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Well-Posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid ⋆

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Abstract: We study a PDE based linearized model for the vertical motion of a solid floating at the free surface of a shallow viscous fluid. The solid is controlled by a vertical force exerted via an actuator. This force is the input of the system, whereas the output is the distance from the solid to the bottom. The first novelty we bring in is that we prove that the governing equations define a well-posed linear system. This is done by considering adequate function spaces and convenient operators between them. Another contribution of this work is establishing that the system is input-output stable. To this aim, we give an explicit form of the transfer function and we show that it lies in the Hardy space H^∞ of the right-half plane.

Keywords: Well-posed systems, input-output stability, transfer function, infinite dimensional systems.

1. INTRODUCTION

In this work we consider an infinite dimensional system describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and flat bottom. This system is motived by the growing interest of wave energy extractors that float on the sea and extract energy by activating a hydraulic pump, which in turn drives a hydraulic motor connected to a generator. In such an arrangement, the torque on the generator can be controlled, leading to a controllable vertical force on the floating object, see for instance Korde and Ringwood (2016) or Pecher and Kofoed (2017). The input of the considered system is the force acting on the solid by an actuator, whereas the output is the distance from the solid to the sea bottom. The novelty brought by this work is twofold:

- The viscous effects are taken in consideration from the beginning of the modelling process, by adapting a method describing viscous free boundary value flows which has been introduced in Maity et al. (2018).
- We give an explicit form of the transfer function, allowing, in particular, to establish the input-output stability of the system. In a future work we aim at using this explicit form to implement simple feedback laws.

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The leading assumptions on the fluid are that it is one dimensional and unbounded in the horizontal direction, that the flow can be described within the shallow water approximation(this mean that the horizontal length scale of motion L is much greater than the perpendicular fluid depth D, i.e. $D/L \ll 1$) and that the viscosity effects cannot be neglected. Concerning the solid, we assume that it has vertical walls, that it can move only vertically and that it is subject to a vertical control force. The output signal is the distance from the bottom of the solid to the sea bottom. More precisely, we consider the model introduced in Maity et al. (2018), with the particularity that the fluid is supposed to be infinite in the horizontal direction, denoting $\mathcal{I} := [a, b]$ the projection on the fluid bottom of the solid domain and setting $\mathcal{E} := \mathbb{R} \setminus [a, b]$. The floating solid is supposed, without loss of generality, to have mass $\mathcal{M} = 1$ and it is constrained to move only in the vertical direction. Given $t > 0$, we denote by $h(t, x)$ the height of the free surface of the fluid, by $q(t, x)$ the flux of viscous fluid in the direction x and by $h_S(t)$ the distance from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal, as described in Fig. 1. We denote by $\overline{h}$ and $\overline{h}_S$ the equilibrium height for the fluid and the solid, respectively. Then, following Maity et al. (2018), we have

$$\overline{h} = \overline{h}_S + \frac{1}{b-a},$$

and for simplicity, we assume that

$$\overline{h} = 1, \quad g = 1, \quad \overline{p} = \frac{1}{b-a}.$$

Hence, Linearizing around the trajectory $(h_S, h, q, p) = (\overline{h}_S, \overline{h}, 0, \overline{p})$ we we obtain the equations

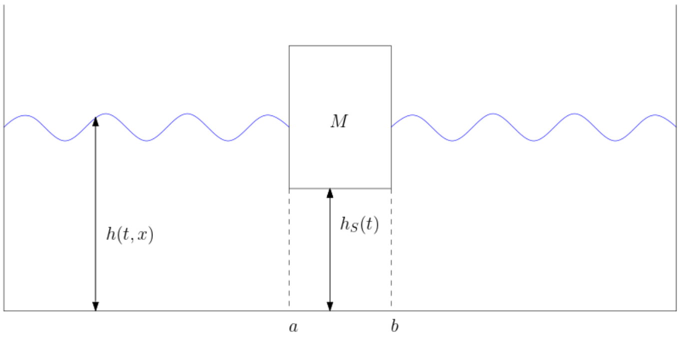

Fig. 1. Graphical sketch of the model. The function $h(t,x)$ denote the height of the free surface of the fluid, and $h_S(t)$ is the function which describes the distance from the bottom of the rigid body to the bottom of the fluid.

