

Solutions of Stable Difference Equations Probably Experience Peak^{*}

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Abstract: From the literature, it is known that solutions of homogenous linear stable difference equations may experience large deviations, or peaks, from the nonzero initial conditions at finite time instants. In this paper we take a probabilistic standpoint to analyze these phenomena by assuming that both the initial conditions and the coefficients of the equation have random nature. Under these assumptions we find the probability for deviations to occur, which turns out very close to unity even for equations of low degree, which means that peak is typical. We also address other issues such as evaluation of the mean magnitude and maximum value of peak.

Keywords: linear difference equations, stability, nonzero initial conditions, non-asymptotic behavior, deviations, probability

1. INTRODUCTION

It is a well known fact (Feldbaum, 1948; Izmailov, 1987; Polyak and Smirnov, 2016; Polyak et al., 2018) that solutions of autonomous linear stable differential or difference equations may experience large deviations from nonzero initial conditions at finite time instants. Many negative implications may follow, such as for instance instability of computational schemes in numerical methods, invalidity of linearized models in the vicinity of an equilibrium point, poor behavior or breakdown of control systems. These deviations are referred as *peaks*, and since exact values of the magnitude of peak generally cannot be found in closed form, computable lower and upper bounds are of large interest. Also, design of control inputs that possibly diminish or minimize peaks is of great importance.

For the continuous-time case, research in this direction has been initiated as early as in Feldbaum (1948), see also Izmailov (1987) for a modern formulation and a bright result. Seemingly, the most recent paper on peak effects in differential equations (continuous-time systems) is Polyak and Smirnov (2016); see bibliography therein. Also see Whidborne and McKernan (2007); Polyak et al. (2015) for the vector case, where LMI-based upper bounds on the magnitude of peak and design of peak-minimizing state feedback are proposed.

Until very recently, no analogous research has been conducted for linear *difference equations* (discrete time). Whereas basic textbooks on difference equations as, e.g., the popular Elaydi (2005), provide numerous results on the behavior of linear and nonlinear equations, stability theory, and applications, they generally do not consider peak phenomena.

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Of immediate interest is the scalar case: to the best of our knowledge the only works directly related to this issue are Shcherbakov (2017); Polyak et al. (2018); Shcherbakov (2019). Similarly, only few results on peak effects are available for vector difference equations; e.g., see Kogan and Krivdina (2011); Shcherbakov and Parsegov (2018); Ahievich et al. (2018); Dudarenko et al. (2019); Polyak and Smirnov (2019).

In the most comprehensive paper (Polyak et al., 2018) related to the *scalar case*, several estimates of peaks in stable linear difference equations are provided under various assumptions on the initial conditions and root locations, and closed-form expressions for peak and peak instant, or bounds on these quantities, are proposed. In particular, a detailed analysis of the non-asymptotic behavior of a special class of trinomial high-order difference equations is performed in (Shcherbakov, 2019). Importantly, these results show that peak may take arbitrarily large values even for low-order equations.

Overall, though being very interesting, the results on peak effects that have been so far obtained in the literature relate to various *particular cases*, or classes of difference equations and initial conditions, whereas the general picture still remains quite unclear, since it is very hard to make conclusions about the non-asymptotic behavior of a “generic” polynomial with “generic” initial conditions.

To make a step towards a deeper understanding of peak phenomena, in this paper we adopt a probabilistic viewpoint, and assume a random nature of the coefficients of stable equations and/or initial conditions. The goal is to answer questions such as: How typical the peak is for a Schur stable polynomial and for initial conditions in the unit cube? What is the mean magnitude of peak? What is the portion of initial conditions in the unit box that yield peak for a given polynomial? and the like. In

the discrete-time case, the set of stable roots and hence, the coefficients of stable polynomials belong to bounded sets, which facilitates probabilistic analysis of the peak phenomena.

Part of the formulations discussed in the current paper were mentioned in a preliminary form in (Shcherbakov, 2019; Polyak et al., 2018).

The outline of the paper is pretty much standard. The next section provides notation and definitions required for the exposition to follow. In Section 3, first main results on the probability of peak and its magnitude are presented, accompanied by discussion and numerical illustrations. In Section 4 we present randomized algorithms for peak estimation and the results of numerical simulations. Finally, Section 5 closes the paper.

