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# Approximate Nash Equilibrium Solutions of Linear Quadratic Differential Games

**Domenico Cappello** * **Thulasi Mylvaganam** *

* *Department of Aeronautics, Imperial College London, London SW7 2AZ, UK (e-mail: {domenico.cappello16, t.mylvaganam}@imperial.ac.uk).*

**Abstract:** It is well known that finding Nash equilibrium solutions of nonzero-sum differential games is a challenging task. Focusing on a class of linear quadratic differential games, we consider three notions of approximate feedback Nash equilibrium solutions and provide a characterisation of these in terms of matrix inequalities which constitute quadratic feasibility problems. These feasibility problems are then recast first as bilinear feasibility problems and finally as rank constrained optimisation problems, *i.e.* a class of static problems frequently encountered in control theory.

*Keywords:* Differential Games, Feedback Nash Equilibria, Linear Control Systems, Optimization Problems, Optimal Control

## 1. INTRODUCTION

*Differential game theory*, which evolved from *static game theory* pioneered in the 1940s by the works of von Neumann and Morgenstern, is the study of the dynamic strategic interactions between individual decision makers, or players, each with its own objective function to minimise (or maximise). In this paper, we consider non-cooperative differential games, which are divided into two main classes, *i.e.* zero-sum and nonzero-sum games, characterised by whether the sum of the objective functions is equal to or different from zero, respectively.

Differential games can be used to model a large number of "competitive" situations arising in *e.g.* economics (Dockner et al. (2000)), military operations (Isaacs (1999)), ecology (Jorgensen et al. (2007)), politics (Morrow (1994)), etc. They also find a wide range of applications in control theory. For instance, zero-sum differential games have been used to solve the $\mathcal{H}_\infty$ control problem in Basar and Bernhard (1995), the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been solved by means of a nonzero-sum differential game formulation in Limebeer et al. (1994) and in Mylvaganam and Astolfi (2016), the multi-agent collision avoidance problem has been solved via a game theoretic formulation in Mylvaganam et al. (2017), multiple local nonzero-sum differential games have been employed for the distributed control of multi-agent systems in Cappello and Mylvaganam (2018, 2019). Despite the widespread use of differential games in control theory, obtaining feedback solutions (*e.g.* in terms of Nash equilibrium solutions) entails solving the associated Hamilton-Jacobi-Isaacs partial differential equation (Basar and Olsder (1982)), which is an intractable problem. Even in the linear quadratic case obtaining solutions, which are characterised by coupled Riccati equations, is not straightforward (Starr and Ho (1969), Engwerda (2005)). In fact, linear quadratic differential games have been extensively studied in the literature. Existence and uniqueness of feedback Nash equilibrium solutions of linear quadratic differential games have been investigated *e.g.* in Basar (1976) and in Papavassilopoulos and Olsder (1979). Necessary and sufficient conditions for the existence of such equilibrium solutions, in the infinite-horizon case, have been derived in Engwerda and Salmah (2013). Many algorithms reported in the literature aim to find, ofter without any guarantees of convergence, one Nash equilibrium solution (see *e.g.* Engwerda (2007) for an overview). In Engwerda (2015) an approach capable of finding all the Nash equilibrium solutions of scalar linear quadratic differential games is provided. In the general case, finding all the Nash equilibrium solutions is more challenging. Some results are available in Possieri and Sassano (2015, 2016).

In this paper we consider $N$-player nonzero-sum differential games defined by integral quadratic cost functionals subject to time-invariant linear system dynamics. Motivated by the difficulties associated with obtaining exact solutions for such games, we consider three notions of approximate equilibrium solutions of differential games and provide a characterisation of these in terms of matrix inequalities. The main contributions of this paper are twofold. First, we introduce two new notions of approximate Nash equilibrium solutions, which include specifications on the convergence rate of the resulting closed-loop system. Such specifications may be of importance in certain applications and are widely adopted in several areas of control theory. Second, we provide alternative formulations of the different notions of approximate solutions in terms of familiar (static) optimisation problems which arise in several control problems (see *e.g.* Fazel et al. (2004)).

The remainder of the paper is organised as follows. In Section 2, we introduce the class of differential games considered herein, and preliminary assumptions and definitions are given. In Section 3, a notion of approximate solutions is recalled, before two new notions of approximate solutions are defined. In Section 4, sufficient conditions for obtaining such solutions are provided, before the problem of obtaining these is reformulated as a Rank Constrained Optimisation Problem (RCOP) in Section 5. Finally, some concluding remarks are given in Section 6.

**Notation.** $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}^-$ denotes the set of complex numbers with negative real part. The $n \times n$ identity matrix is denoted by $I_n$. Given a square matrix $A \in \mathbb{R}^{n\times n}$, $\sigma(A)$, $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$ denote the spectrum, the minimum singular value and the maximum singular value of $A$, respectively. The induced 2-norm of the matrix $A$ is denoted by $\|A\|$, or equivalently by $\bar{\sigma}(A)$. The 2-norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$. $\text{blockdiag}(A_1, \ldots, A_N)$ denotes the block diagonal matrix with diagonal blocks $A_1, \ldots, A_N$.