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \tag{1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \tag{2}$$

$$h(t,a^-) - \mu \frac{\partial q}{\partial x}(t,a^-) = p(t,a^+) + h_S(t) - \mu \frac{\partial q}{\partial x}(t,a^+), \tag{3}$$

$$h(t,b^+) - \mu \frac{\partial q}{\partial x}(t,b^+) = p(t,b^-) + h_S(t) - \mu \frac{\partial q}{\partial x}(t,b^-), \tag{4}$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{5}$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{6}$$

$$\ddot{h}_S(t) = \int_a^b p(t,x)dx + u(t) \qquad (t > 0), \tag{7}$$

where p is a Lagrange multiplier, similar to a pressure term (which is obtained in the Hamiltonian modelling process), u is the input function whereas the output is

$$y(t) = h_S(t) \qquad (t \geq 0). \tag{8}$$

Our first main result is the following reformulation of the system. Set

$$X := \mathbb{C} \times H^1(\mathcal{E}) \times L^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C}. \tag{9}$$

Theorem 1. Equations (1)-(8) can be recast as

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz, \end{aligned} \tag{10}$$

where the components of the vector $z(t)$ are $h_S(t)$, $h(t,\cdot)$, $q(t,\cdot)$, $q(t,a)$ and $q(t,b)$, B is in $\mathcal{L}(\mathbb{C}, X)$, C is in $\mathcal{L}(X,\mathbb{C})$ and A is the generator of an analytic semigroup on X.

Using the classical definition of well-posed linear systems (see for instance the survey paper Tucsnak and Weiss (2014), or also the book Tucsnak and Weiss (2009)), the above theorem implies the following result:

Corollary 2. Equations (1)-(8) define a well-posed linear system with state space X defined in (9) and input and output spaces $U = Y = \mathbb{C}$.

Remark 3. From the above results it follows, in particular, that for every

$$z_0 = \begin{bmatrix} h_{S,0} \\ h_0 \\ q_0 \\ q_{a,0} \\ q_{b,0} \end{bmatrix} \in X$$

and every $u \in L^2[0,\infty)$, the initial value problem formed of (1)-(8) and the initial condition $z(0) = z_0$ admits a unique solution

$$z(t,x) = \begin{bmatrix} h_S(t) \\ h(t,x) \\ q(t,x) \\ q(t,a) \\ q(t,b) \end{bmatrix},$$

in $C([0,\infty); X)$. Moreover, it is easily checked that $\tilde{z}$ defined by

$$\tilde{z}(t,x) = \begin{bmatrix} h_S(t) \\ h(t,a+b-x) \\ -q(t,a+b-x) \\ -q(t,b) \\ -q(t,a) \end{bmatrix},$$

satisfies (1)-(8). Moreover, if we assume that

$$q_0(x) = -q_0(a+b-x), \quad h_0(x) = h_0(a+b-x) \quad (x \in \mathcal{E}), \tag{11}$$

then $\tilde{z}(0,\cdot) = z_0$, thus $\tilde{z}$ satisfies the same initial value problem as z. Using the uniqueness of solutions of this initial value problem we deduce that $\tilde{z} = z$. This means, in particular, that for initial data satisfying (11) we have

$$q(t,a) = -q(t,b), \qquad h(t,a) = h(t,b) \qquad (t \geqslant 0).$$

Continuing with our results, we remember that a well-posed linear system of the form (10) is said input-output stable if equations (10) define, for $z(0) = 0$ a bounded map from $L^2([0,\infty); U)$ to $L^2([0,\infty); Y)$. Considering this our second main result can be stated as:

Theorem 4. The system described by (1)-(8) is input-output stable.