2. NOTATION AND DEFINITIONS

We consider generic n th order scalar linear difference equation

$$x_k + a_1x_{k-1} + \dots + a_nx_{k-n} = 0, \quad k = n, n+1, \dots, \quad (1)$$

with real coefficients $a_i \in \mathbb{R}$ and initial conditions

$$x^{(0)} = (x_0, \dots, x_{n-1})^\top \in \mathbb{R}^n. \quad (2)$$

The characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n \quad (3)$$

is assumed to be Schur stable, i.e., $|\lambda_i| < 1$ for all roots λ_i of $p(\lambda)$; denote

$$a = (a_1, \dots, a_n)^\top \in \mathbb{R}^n, \quad \Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}.$$

We use a to denote both the polynomial and the vector $a = (a_1, \dots, a_n)^\top$ of its coefficients. Also denote by

$$\mathcal{S}_n = \{a \in \mathbb{R}^n : (3) \text{ is Schur stable}\} \quad (4)$$

the *Schur domain* in the coefficient space, and by

$$\mathcal{C}_n = \{a \in \mathbb{R}^n : \|a\|_1 < 1\} \subset \mathcal{S}_n \quad (5)$$

the so-called *Cohn domain*; the condition $a \in \mathcal{C}_n$ is a well-known simple sufficient condition for (3) to be Schur stable; e.g., see Elaydi (2005).

Unless otherwise stated, we assume that $x^{(0)}$ belongs to the unit box:

$$x^{(0)} \in \mathcal{B}_n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}.$$

This is without loss of generality, since the solution of (1), (2) depends linearly on the initial conditions, and the assumption above is just a matter of scaling.

Of our interest is the following quantity:

$$\eta(a, x^{(0)}) = \max_{k=n, n+1, \dots} |x_k|, \quad (6)$$

and we say that the trajectory *experiences peak* if $\eta(a, x^{(0)}) > \|x^{(0)}\|_\infty$.

We say that $x_*^{(0)} \in \mathcal{B}_n$ is the *worst-case initial condition* if

$$x_*^{(0)} = \arg \max_{x^{(0)} \in \mathcal{B}_n} \eta(a, x^{(0)}).$$

By Assertion 2 in (Shcherbakov, 2019), the worst-case initial condition is attained at a vertex of \mathcal{B}_n , and sometimes we will pay special attention to the vertices as well as to the whole surface of \mathcal{B}_n . In the latter case, peak takes place if $\eta(a, x^{(0)}) > 1$.

Finally, the notation $\xi \sim U(\mathcal{X})$ is intended to mean that the random vector/scalar ξ has the uniform distribution over the set $\mathcal{X} \subset \mathbb{R}^n$, and $\text{Prob}\{A\}$ denotes the probability of the event A .

3. MAIN RESULTS

In this section we present several results on evaluation of peak by assuming the random nature of the coefficients a and/or the initial conditions $x^{(0)}$.

3.1 Probability of potential peak

Let us define the following *potential peak domain*:

$$\mathcal{P}_n = \{a \in \mathcal{S}_n : \exists x^{(0)} \in \mathcal{B}_n : \eta(a, x^{(0)}) > \|x^{(0)}\|_\infty\} \quad (7)$$

which represents the set of all n th order stable difference equations that experience peak for *some* initial conditions.

We have the following result (due to space limitations, proofs of all assertions are omitted).

Theorem 1. The condition $\|a\|_1 > 1$ is necessary and sufficient for the existence of $x^{(0)}$ such that $\eta(a, x^{(0)}) > \|x^{(0)}\|_\infty$. That is

$$a \in \mathcal{P}_n \iff \|a\|_1 > 1.$$

From this theorem, the set of stable difference equations which do not experience peak, no matter what $x^{(0)}$ is, is seen to coincide with the Cohn domain \mathcal{C}_n defined in (5). In other words, we have

$$\mathcal{P}_n = \mathcal{S}_n \setminus \mathcal{C}_n.$$

In the following, we are interested in determining the Lebesgue measure (volume, Vol) of the set \mathcal{P}_n . To this end, we first highlight that there exists a recursive formula for $\text{Vol}(\mathcal{S}_n)$, originally proposed in Fam (1989) (also see Tempo et al. (2013); Polyak and Halpern (2001), and Shcherbakov and Dabbene (2011) for various applications):