## 2. PRELIMINARIES

Consider a linear dynamical system influenced by $N$ players, *i.e.* consider a system with dynamics of the form

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system and $u_i \in \mathbb{R}^{p_i}$ is the control input (also referred to as *strategy*) corresponding to player $i$, $i = 1, \ldots, N$. $A$ and $B_i$, $i = 1, \ldots, N$, are constant matrices of appropriate dimensions. Each player $i$, $i = 1, \ldots, N$, seeks to minimise its (individual) cost functional

$$J_i(x_0, u_1, \ldots, u_N) = \frac{1}{2}\int_0^\infty (x^\top Q_i x + u_i R_i u_i) dt, \tag{2}$$

where $Q_i = Q_i^\top \geq 0$ and $R_i = R_i^\top > 0$, via the selection of its control strategy $u_i$.

Consider the following standard assumption.

*Assumption 1.* The matrices $Q_i$, for $i = 1, \ldots, N$, are such that $\sum_{i=1}^N Q_i > 0$.

In this paper we consider linear state-feedback strategies of the form

$$u_i = K_i x, \tag{3}$$

for $i = 1, \ldots, N$. Note that the cost functionals (2) are defined on an infinite-horizon and that Assumption 1 implies detectability of the couple $(A, \sum_{i=1}^N Q_i)$. Therefore, if the closed-loop system (1) were unstable then, for infinitely many initial conditions $x(0)$, at least one player would incur an infinite cost. In the following we recall the concept of admissible and $\alpha$-admissible strategies (introduced in Mylvaganam et al. (2015)). The latter is used in notions of approximate Nash equilibria in the following section.

*Definition 1.* ($\alpha$-admissible strategies). A set of feedback strategies $\{u_1, \ldots, u_N\}$ is $\alpha$-admissible if it renders the closed-loop system (1) asymptotically stable with rate of convergence greater than or equal to $\alpha \geq 0$, *i.e.* $\sigma(A_{cl} + \alpha I_n) \in \mathbb{C}^-$, where $A_{cl}$ is the system matrix of the closed-loop system (1), *i.e.* $A_{cl} \triangleq A + \sum_{i=1}^N B_i K_i$. If a set of strategies is $\alpha$-admissible with $\alpha = 0$, it is said to be admissible.

## 3. PROBLEM FORMULATION

In this section, we first recall the "classical" Nash equilibrium solution concept and the notion of $\epsilon_\alpha$-Nash equilibrium solution - a relaxation of the classical solution introduced in Mylvaganam et al. (2015). We then introduce two new solution concepts for differential games, which include a desired minimum convergence rate for the closed-loop system.

Let $u_{-i}$ denote the set of strategies of every player excluding player $i$, *i.e.* the set of strategies $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N\}$. The most common solution concept of nonzero-sum differential games is the Nash equilibrium, *i.e.* a set of control strategies such that none of the players would have any incentive, quantified in terms of their cost functional, for unilaterally deviating from their equilibrium strategy. Namely, a Nash equilibrium solution is an admissible set of strategies $\{u_i^*, u_{-i}^*\}$ such that

$$J_i(x_0, u_i^*, u_{-i}^*) \leq J_i(x_0, \hat{u}_i, u_{-i}^*), \tag{4}$$

holds, for every admissible set of strategies $\{\hat{u}_i, u_{-i}^*\}$, for all $i = 1, \ldots, N$. Considering linear quadratic differential games defined by the dynamics (1) and cost functionals (2), $i = 1, \ldots, N$, feedback Nash equilibrium solutions (considering linear feedback strategies (3)) are of the form

$$u_i^* = -R_i^{-1} B_i^\top P_i x, \tag{5}$$

for $i = 1, \ldots, N$, where the matrices $P_i = P_i^\top \geq 0$, $i = 1, \ldots, N$, are solutions of the coupled Algebraic Riccati equations (cARE)

$$\begin{aligned} Q_i + P_i B_i R_i^{-1} B_i^\top P_i + P_i(A - \sum_{j=1}^N B_j R_j^{-1} B_j^\top P_j) \\ + (A - \sum_{j=1}^N B_j R_j^{-1} B_j^\top P_j)^\top P_i = 0, \end{aligned} \tag{6}$$

for $i = 1, \ldots, N$. Obtaining a solution of (6), if it exists, is typically not straightforward (Starr and Ho (1969)). For this reason, approximate Nash equilibrium solutions are often sought for (Lin (2013); Mylvaganam et al. (2015)). Considering general (linear and nonlinear) differential games, in Mylvaganam et al. (2015) the authors introduce the concept of $\epsilon_\alpha$-Nash equilibrium. The problem of solving a differential game in terms of $\epsilon_\alpha$-Nash equilibrium solutions is defined as follows.