The remaining part of this paper is organized as follows. In Section 2 we prove Theorem 1. Finally, Section 3 is devoted to the proof of our second main result, asserting the input-output stability of the considered system.

2. PROOF OF THEOREM 1

For $t \geq 0$, we set $q_a(t) := q(t,a)$ and $q_b(t) := q(t,b)$. Since (5) implies that q is a linear function of x on $\mathcal{I}$, for every $t \geq 0$ and $x \in \mathcal{I}$,

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b-a}, \tag{12}$$

$$q(t,x) = q_a(t)\Big(\frac{x-b}{a-b}\Big) + q_b(t)\Big(\frac{x-a}{b-a}\Big), \tag{13}$$

$$\frac{\partial q}{\partial x}(t,x) = \frac{q_b(t) - q_a(t)}{b-a}. \tag{14}$$

We differentiate (6) with respect to x and use (3)-(5) to arrive at

$$\begin{aligned} &\frac{\partial^2 p}{\partial x^2}(t,x) = \ddot{h}_S(t) \quad (x \in \mathcal{I}), \\ &p\left(t,a^+\right) = p_a(t),\ p\left(t,b^-\right) = p_b(t), \end{aligned} \tag{15}$$

where

$$p_a(t) := h\left(t,a^-\right) - \mu \frac{\partial q}{\partial x}\left(t,a^-\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t), \tag{16}$$

$$p_b(t) := h\left(t,b^+\right) - \mu \frac{\partial q}{\partial x}\left(t,b^+\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t). \tag{17}$$

Moreover, the first equation in (15) implies that, for every $t \geq 0$, $p(t,x)$ is a second order polynomial in x so that

$$\int_a^b p(t,x)\, dx = p(t,a)l - \dot{q}_a(t)\frac{l^2}{3} - \dot{q}_b(t)\frac{l^2}{6} = p(t,b)l + \dot{q}_a(t)\frac{l^2}{6} + \dot{q}_b(t)\frac{l^2}{3},$$

where we set $l := b - a$. Combining this with (7) and (12) we deduce that

$$\left[1 + \frac{l^3}{3}\right]\dot{q}_a(t) - \left[1 - \frac{l^3}{6}\right]\dot{q}_b(t) = p(t,a)l^2 + lu(t),$$
$$-\left[1 - \frac{l^3}{6}\right]\dot{q}_a(t) + \left[1 + \frac{l^3}{3}\right]\dot{q}_b(t) = -p(t,b)l^2 - lu(t).$$

Inverting the above linear system, we get

$$\begin{bmatrix}\dot{q}_a(t)\\ \dot{q}_b(t)\end{bmatrix} = M\begin{bmatrix}p(t,a)\\ -p(t,b)\end{bmatrix} + \frac{1}{l}M\begin{bmatrix}u(t)\\ -u(t)\end{bmatrix} \tag{18}$$

where M is the matrix given by

$$M := \frac{1}{l(1+\frac{l^3}{12})}\begin{pmatrix}1+\frac{l^3}{3} & 1-\frac{l^3}{6}\\ 1-\frac{l^3}{6} & 1+\frac{l^3}{3}\end{pmatrix}. \tag{19}$$

Considering equation (3)-(4) together with (14) we deduce that

$$p(t,a) = h(t,a^-) - \mu\frac{\partial q}{\partial x}(t,a^-) - h_S(t) + \mu\frac{q_b - q_a}{b-a}, \tag{20}$$

and

$$p(t,b) = h(t,b^+) - \mu\frac{\partial q}{\partial x}(t,b^+) - h_S(t) + \mu\frac{q_b - q_a}{b-a}. \tag{21}$$