$$\begin{aligned} \text{Vol}(\mathcal{S}_1) &= 2, & \text{Vol}(\mathcal{S}_2) &= 4, & \text{Vol}(\mathcal{S}_3) &= 16/3, \\ \text{Vol}(\mathcal{S}_{n+1}) &= \begin{cases} \frac{\text{Vol}(\mathcal{S}_n)^2}{\text{Vol}(\mathcal{S}_{n-1})} & \text{for } n \text{ odd,} \\ \frac{n\text{Vol}(\mathcal{S}_n)\text{Vol}(\mathcal{S}_{n-1})}{(n+1)\text{Vol}(\mathcal{S}_{n-2})} & \text{for } n \text{ even.} \end{cases} \end{aligned} \quad (8)$$

Now, since

$$\text{Vol}(\mathcal{C}_n) = \frac{2^n}{n!},$$

the *exact* probability of peak (in the sense of (7)) can be computed. Specifically, for a given $x^{(0)} \in \mathcal{B}_n$ and the coefficient vector $a \sim U(\mathcal{S}_n)$, consider the random variable

$$\eta_{x^{(0)}}(a) = \max_k \{|x_k| : (1), (2)\};$$

then

$$\begin{aligned} P_a(n) &\doteq \text{Prob}\{a \in \mathcal{S}_n : \exists x^{(0)} \in \mathcal{B}_n : \eta(a, x^{(0)}) > \|x^{(0)}\|_\infty\} \\ &= \text{Prob}\{\mathcal{P}_n\} = 1 - \frac{\text{Vol}(\mathcal{C}_n)}{\text{Vol}(\mathcal{S}_n)}. \end{aligned}$$

Numerical values of this probability are given in Table 1. The probability of peak is seen to be rather high already

n	1	2	3	4	5	6	7	8	9	10
$\text{Vol}(\mathcal{S}_n)$	2	4	5.3333	7.1111	7.5852	8.0909	7.3974	6.7633	5.4965	4.4670
$P_a(n)$	0	0.5	0.75	0.9062	0.9648	0.9890	0.9966	0.9991	0.9997	0.9999

Table 1. Probability of peak for increasing degree n

for second ($P_a(2) = 1/2$) and third ($P_a(3) = 2/3$) order equations, and it rapidly converges to unity for increasing values of n . Observe that $P_a(20) \approx 1 - 10^{-11}$.

In other words, when sampling the coefficients of polynomials randomly uniformly in \mathcal{S}_n , peak is a typical phenomenon if understood in the sense of (7).

3.2 A special case

The conclusion of the previous analysis is that high-order difference equations generically experience peak. However, this is not the case if we restrict the analysis to specific classes of high-order difference equations. For instance, we consider next the $(n + 1)$ st order trinomial equation

$$x_{k+1} + a_1 x_k - a_2 x_{k-n} = 0, \quad k = n + 1, n + 2, \dots, \quad (9)$$

where $a = (a_1, a_2)^\top \in \mathbb{R}^2$ are the only two nonzero coefficients (while all other coefficients are zero). The associated characteristic polynomial is

$$p(\lambda) = \lambda^{n+1} - a_1 \lambda^n + a_2.$$

This equation represents one of the commonly used linearized models of the population dynamics; it was first analyzed in (Kuruklis, 1994), where the boundary of its Schur domain $\mathcal{S}_n(a_1, a_2)$ on the plane (a_1, a_2) was computed in ‘‘closed form.’’ Later, it became the subject of numerous generalizations in (Dannan, 2004; Matsunaga, 2007), etc.; also, see (Elaydi, 2005) for discussions.

The set $\mathcal{S}_n(a_1, a_2)$ is depicted in Fig. 1 for $n = 3$, where its parts, the peak domains and the Cohn domain are denoted by \mathcal{P}_n (the winglets) and \mathcal{C}_n , respectively. Modulo certain symmetry, this domain has the same shape for even values of n .

As in the generic case, it is possible to evaluate the probability $P_a(n)$ of potential peak.

Theorem 2. For equation (9) and the set \mathcal{P}_n defined in (7), the following estimate holds:

$$\text{Prob}\{a \in \mathcal{P}_n\} < \frac{1}{n + 1}.$$

In other words, in contrast to the generic case above, this probability *decreases* as the degree n grows.