*Problem 1.* Determine an $\epsilon_\alpha$- Nash equilibrium of the nonzero-sum differential game defined by cost functionals (2), for $i = 1, \ldots, N$, subject to the dynamics (1), *i.e.* find an *admissible* set of feedback strategies $\{u_1^*, \ldots, u_N^*\}$ such that there exists a non-negative constant $\epsilon_{\alpha,x_0}$ parametrised with respect to both the initial state $x(0) = x_0$ and $\alpha > 0$, such that

$$J_i(x_0, u_i^*, u_{-i}^*) \leq J_i(x_0, \hat{u}_i, u_{-i}^*) + \epsilon_{\alpha,x_0}, \tag{7}$$

for every *$\alpha$-admissible* set of strategies $\{\hat{u}_i, u_{-i}^*\}$, for all $i = 1, \ldots, N$.

The solution of Problem 1 proposed in (Mylvaganam et al., 2015, Prop. 2) is such that the non-negative constant $\epsilon_{\alpha,x_0}$ can only be calculated *a posteriori*, *i.e.* after a solution of Problem 1 is found. Note also that the equilibrium strategies defined in Problem 1 are not themselves $\alpha$-admissible, with $\alpha > 0$. To capture, instead, scenarios in which we are interested in finding an equilibrium solution which guarantees a minimum rate of convergence $\alpha$ of the closed-loop system (1), we introduce a different problem, in terms of the $\epsilon_\alpha$-Nash equilibrium with guaranteed convergence rate $\alpha$, defined as follows.

*Problem 2.* Given a positive constant $\alpha$, determine an $\epsilon_\alpha$-Nash equilibrium of the nonzero-sum differential game defined by cost functionals (2), for $i = 1, \dots, N$, subject to the dynamics (1), such that the system in closed-loop with the equilibrium strategies has a guaranteed minimum convergence rate $\alpha \geq 0$, *i.e.* find an $\alpha$-*admissible* set of feedback strategies $\{u_1^*, \dots, u_N^*\}$ such that there exists a non-negative constant $\epsilon_{\alpha,x_0}$, parametrised with respect to both the initial state $x(0) = x_0$ and $\alpha$, such that (7) holds, for every $\alpha$-*admissible* set of strategies $\{\hat{u}_i, u_{-i}^*\}$, for all $i = 1, \dots, N$.

As will be illustrated in Section 4, the proposed solution of Problem 2 shares the same limitation, regarding the calculation of $\epsilon_{\alpha,x_0}$, of the solution of Problem 1. Namely, $\epsilon_{\alpha,x_0}$ cannot be calculated before the linear feedback matrices $P_i$, $i = 1, \dots, N$, are computed. Problem 2 can be regarded as an intermediate problem included, for ease of exposition, before introducing the problem of obtaining an $\epsilon$-Nash equilibrium with guaranteed convergence rate $\alpha$, defined as follows.

*Problem 3.* Let the constants $\alpha > 0$ and $b > 0$ be the desired minimum convergence rate of system (1) and the upper bound on the norm of the feedback matrices $K_i$, for $i = 1, \dots, N$, respectively. Let $\epsilon_{x_0} = c \|x_0\|^2$, with $c > 0$, be a non-negative constant parametrised with respect to $x(0) = x_0$. Find an $\epsilon$-Nash equilibrium with guaranteed convergence rate $\alpha$ of the nonzero-sum differential game defined by cost functionals (2), for $i = 1, \dots, N$, subject to the dynamics (1), *i.e.* find an $\alpha$-*admissible* set of feedback strategies $\{u_1^*, \dots, u_N^*\}$ such that

$$J_i(x_0, u_i^*, u_{-i}^*) \leq J_i(x_0, \hat{u}_i, u_{-i}^*) + \epsilon_{x_0}, \tag{8}$$

holds, for every $\alpha$-*admissible* set of strategies $\{\hat{u}_i, u_{-i}^*\}$, for all $i = 1, \dots, N$.

The $\epsilon$-Nash equilibrium is a standard and well studied concept of approximate solution in game theory (see *e.g.* Nisan et al. (2007)).

## 4. SOLUTION

In this section we characterise and present certain properties of the solutions of Problems 1, 2 and 3 presented in Section 3. For brevity, we define the symmetric positive semidefinite matrices $S_i \triangleq B_i R_i^{-1} B_i^\top$, for $i = 1, \dots, N$.

### 4.1 Solution of Problem 1

We briefly recall the solution of Problem 1 as given in (Mylvaganam et al., 2015, Prop. 2), here in the general $N$-player case.