Finally, the system (1)-(7) writes in the equivalent form

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b-a} \qquad (t \geq 0), \tag{22}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{E}), \tag{23}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu\frac{\partial^2 q}{\partial x^2} = 0 \quad (x \in \mathcal{E}), \tag{24}$$

$$\begin{bmatrix}\dot{q}_a(t)\\ \dot{q}_b(t)\end{bmatrix} = \frac{M}{b-a}\begin{bmatrix}u(t)\\ -u(t)\end{bmatrix} + M\begin{bmatrix}h(t,a^-) - \mu\frac{\partial q}{\partial x}(t,a^-) - h_S(t) + \mu\frac{q_b-q_a}{b-a}\\ -h(t,b^+) + \mu\frac{\partial q}{\partial x}(t,b^+) + h_S(t) - \mu\frac{q_b-q_a}{b-a}\end{bmatrix}. \tag{25}$$

Let X be defined by (9), set

$$W := \mathbb{C} \times H^1(\mathcal{E}) \times H^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C},$$

and denote by $z := [h_S\ h\ q\ q_a\ q_b]^T$ a generic element of X. Consider the operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) := \{z \in W \mid q(a) = q_a,\, q(b) = q_b\}, \tag{26}$$

$$Az := \begin{bmatrix}-\frac{q(b)-q(a)}{b-a}\\ -\frac{dq}{dx}\\ -\frac{dh}{dx} + \mu\frac{d^2q}{dx^2}\\ R_1 z\\ R_2 z\end{bmatrix}, \tag{27}$$

where

$$\begin{bmatrix}R_1 z\\ R_2 z\end{bmatrix} := M\begin{bmatrix}h(a^-) - \mu\frac{dq}{dx}(a^-) - h_S + \mu\frac{q_b-q_a}{b-a}\\ -h(b^+) - \mu\frac{dq}{dx}(b^+) - h_S + \mu\frac{q_b-q_a}{b-a}\end{bmatrix}.$$

In the situation when $\mathcal{E}$ is supposed to be bounded (which means that the fluid is contained in a container), it has been proved in (Maity et al., 2018, Section 6) that the corresponding version of A defined in (26)-(27) generates an analytic semigroup. This proof can be transposed with obvious modifications to our case so that the operator A generates an analytic semigroup on X. We set

$$Bu := [0,0,0,\frac{lu}{2\left(1+\frac{l^3}{12}\right)}, -\frac{lu}{2\left(1+\frac{l^3}{12}\right)}]^T \text{ and } Cz := h_S, \tag{28}$$

and we observe that $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$. Hence the proof of Theorem 1 is completed.

3. PROOF OF THEOREM 4

It has been shown in Maity et al. (2018) that in the case of a bounded container, the linearized system describing the motion of the floating body is exponentially stable. It is not difficult to check that in our case we have that 0 lies in the spectrum of A, thus the system is no longer exponentially stable. However, we have the following result where $\mathbb{C}_0$ denotes the open right-half plane

$$\mathbb{C}_0 := \{s \in \mathbb{C} \ :\ \mathrm{Re}\, s > 0\}. \tag{29}$$

Proposition 5. The resolvent set $\rho(A)$ contains $\mathbb{C}_0$.

Proof. Let $\lambda \in \mathbb{C}_0$ and $F = [f_1, f_2, f_3, f_4, f_5]^T$ in X. The equation $(\lambda I - A)z = F$ for $z \in \mathcal{D}(A)$ reads

$$\lambda h_S + \frac{q(b)-q(a)}{b-a} = f_1, \tag{30}$$

$$\lambda h(x) + \frac{dq}{\mathrm{d}x} = f_2(x) \quad (x \in \mathcal{E}), \tag{31}$$

$$\lambda q(x) + \frac{dh}{dx} - \mu\frac{d^2q}{dx^2} = f_3(x) \quad (x \in \mathcal{E}), \tag{32}$$