3.3 Probability of peak for a given equation

In the previous sections we considered random coefficients in \mathcal{S}_n , and evaluated the probability of the equation to expose peak.

Now, conversely, we let the coefficient vector a in (1) be fixed, let $x^{(0)}$ be uniformly distributed in \mathcal{B}_n , and consider the random variable

$$\eta_a(x^{(0)}) = \max_k \{|x_k| : (1), (2)\}.$$

Our goal is then to estimate

$$P_{x^{(0)}}(a, n) \doteq \text{Prob}\{x^{(0)} \in \mathcal{B}_n : \eta_a(x^{(0)}) > \|x^{(0)}\|_\infty\}, \quad (10)$$

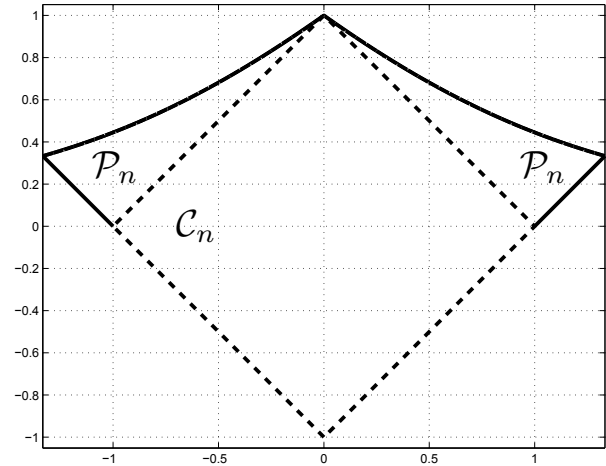


Fig. 1. Stability domain and peak domains in the coefficient space of (9) for $n = 3$

i.e., the portion of initial conditions in \mathcal{B}_n that yield peak for a given equation.

We first remark that a simple lower bound for $P_{x^{(0)}}$ is immediate to obtain. Indeed, from (1) we have

$$x_n = -(a_1 x_{n-1} + \dots + a_n x_0) = -\bar{a}^\top x^{(0)},$$

where $\bar{a} \doteq \text{fliplr}(a)$, the reverse-order vector. Hence, $|\bar{a}^\top x^{(0)}| > \|x^{(0)}\|_\infty$ specifies all initial conditions in \mathcal{B}_n that yield peak at the first iteration. Note however that computing the volume of this portion, we obtain just a lower bound on the probability.

Remark 1. (Lower bound computation). To compute the lower bound of (10), we need to solve the following problem: Given fixed $\bar{a} \in \mathbb{R}^n$, find the volume of the set

$$\mathcal{D} = \{x \in \mathcal{B}_n : |\bar{a}^\top x| > \|x\|_\infty\} \subseteq \mathcal{B}_n.$$

Obviously, we have $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$, where

$$\mathcal{D}_+ = \{x \in \mathcal{B}_n : \bar{a}^\top x > \|x\|_\infty\},$$

and \mathcal{D}_- is its symmetric with respect to the origin. Hence, $\text{Vol}(\mathcal{D}) = 2\text{Vol}(\mathcal{D}_+)$.

We note then that \mathcal{D}_+ consists of two convex hyperpyramids \mathcal{R}_0 and \mathcal{R}_1 with common base and apexes $v_0 = (0, \dots, 0)$ and $v_1 = \text{sign}(\bar{a})$ (proper vertex of \mathcal{B}_n), respectively. The rest of the vertices v_i (those of the base) are the intersection points of the hyperplane

$$\mathcal{H} = \{x \in \mathbb{R}^n : \bar{a}^\top x = 1\}$$

and the edges of the hyperbox \mathcal{B}_n . Note that the intersection between a hyperbox and a hyperplane describes a so-called hyperpolygon. The problem of computing the vertices of that hyperpolygon has been addressed in the literature, and computationally efficient algorithms are available. For instance, the algorithm in Lara et al. (2009) exhibits time complexity of $O(mn)$, where m is the number of solutions (intersection points). Note also that the volume of a hyperpyramid is given by the formula

$$\text{Vol}(\mathcal{R}) = \frac{1}{n} Ah$$

where A is the the $(n - 1)$ -dimensional volume of the base and h is the height.