*Proposition 1.* Consider the nonzero-sum differential game defined by the cost functionals (2), for $i = 1, \dots, N$, subject to the linear system dynamics (1), and let Assumption 1 hold. Suppose we can find matrices $P_i = P_i^\top \geq 0$, for $i = 1, \dots, N$, such that $\sum_{i=1}^N P_i > 0$, and such that

$$\begin{aligned} -\Upsilon_i \triangleq Q_i + P_i S_i P_i + P_i(A - \sum_{j=1}^N S_j P_j) \\ + (A - \sum_{j=1}^N S_j P_j)^\top P_i \leq 0, \end{aligned} \tag{9}$$

for $i = 1, \dots, N$, hold. Then the set of strategies (5), $i = 1, \dots, N$, is admissible and yields an $\epsilon_\alpha$-Nash equilibrium, for any $\alpha > 0$, of the nonzero-sum differential game.

**Proof.** The claim can be proved by the same reasoning as for the 2-player case of (Mylvaganam et al., 2015, Prop. 2), by noting that the set of strategies (5), for $i = 1, \dots, N$, constitutes a Nash equilibrium of the nonzero-sum differential game defined by the modified cost functionals

$$\tilde{J}_i(x_0, u_i, u_{-i}) \triangleq \frac{1}{2} \int_0^\infty (x^\top (Q_i + \Upsilon_i) x + u_i^\top R_i u_i) dt, \tag{10}$$

for $i = 1, \dots, N$, subject to the dynamics (1). Admissibility of the set of strategies (5), $i = 1, \dots, N$, follows from Lyapunov arguments based on the Lyapunov candidate function $W(x) = \sum_{i=1}^N J_i(x, u_i^*, u_{-i}^*) = \frac{1}{2} \sum_{i=1}^N x^\top P_i x$ and on Assumption 1.

Note that, since the modified cost functionals (10) are such that $\tilde{J}_i(x_0, u_i, u_{-i}) = J_i(x_0, u_i, u_{-i}) + \frac{1}{2} \int_0^\infty (x^\top \Upsilon_i x) dt$, and since $J_i(x_0, u_i^*, u_{-i}^*) \leq \tilde{J}_i(x_0, u_i^*, u_{-i}^*)$, then (7) is satisfied, for $i = 1, \dots, N$, with $\epsilon_{\alpha,x_0} \geq \max_i \frac{1}{2} \int_0^\infty (x^\top \Upsilon_i x) dt$. Let $\hat{u}_i = \hat{K}_i x$ be such that the set of strategies $\{\hat{u}_i, u_{-i}^*\}$ is $\alpha$-admissible and yields the maximum value of the integral $\frac{1}{2} \int_0^\infty (x^\top \Upsilon_i x) dt$. In particular, (7) is also satisfied with $\epsilon_{\alpha,x_0} = \frac{1}{2} \max_i \{x_0^\top P_{i,\epsilon} x_0\}$, where $P_{i,\epsilon} = P_{i,\epsilon}^\top > 0$ solves $P_{i,\epsilon} A_{K_i} + A_{K_i}^\top P_{i,\epsilon} + \Upsilon_i = 0$, with $A_{K_i} \triangleq A - \sum_{j=1, j \neq i}^N B_j R_j^{-1} B_j^\top P_j + B_i \hat{K}_i$. □

Note that, as previously remarked, the matrices $A_{K_i}$, and therefore also the non-negative constant $\epsilon_{\alpha,x_0}$, depend on the matrices $P_j$, for $j = 1, \dots, N$, $j \neq i$, for $i = 1, \dots, N$. The following result provides sufficient conditions in terms of structural properties of the system (1), for the existence of solutions of Problem 1.

*Proposition 2.* Consider the nonzero-sum differential game defined by the cost functionals (2), for $i = 1, \dots, N$, subject to the dynamics (1). Under Assumption 1, if the pair of matrices $(A, [B_1, \dots, B_N])$ is stabilisable then there exists a solution of Problem 1, *i.e.* there exists an $\epsilon_\alpha$-Nash solution of the differential game.

### 4.2 Solution of Problem 2

In the next statement a sufficient condition for obtaining a solution of Problem 2 is given.

*Theorem 1.* Consider the nonzero-sum differential game defined by the cost functionals (2), for $i = 1, \dots, N$, subject to the linear system dynamics (1), and let Assumption 1 hold. Let the constant $\alpha > 0$ describe the desired minimum rate of convergence of system (1). Suppose we can find matrices $P_i = P_i^\top \geq 0$, for $i = 1, \dots, N$, such that $\sum_{i=1}^N P_i > 0$, and such that

$$\begin{aligned} -\Upsilon_i \triangleq Q_i + P_i S_i P_i + P_i(A - \sum_{j=1}^N S_j P_j) \\ + (A - \sum_{j=1}^N S_j P_j)^\top P_i \leq -2\alpha P_i, \end{aligned} \tag{11}$$

for $i = 1, \dots, N$, hold. Then the set of strategies $u_i^* = -R_i^{-1} B_i^\top P_i x$, for $i = 1, \dots, N$, is $\alpha$-admissible and yields an $\epsilon_\alpha$-Nash equilibrium with guaranteed convergence rate $\alpha$ of the nonzero-sum differential game.