$$M\begin{bmatrix}h(a^-) - \mu\frac{dq}{dx}(a^-) - h_S + \mu\frac{q(b)-q(a)}{b-a}\\ -h(b^+) + \mu\frac{q}{dx}(b^+) + h_S - \mu\frac{q(b)-q(a)}{b-a}\end{bmatrix} = \begin{bmatrix}f_4\\ f_5\end{bmatrix} - \lambda\begin{bmatrix}q_a\\ q_b\end{bmatrix}, \tag{33}$$

$$\lim_{x\to-\infty} q(x) = \lim_{x\to\infty} q(x) = 0, \tag{34}$$

$$q(a) = q_a, \quad q(b) = q_b. \tag{35}$$

From (33) and (25), it follows that

$$\mu\frac{q(b)-q(a)}{l} + h\left(a^-\right) - h_S - \mu\frac{\mathrm{d}q}{\mathrm{d}x}\left(a^-\right) = \left[1 + \frac{l^3}{3}\right]\frac{(\lambda q_{a,0} - f_4)}{l^2} + \left[1 - \frac{l^3}{6}\right]\frac{(f_5 - \lambda q_{b,0})}{l^2}, \tag{36}$$

$$-\mu\frac{q(b)-q(a)}{l} - h\left(b^+\right) + h_S + \mu\frac{\mathrm{d}q}{\mathrm{d}x}\left(b^+\right) = \left[1 - \frac{l^3}{6}\right]\frac{(f_4 - \lambda q_{a,0})}{l^2} + \left[1 + \frac{l^3}{3}\right]\frac{(\lambda q_{b,0} - f_5)}{l^2}. \tag{37}$$

We next transform (30)-(35) into a boundary value problem for q by eliminating h, h_S, q_a, q_b from the above mentioned equations. First, from (31) and (32), we deduce

$$\lambda q - \left(\mu + \frac{1}{\lambda}\right)\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = \phi_1 \quad (x \in \mathcal{E}), \tag{38}$$

where

$$\phi_1 := f_3 - \frac{1}{\lambda}\frac{\mathrm{d}f_2}{\mathrm{d}x} \in L^2(\mathcal{E}).$$

Next, using (30), (31) and (35) in (36) and (37) it follows that

$$\left(\mu + \frac{1}{\lambda}\right)\frac{\mathrm{d}q}{\mathrm{d}x}\left(a^-\right) = \left(\mu + \frac{1}{\lambda} + \frac{\lambda}{l}\right)\frac{q(b) - q(a)}{l} - \frac{\lambda l}{6}\left(2q(a) + q(b)\right) + \phi_2 \tag{39}$$

$$\left(\mu + \frac{1}{\lambda}\right)\frac{\mathrm{d}q}{\mathrm{d}x}\left(b^+\right) = \left(\mu + \frac{1}{\lambda} + \frac{\lambda}{l}\right)\frac{q(b) - q(a)}{l} + \frac{\lambda l}{6}\left(q(a) + 2q(b)\right) + \phi_3 \tag{40}$$

with

$$\phi_2 := \left(\frac{f_2\left(a^-\right)}{\lambda} - \frac{f_1}{\lambda}\right) + \left[1 + \frac{l^3}{3}\right]\frac{f_4}{l^2} - \left[1 - \frac{l^3}{6}\right]\frac{f_5}{l^2},$$
$$\phi_3 := \left(\frac{f_2\left(b^+\right)}{\lambda} - \frac{f_1}{\lambda}\right) + \left[1 - \frac{l^3}{6}\right]\frac{f_4}{l^2} - \left[1 + \frac{l^3}{3}\right]\frac{f_5}{l^2}.$$