An illustration of the sets of interest is given in Figure 2 for $n = 3$ and $\bar{a} = (1, 0.5, 0.25)^\top$; for ease of visualization, only the set \mathcal{D}_+ is depicted. The main vertices v_0 and v_1 are given in green, the vertices of the base are plotted in black.

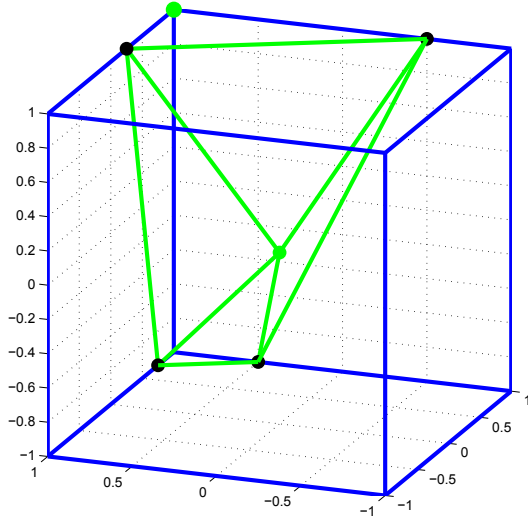


Fig. 2. A 3D illustration of the set \mathcal{D}_+ in Section 3.3

3.4 Expected peak magnitude

In this section, we shift our focus from the evaluation of the probability of experiencing peak to the evaluation of the possible *magnitude* of peak. We still take a probabilistic perspective and we aim at evaluating the expected value of the peak magnitude.

We concentrate our analysis on the special case of equal roots $\lambda_i \equiv \rho$ in (1), for which an explicit formula for the solution exists (Polyak et al., 2018):

$$x_k = x_0 P_0(k) \rho^k + x_1 P_1(k) \rho^{k-1} + \dots + x_{n-1} P_{n-1}(k) \rho^{k-n+1}, \quad (11)$$

where

$$P_i(k) = \prod_{j \neq i} \frac{k-j}{i-j}, \quad i = 0, 1, \dots, n-1, \quad j = 0, 1, \dots, n-1.$$

In particular, for $n = 2$ we have

$$x_k = -x_0(k-1)\rho^k + x_1 k \rho^{k-1},$$

and for $n = 3$ the explicit expression writes

$$x_k = x_0 \frac{(k-1)(k-2)}{2} \rho^k - x_1 k(k-2) \rho^{k-1} + x_2 \frac{k(k-1)}{2} \rho^{k-2}.$$

We let the vector $x^{(0)} = (x_0, \dots, x_{n-1})^\top$ of initial conditions be uniformly distributed over the unit box \mathcal{B}_n . Since $x_i, i = 0, \dots, n-1$, are independent and $\mathbb{E}\{x_i\} = 0$, from (11) we have

$$\mathbb{E}\{x_k\} = 0$$

and

$$\mathbb{E}\{x_k^2\} = \frac{1}{3} \sum_{i=0}^{n-1} P_i^2(k) \rho^{2(k-i)}, \quad (12)$$

i.e., a closed-form expression for the expectation and the variance of the random variables x_k .

It is hardly possible to obtain closed-form expression for the *peak magnitude* of this quantity, but simulations show that it can take very large values. Let $\eta_{x^{(0)}}(n, \rho)$ denote the random variable, value of peak for the n th order equation with all roots $\lambda_i = \rho$ for random initial conditions $x^{(0)} \sim U(\mathcal{B}_n)$. Then numerically, for the averaged squared peak we have

$$\mathbb{E}\{\eta_{x^{(0)}}^2(2, 0.9)\} > 8 \quad \text{and} \quad \mathbb{E}\{\eta_{x^{(0)}}^2(4, 0.9)\} \approx 2.5 \cdot 10^5,$$

the latter being a very large number (note however that we are considering squared quantities).

It should be observed that expression (12) gives the “absolute value” of the averaged squared x_k s, whereas finding the associated $\|x^{(0)}\|_\infty$ -normalized quantity would be more ostensive. To this end, the following result holds.