**Proof.** The proof is based on an approach similar to that adopted in Grasselli and Galeani (2015) for the linear quadratic optimal control problem. Given a scalar $\alpha > 0$, consider the reverse-discounted modified cost functionals

$$\tilde{J}_{\alpha,i}(x_0, u_i, u_{-i}) \triangleq \frac{1}{2}\int_0^\infty e^{2\alpha t}(x^\top(Q_i + \bar{\Upsilon}_i)x + u_i^\top R_i u_i)dt, \quad (12)$$

for $i = 1, \dots, N$, where the importance of the stage cost increases exponentially with time. It is easy to show that any set of strategies $\{\hat{u}_1, \dots, \hat{u}_N\}$ such that $\tilde{J}_{\alpha,i}(x_0, \hat{u}_i, \hat{u}_{-i})$ is finite, for $i = 1, \dots, N$, is $\alpha$-admissible. Let $x_\alpha(t) \triangleq e^{\alpha t}x(t)$ and $u_{\alpha,i}(t) \triangleq e^{\alpha t}u_i(t)$, denote the state of an auxiliary system and the modified control strategy of player $i$, for $i = 1, \dots, N$, respectively. Note that the auxiliary system, described by dynamics

$$\dot{x}_\alpha = (\alpha I_n + A)x_\alpha + \sum_{j=1}^N B_j u_{\alpha,i}, \quad (13)$$

is such that $(A + \alpha I_n, \sum_{j=1}^N B_j)$ is stabilisable if and only if $(A, \sum_{j=1}^N B_j)$ is $\alpha$-stabilisable [1], and that $(A + \alpha I_n, \sum_{j=1}^N Q_j)$ is detectable if and only if $(A, \sum_{j=1}^N Q_j)$ is $\alpha$-detectable [2]. The reverse-discounted modified cost functional (12) can be written, in terms of the state of the auxiliary system, as

$$\tilde{J}_{\alpha,i}(x_0, u_i, u_{-i}) \triangleq \frac{1}{2}\int_0^\infty (x_\alpha^\top(Q_i + \bar{\Upsilon}_i)x_\alpha + u_{\alpha,i}^\top R_i u_{\alpha,i})dt, \quad (14)$$

for $i = 1, \dots, N$. Note that, for $x(0) = x_0$, the initial state of the auxiliary system (13) is $x_\alpha(0) = x_0$. The claim follows from the same arguments of Proposition 1 applied to the nonzero-sum differential game defined by the cost functionals (14), for $i = 1, \dots, N$, subject to the auxiliary system dynamics (13). In fact, a (feedback) Nash equilibrium solution of this nonzero-sum differential game, namely $u_{\alpha,i} = -R_i^{-1}B_i^\top P_i x_\alpha$, for $i = 1, \dots, N$, corresponds to the $\epsilon_\alpha$-Nash equilibrium solution, given by $u_i = -R_i^{-1}B_i^\top P_i x$, for $i = 1, \dots, N$, of the nonzero-sum differential game defined by the cost functionals (2), for $i = 1, \dots, N$, subject to the linear system dynamics (1), *i.e.* to a solution of (9), wherein $\Upsilon_i = \bar{\Upsilon}_i + 2\alpha P_i$, for $i = 1, \dots, N$. Moreover, since $\tilde{J}_{\alpha,i}(x_0, u_i, u_{-i})$, for $i = 1, \dots, N$, are finite, it follows that the closed-loop system has a guaranteed convergence rate $\alpha$, *i.e.* that the set of strategies (5), $i = 1, \dots, N$, constitutes a solution of Problem 2. □

As already observed in Section 3, the solutions of the two approximate Nash equilibria defined in Problem 1 and 2 do not render possible to quantify their degree of approximation before solving (11), $i = 1, \dots, N$. Namely, for both solution concepts, the constant $\epsilon_{\alpha,x_0}$, parametrised with respect to the desired convergence rate $\alpha$ and to the initial state of system (1), is also dependent on the matrices $P_i$, $i = 1, \dots, N$.

*Remark 1.* Following the same reasoning as Proposition 2, it can easily be shown that, under Assumption 1, a solution of Problem 2 exists if the pair of matrices $(A, [B_1, \dots, B_N])$ is $\alpha$-stabilisable.

