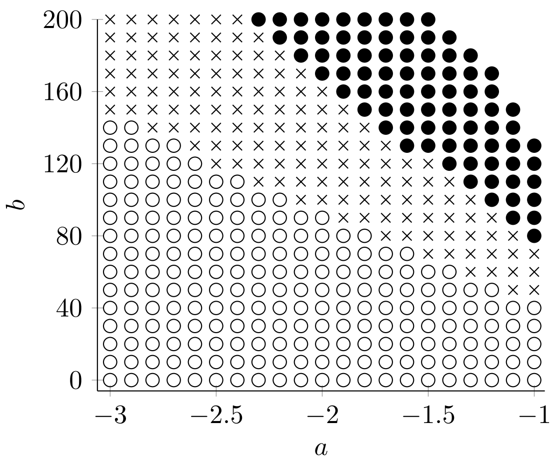

Fig. 1. Feasible regions of Theorem 1 (○), Theorem 2 (○, ×) and Theorem 3 (○, ×, ●).

By Fig. 1, the stable region of Theorem 3 covers the stable regions of Theorems 1 and 2, ensuring that LMIs (34) are less conservative. The LMI relaxation was obtained reducing the number of columns in matrix $\tilde{\mathbf{G}}$ by using property (30) (see Table 1, for $p = 2$).

Another advantage of Theorem 3 regarding Theorems 1 and 2 is the basin of attraction (10). For instance, taking parameters $a = -2$ and $b = 90$, by Fig.1 all theorems are feasible. However, they can ensure the feasibility using different maximum values for parameter $\tilde{\phi}$. The maximum value $\tilde{\phi}$ of Theorems 1, 2 and 3, respectively, are $0.94, 1.65$ and 3.77.

The impact of parameter $\tilde{\phi}$ in the estimate of the basin of attraction is showed in Fig. 2.

By Fig. 2, the best estimate of the basin of attraction $\mathcal{B}$ is given by Theorem 3. Moreover, Fig. 2 shows the time response of system for initial condition $\mathbf{x}_0 = [-0.2 \quad 0.35]$.

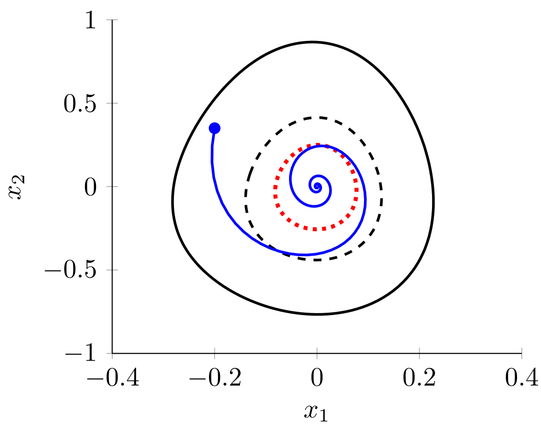

Fig. 2. Basin of attractions of Theorem 1 (dotted line), Theorem 2 (dashed line) and Theorem 3 (solid line).

The dynamical behavior of the Lyapunov candidate function (27) with switching law (28) are illustrated in Fig. 3. Since $V_\sigma(\mathbf{x}(t))$ is positive and decreasing, than it represents a Lyapunov function for system (2).

4. CONCLUSIONS

LMI conditions for the stability analysis of TS fuzzy systems were proposed. An alternative way to describe the time derivative of the membership functions combined with a polynomial Lyapunov function gave less conservative conditions for the feasibility of the LMI constraints. Futhermore, the results can also provide better estimation of the attraction basin.

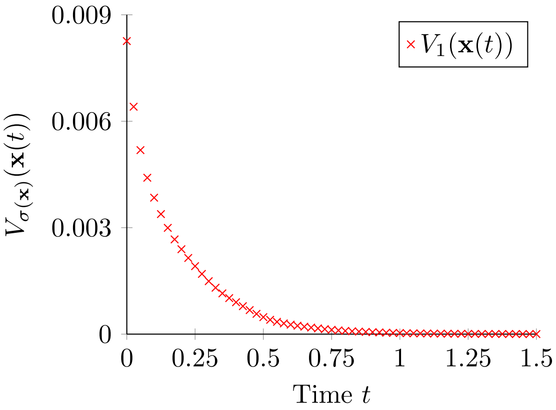

Fig. 3. Dynamical behavior of (27) with switching law (28).

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