Theorem 3. Let $x_k, k = n, n+1, \dots$ be a solution of the stable n th order equation (1)–(2) with all equal roots $\lambda_i \equiv \rho$ and random initial conditions $x^{(0)} \sim U(\mathcal{B}_n)$. Then the mean value of the squared normalized solution is given by

$$\mathbb{E}\left\{\frac{x_k^2}{\|x^{(0)}\|_\infty^2}\right\} = \frac{n+2}{3n} \sum_{i=0}^{n-1} P_i^2(k) \rho^{2(k-i)}. \quad (13)$$

Note that our final goal would be to find the maximum over k in (13), since $\eta_{x^{(0)}}^2(n, \rho) = \max_k \frac{x_k^2}{\|x^{(0)}\|_\infty^2}$.

This is the subject of ongoing research. However, we observe that the result in (13) may provide some insight: for instance, we see that the expected peak would have a direct dependence on the root magnitude ρ and on the degree n , i.e., the larger ρ and/or the degree n , the larger the peak. This is indeed confirmed by our numerical observations, and is in concordance with the deterministic results for polynomials with all equal roots (Polyak et al., 2018).

Example 2. To illustrate the result in Theorem 3, we perform the following experiment. Let us fix n, ρ and numerically find the maximum of the right-hand side of (13) over k ; this will give us the expected value of peak in the sense of (13). We then vary ρ from 0.4 to 0.99 and repeat for $n = 2, 3, 4, 5$. The results are presented in Fig. 3 where the values of (13) are plotted in the logarithmic scale for the indicated values of ρ and n .

These results show that, even for moderate degrees, the averaged peak take huge values.

4. RANDOMIZED ALGORITHMS FOR PEAK ESTIMATION

4.1 Random coefficients and initial conditions

In this section we propose randomized algorithms for evaluating both the probability of peak and the magnitude of it.

More specifically, we assume that both the coefficients of the equation and the initial conditions are random, uniformly distributed in the respective domains, and we are interested in estimating the “total” probability of peak, defined as follows:

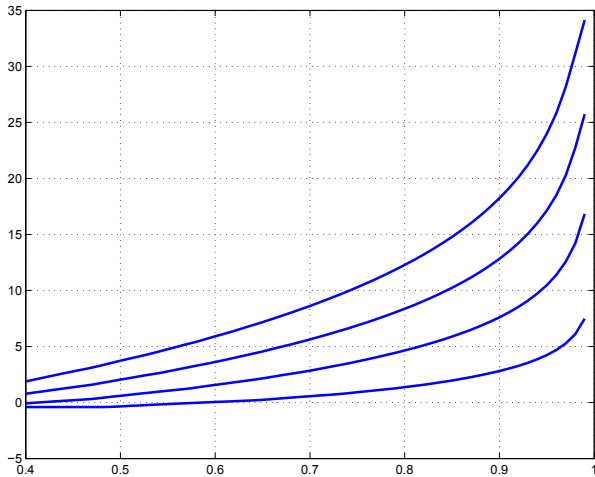


Fig. 3. Expected values of the mean squared peak for $\rho \in [0.4, 0.99]$ and $n = 2, 3, 4, 5$ (bottom up)

$$P_{a,x^{(0)}}(n) = \text{Prob}\{a \in \mathcal{S}_n, x^{(0)} \in \mathcal{B}_n: \eta_a(x^{(0)}) > \|x^{(0)}\|_\infty\}. \quad (14)$$

Also, we are interested in deriving estimates on the magnitude of such peak.

To this end, we make recourse to randomized estimation schemes (Tempo et al., 2013).

In particular, for fixed degree n , we fix accuracy value $\epsilon \in (0, 1)$, confidence value $\delta \in (0, 1)$, and sample N pairs $(a[i], x^{(0)}[i])$, with $a[i] \sim U(\mathcal{S}_n)$ and $x^{(0)}[i] \sim U(\mathcal{B}_n)$, $i = 1, \dots, N$. Note that sampling uniformly Schur stable equations is doable by using the technique from (Andrieu and Doucet, 1999); also see (Tremba et al., 2008) for implementation.

For every i -th pair, we compute numerically the solution x_k for $k = 1, \dots, k_{\max}$ and let $\eta[i] \doteq \max_k |x_k| / \|x^{(0)}\|_\infty$, $i = 1, \dots, N$.

Then, we can immediately estimate the total peak probability in (14) by the *empirical probability*

$$\hat{P}_{a,x^{(0)}}(n) = \frac{N_{\text{peaks}}}{N},$$

where N_{peaks} is the overall amount of peaks (those for which $\eta[i] > 1$).