### *4.3 Solution of Problem 3*

For clarity of presentation, we define some notation (via the introduction of an optimal control problem) used in the consideration of solutions of Problem 3. To this end, consider a scenario in which the $N$ players cooperate with the aim of minimising the sum of their cost functionals (2), $i = 1, \dots, N$, *i.e.* with the aim of minimising the cost functional

$$J_{oc}(x_0, u_1, \dots, u_N) = \frac{1}{2}\int_0^\infty (x^\top \bar{Q}x + u^\top \bar{R}u)dt, \quad (15)$$

wherein $u \triangleq [u_1^\top, \dots, u_N^\top]^\top$, and $\bar{Q} \triangleq \sum_{i=1}^N Q_i$ and $\bar{R} \triangleq \text{blockdiag}(R_1, \dots, R_N) > 0$, subject to the linear system dynamics (1), with $x(0) = x_0$. This is a classical linear quadratic optimal control problem which admits a solution if the pair of matrices $(A, [B_1, \dots, B_N])$ is stabilisable (note that $(A, \bar{Q})$ is detectable by Assumption 1). Let $u_{oc}^* \triangleq [\bar{u}_1^{*\top}, \dots, \bar{u}_N^{*\top}]^\top$ denote the (unique) solution of this optimal control problem, with the respective value function denoted by

$$J_{oc}^*(x_0, u_{oc}^*) = \frac{1}{2}x_0^\top P_{oc}x_0, \quad (16)$$

wherein the matrix $P_{oc}$ is symmetric and positive definite.

In the following theorem, sufficient conditions for finding a solution of Problem 3 are given.

*Theorem 2.* Consider the nonzero-sum differential game defined by the cost functionals (2), for $i = 1, \dots, N$, subject to the linear system dynamics (1), and let Assumption 1 hold. Let the constants $\alpha > 0$ and $b > 0$ be the desired minimum convergence rate of system (1) and the upper bound on the 2-norm of the state-feedback matrices $K_i$, for $i = 1, \dots, N$, respectively. In addition, let the constant $k_1 > 0$ be the minimum singular value of the matrix $P_{oc}$ defined in (16), *i.e.* $k_1 = \underline{\sigma}(P_{oc})$, and let the constant $k_2 \triangleq \sum_{i=1}^N b_i$, where $b_i \triangleq b \cdot \left\|B_i R_i^{-1}\right\|^{-1}$, for $i = 1, \dots, N$. Given $\epsilon_{x_0} = c \cdot \|x_0\|^2$, with $c > 0$, suppose we can find symmetric matrices $0 \le P_i \le b_i I_n$, for $i = 1, \dots, N$, such that $\sum_{i=1}^N P_i > 0$, and such that (11) holds, with $\|\Upsilon_i\| \le 4\alpha c k_1/k_2$, for $i = 1, \dots, N$. Then the set of strategies (5), for $i = 1, \dots, N$, is $\alpha$-admissible and yields an $\epsilon$-Nash equilibrium with guaranteed convergence rate $\alpha$ of the nonzero-sum differential game.

**Proof.** The $\alpha$-admissibility of the set of strategies follows from Theorem 1 since $P_i$, $i = 1, \dots, N$, satisfy (11). We need to prove that these strategies constitute also a solution of Problem 3. To this end, let $W(x) = \sum_{i=1}^N \tilde{J}_i(x, u_i^*, u_{-i}^*)$, with $\tilde{J}_i$ defined in (10), be a Lyapunov candidate function of the closed-loop system (1). Note that, by definition, $W(x) = J_{oc}(x, u_i^*, \dots, u_N^*) + \frac{1}{2}\int_0^\infty \sum_{i=1}^N (x^\top \Upsilon_i x)dt$, and that $J_{oc}(x, u_{oc}^*) \le W(x)$, for all $x \in \mathbb{R}^n$. Therefore, given the bounds on the matrices $P_i$, for $i = 1, \dots, N$, the Lyapunov candidate function

[1] $(A, B)$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, is $\alpha$-stabilisable if and only if $\text{rank}([A - \lambda I_n \quad B]) = n$, for all $\lambda \in \mathbb{C} : re[\lambda] \ge -\alpha$.

[2] $(A, Q)$, where $A \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{q\times n}$, is $\alpha$-detectable if and only if $\text{rank}\left(\begin{bmatrix} A - \lambda I_n \\ Q \end{bmatrix}\right) = n$, for all $\lambda \in \mathbb{C} : re[\lambda] \ge -\alpha$.

can be bounded as $\frac{1}{2}k_1 \|x\|^2 \leq W(x) \leq \frac{1}{2}k_2 \|x\|^2$, where $k_1 = \underline{\sigma}(P_{oc})$ and $k_2 = \sum_{i=1}^{N} b_i$. Note that both $k_1$ and $k_2$ are independent from the matrices $P_i$, for $i = 1, \ldots, N$. The evolution of the state of the closed-loop system(1) is thus norm-bounded as $\|x(t)\| \leq \sqrt{\frac{k_2}{k_1}} e^{-\alpha t} \|x(0)\|$, for all $t \geq 0$ (Khalil, 2002, Thm 4.10). Therefore, the constant $\epsilon_{x_0}$ defined in (8) can be bounded as

$$\epsilon_{x_0} \leq \frac{k_2}{4\alpha k_1} \max_i \|\Upsilon_i\| \|x_0\|^2 \leq c \|x_0\|^2 , \quad (17)$$

thus proving the claim. Namely, the set of strategies (5), $i = 1, \ldots, N$, render the closed-loop system asymptotically stable with guaranteed convergence rate $\alpha$ and are such that (8) holds, with $\epsilon_{x_0} = c \|x_0\|^2$, for every *$\alpha$-admissible* set of strategies $\{\hat{u}_i, u_{-i}^*\}$, for all $i = 1, \ldots, N$. □

*Remark 2.* It is possible to find less conservative bounds on $\|\Upsilon_i\|$, introduced in Theorem 2, by considering, in place of the given optimal control problem described at the beginning of Subsection 4.3, the problem of finding *Pareto*-efficient solutions (see *e.g* Engwerda (2005)) maximising $k_1$. Namely, by finding the set of strategies (5), $i = 1, \ldots, N$, minimising the cost functional $J_p \triangleq \sum_{i=1}^{N} \alpha_i J_i(x_0, u_i, u_{-i})$, subject to dynamics (1), with non-negative constants $\alpha_i$, for $i = 1, \ldots, N$, such that $\sum_{i=1}^{N} \alpha_i = 1$, $\sum_{i=1}^{N} \alpha_i Q_i > 0$ and such that $k_1 = \underline{\sigma}(\sum_{i=1}^{N} \alpha_i P_i)$ is maximised.

## 5. COMPUTATION OF THE EQUILIBRIA

As illustrated in Section 4, a solution of each of the problems formulated in Section 3 can be obtained by solving a system of coupled matrix inequalities which constitute a quadratic feasibility problem (FP). In this section, a reformulation of these FPs is given. The aim of this reformulation is to cast the problem of obtaining (approximate) solutions of differential games in terms of alternative problem formulations, frequently encountered in control theory.

The Rank Minimisation Problem (RMP), which can be expressed as the minimisation of the rank of a decision matrix variable subject to a set of convex constraints, has received considerable attentions in the last decades (see *e.g.* Sun and Dai (2017) and references therein) due to its wide range of applications in *e.g.* system identification, control, statistics and signal processing (Fazel et al. (2004)). In Sun and Dai (2017), the equivalence between RMPs and RCOPs, which consist in the minimisation of a convex function subject to a set of convex constraints and to rank constraints on semidefinite matrices, has been shown. RMPs and RCOPs are known to be NP-hard in general, but their importance in many engineering applications has resulted in extensive studies to develop efficient optimisation algorithms for their solution (Sun and Dai (2017)). For this reason, the aim of this section is to reformulate the FPs introduced in Section 4 as equivalent RCOPs. First, for ease of exposition, intermediate Bilinear Feasibility Problem (BFP) reformulations are presented.

In the remainder of this paper we assume the following condition holds.

*Assumption 2.* The inequalities (9) and (11), admit positive definite solutions $P_i > 0$, $i = 1, \ldots, N$.

In the following, we consider the 2-player case for ease of presentation. The same exact reasoning could be applied for the general $N$-player case. Under Assumption 2, Problem 1 and 2 can be equivalently recast as BFPs. In fact, in the 2-player case, a solution of Problem 1 [Problem 2] can be calculated by finding the matrix variables $L_i \in \mathbb{R}^{n\times n}$, $L_i = L_i^\top > 0$ and $F_{ji} \in \mathbb{R}^{n\times n}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, such that

$$L_j F_{ji} = L_i, \quad (18)$$

and such that

$$\begin{bmatrix} S_i + S_j F_{ji} + F_{ji}^\top S_j - A L_i - L_i A^\top + \beta_i & L_i \\ L_i & (Q_i + \Gamma)^{-1} \end{bmatrix} \geq 0 \quad (19)$$

where $\beta_i = 0$ [$\beta_i = -2\alpha L_i$], for $i = 1, 2$, $j = 1, 2$, $j \neq i$, for any $\Gamma > 0$. A solution of the original problem, *i.e.* of Problem 1 [Problem 2] can be calculated as $u_i^* = -R_i^{-1} B_i^\top P_i x$, where $P_i = L_i^{-1}$, for $i = 1, 2$.