It follows from the well-known Chernoff Bound (Chernoff, 1952; Tempo et al., 2013) that if we choose N such that

$$N \geq N_{\text{Cher}}(\epsilon, \delta) \doteq \frac{1}{2\epsilon^2} \log \frac{2}{\delta},$$

then, with probability greater than $1 - \delta$, we have

$$\left| P_{a,x^{(0)}}(n) - \hat{P}_{a,x^{(0)}}(n) \right| \leq \epsilon.$$

To estimate the *magnitude* of the “total” peak, we instead make recourse to a recent result by Alamo et al. (2018) that introduces a generalized notion of max function and shows how it can be used to provide a probabilistic upper bound on a given random variable.

To this end, we sort the N values $\eta[i]$, $i = 1, \dots, N$, in the nonincreasing order and define the *r-generalized maximum* η_{\max}^r as the r th entry of this ordered sequence.

In other words, define η_{\max}^r as the largest value of $\eta[i]$ after discarding the $r - 1$ largest values.

Clearly, for $r = 1$ one recovers the classical definition of maximum, i.e., $\eta_{\max}^1 = \eta_{\max} = \max_{i=1, \dots, N} \eta[i]$.

The following result is a direct consequence of (Alamo et al., 2018, Property 3).

Lemma 1. With the quantities defined above, for given $\epsilon, \delta \in (0, 1)$, and $r \in 1, \dots, N$, if

$$\sum_{k=0}^{r-1} \binom{N}{k} \epsilon^k (1 - \epsilon)^{N-k} \leq \delta, \quad (15)$$

then, with probability no smaller than $1 - \delta$, the following estimate holds

$$\text{Prob}\{\eta_a(x^{(0)}) > \eta_{\max}^r\} \leq \epsilon.$$

In words, almost certainly¹, the probability of obtaining a peak magnitude larger than the estimate η_{\max}^r can be made arbitrarily small by appropriately choosing ϵ and r . In particular, in Alamo et al. (2018) it is shown that (15) is satisfied if we choose

$$N \geq N_{\text{Tco}}(\epsilon, \delta, r) \doteq \frac{1}{\epsilon} \left(r - 1 + \ln \frac{1}{\delta} + \sqrt{2(r-1) \ln \frac{1}{\delta}} \right). \quad (16)$$

In our experiments, we let $\epsilon = 0.01$, $\delta = 10^{-6}$ and chose $r = 500$, obtaining $N_{\text{Cher}} = 65,612$ and $N_{\text{Tco}} = 63,024$, respectively. Hence, we draw $N = 66,000$ random pairs of equation/initial condition for different values of $n = 2, \dots, 7$. Since the considered degrees n are relatively low, we took $k_{\max} = 100$ “to make sure” we grasp the peak instant. The results are presented in Table 2.

From the first row of the table, we observe that the “total” peak probability $\hat{P}_{a,x^{(0)}}(n)$ grows as n grows. Also, whereas very large peaks are observed (see the second row), these are sort of exotic, since they have a rather low probability of occurring, as the last row clearly shows.

5. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we studied peak phenomena in stable difference equations from the probabilistic point of view, by assuming random nature of the characteristic polynomial and initial conditions. We showed that peak is “typical” for a generic equation and its magnitude may potentially take huge values, whereas on average, large peaks are not likely to occur.

Directions of future research are numerous and diverse. For instance, more specific results are desired on the evaluation of the probability of peak for a given polynomial, as in Section 3.3; a closed-form estimate of the mean value of peak in the spirit of the results in Section 3.4; evaluation of peak domains in the coefficient space for a specific initial conditions, and the like. Probabilistic analysis of classes of equations (e.g., those similar to (9)) are also of interest.

Beyond the scalar case considered here, analysis of peaks of norms of Schur stable *matrices* is highly demanded as well as the issues of robustness and peak-minimizing design.

¹ That is, with probability $1 - \delta$, but note that, due to the logarithm in (16), the level δ can be chosen very small.

n	2	3	4	5	6	7
$\widehat{P}_{a,x(0)}(n)$	0.2268	0.2652	0.3549	0.4008	0.4628	0.5052
η_{\max}	28.2440	54.2223	61.0974	97.4572	124.3977	162.7770
η_{\max}^r	3.4605	4.4748	5.9160	6.6763	7.7265	8.5282

Table 2. Results of the numerical experiment in Section 4.1

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