A solution of Problem 3 can be similarly computed by solving a BFP, *i.e.* by finding the matrix variables $L_i \in \mathbb{R}^{n\times n}$, $L_i = L_i^\top > 0$ and $F_{ji} \in \mathbb{R}^{n\times n}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, such that (18) and (19) hold, with $\beta_i = -2\alpha L_i$, subject to the additional constraints

$$L_i P_i = I_n, \quad (20)$$

and

$$\begin{bmatrix} Q + \bar{c}_i I_n & (A - S_j P_j)^\top \\ A - S_j P_j & \bar{c}_i I_n - S_i \end{bmatrix} \geq 0, \quad (21)$$

where $\bar{c}_i \triangleq \frac{4\alpha c k_1}{k_2(1+b_i^2)}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$. Differently from Problem 1 and 2, this BFP and the FP given in Subsection 4.3 are not equivalent, even under Assumption 2. Nevertheless, every solution of the BFP constitutes a solution of the FP in Subsection 4.3.

*Remark 3.* Despite being a rather conservative condition, the linear matrix inequality (21) always admits a solution, due to diagonal dominance arguments, for a large enough constant $\bar{c}_i > 0$, *i.e.* for a large enough constant $c > 0$ which is proportional to the degree of approximation $\epsilon_{x_0} = c \|x_0\|^2$ in (8).

The inequality (18) holds if and only if (see the *semidefinite embedding lemma* in Fazel et al. (2003)) there exist symmetric semidefinite matrices $Y_i \in \mathbb{R}^{2n\times 2n}$ and $Z_i \in \mathbb{R}^{2n\times 2n}$ such that

$$\operatorname{rank}(\operatorname{blockdiag}(Y_i, Z_i)) \leq 2n,$$
$$\begin{bmatrix} Y_i & \begin{bmatrix} F_{ji} & I_n \\ L_i & L_j \end{bmatrix} \\ \begin{bmatrix} F_{ji}^\top & L_i \\ I_n & L_j \end{bmatrix} & Z_i \end{bmatrix} \geq 0, \quad (22)$$

for $i = 1, 2$. Similarly, (20) holds if and only if there exist symmetric semidefinite matrices $W_i \in \mathbb{R}^{2n\times 2n}$ and $M_i \in \mathbb{R}^{2n\times 2n}$ such that

$$\operatorname{rank}(\operatorname{blockdiag}(W_i, M_i)) \leq 2n,$$
$$\begin{bmatrix} W_i & \begin{bmatrix} L_i & I_n \\ I_n & P_i \end{bmatrix} \\ \begin{bmatrix} L_i & I_n \\ I_n & P_i \end{bmatrix} & M_i \end{bmatrix} \geq 0, \quad (23)$$

respectively, for $i = 1, 2$. Finally, the BFPs defined in this section can be equivalently formulated as follows.

*Proposition 3.* A solution of Problem 1 [Problem 2] can be computed by finding the matrix variables $L_i \in \mathbb{R}^{n\times n}$,

$L_i = L_i^\top > 0$, $F_{ji} \in \mathbb{R}^{n\times n}$, and symmetric semidefinite matrix variables $Y_i \in \mathbb{R}^{2n\times 2n}$ and $Z_i \in \mathbb{R}^{2n\times 2n}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, such that (22) and (19) hold, with $\beta_i = 0$ $[\beta_i = -2\alpha L_i]$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, for any $\Gamma > 0$. □

*Proposition 4.* A solution of Problem 3 can be computed by finding the matrix variables $L_i \in \mathbb{R}^{n\times n}$, $L_i = L_i^\top > 0$, $F_{ji} \in \mathbb{R}^{n\times n}$, and symmetric semidefinite matrix variables $Y_i \in \mathbb{R}^{2n\times 2n}$, $Z_i \in \mathbb{R}^{2n\times 2n}$, $W_i \in \mathbb{R}^{2n\times 2n}$ and $M_i \in \mathbb{R}^{2n\times 2n}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, such that (22), (23), (19) and (21) hold, with $\bar{c}_i \triangleq \frac{4\alpha c k_1}{k_2(1+b_i^2)}$, for $i = 1, 2$, $j = 1, 2$, $j \neq i$, for any $\Gamma > 0$. □

*Remark 4.* The FPs defined in Propositions 3 and 4 can be regarded as RCOPs with constant objective functions, subject to the rank and linear matrix inequality constraints given in the respective statements. A different choice of the objective function could be made *e.g.* for finding an $\epsilon$-Nash equilibrium with guaranteed convergence rate $\alpha$ ensuring the smallest achievable degree of approximation $\epsilon_{x_0} = c\,\|x_0\|^2$. This could be obtained by minimising the objective function $J = c$, subject to the constraints specified in Proposition 4.

## 6. CONCLUSION

In this paper, motivated by challenges associated with solving cAREs arising in the search for (exact) Nash equilibrium solutions of linear quadratic differential games, we consider three notions of approximate Nash equilibrium solutions for nonzero-sum differential games. These include two novel solution concepts which guarantee a certain desired convergence rate of the closed-loop system. Sufficient conditions for their existence are given and, in addition, the problem of calculating such solutions is recast as static optimisation problems often encountered in control theory.

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