We first prove the existence of a weak solution to (38)-(40) with the condition (34) at infinity. We thus set $V := H^1(\mathcal{E})$. The weak formulation of (38)-(40) with the condition (34) is to find $q \in V$ such that

$$B_\lambda(q, \psi) = \int_{\mathcal{E}} \phi_1 \bar{\psi}\mathrm{d}x + \phi_2\bar{\psi}(a) - \phi_3\bar{\psi}(b) \quad (\psi \in V), \tag{41}$$

where

$$B_\lambda(q,\psi) := \int_{\mathcal{E}}\left[\lambda q\bar{\psi} + \left(\mu + \frac{1}{\lambda}\right)\frac{\mathrm{d}q}{\mathrm{d}x}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}x}\right]\mathrm{d}x - \left[\left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l}\right)\frac{q(b) - q(a)}{l} - \frac{\lambda l}{6}\left[2q(a) + q(b)\right]\right]\bar{\psi}(a) + \left[\left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l}\right)\frac{q(b) - q(a)}{l} + \frac{\lambda l}{6}\left[q(a) + 2q(b)\right]\right]\bar{\psi}(b). \tag{42}$$

For $q, \psi \in V$,

$$B_\lambda(q,\psi) = \int_{\mathcal{E}}\left[\lambda q\bar{\psi} + \left(\mu + \frac{1}{\lambda}\right)\frac{\mathrm{d}q}{\mathrm{d}x}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}x}\right]\mathrm{d}x + \left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l}\right)\frac{\left(q(b) - q(a)\right)\overline{\left(\psi(b) - \psi(a)\right)}}{l} + \frac{\lambda l}{6}\left[q(a)\bar{\psi}(a) + q(b)\bar{\psi}(b) + \left(q(b) + q(a)\right)\left(\bar{\psi}(b) + \bar{\psi}(a)\right)\right]. \tag{43}$$

From the above formula, it follows that for any $\lambda \in \mathbb{C}_0$ we can consider positive constants $C = C(\lambda)$, $\alpha = \min\{\mathrm{Re}\ \lambda, \mathrm{Re}\ 1/\lambda\}$ such that

$$|B_\lambda(q,\psi)| \leqslant C\|q\|_V\|\psi\|_V, \mathrm{Re}\, B_\lambda(q,q) \geqslant \alpha\|q\|_V^2 \tag{44}$$

where $q, \psi \in V$, thus B_λ is a bounded and coercive form on V. Moreover, the right hand side of (41) clearly defines a bounded linear functional on V. Thus, the conclusion follows by the complex version of the Lax-Milgram Lemma (see, for instance, Lemma 5.4 on Arendt et al. (2014)).

3.1 Transfer function

From Proposition 5 it follows that the transfer function

$$G(s) = C(sI - A)^{-1},$$

of the system (1)-(8) is defined for every $s \in \mathbb{C}_0$. In this subsection we compute this transfer function and we show that it lies in the Hardy space $H^\infty(\mathbb{C}_0)$ and thus obtain the main ingredient of the proof of Theorem 4. In other terms, we compute the Laplace transform of the solution of (1)-(8) with zero initial data. More precisely, we have:

Proposition 6. The transfer function of the system (1)-(8) is given by

$$G(s) := \frac{1}{\left(1 + \frac{l^3}{12}\right)s^2 + \frac{l^2}{2}s\sqrt{1 + \mu s} + \mu l s + l} \qquad (s \in \mathbb{C}_0). \tag{45}$$

Proof. We first express $h\left(t, a^-\right) - \mu\frac{\partial q}{\partial x}(t, a^-)$ in terms of h_S and $\dot{h}_S$. To this end, for $x \in \mathcal{I}$, we first note that

$$q(t,b) - q(t,a) = -l\dot{h}_S(t). \tag{46}$$

Moreover, using Remark 3 we obtain

$$h(t,a^-) = h(t,b^+), \quad -q(t,a^-) = q(t,b^+), \tag{47}$$

thus

$$q(t,a) = \frac{l}{2}\dot{h}_S, \quad q(t,b) = -\frac{l}{2}\dot{h}_S. \tag{48}$$

From equations (22)-(23) it follows that for $x \in (-\infty, a]$ we have

$$\begin{aligned}&\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} - \mu\frac{\partial^3 q}{\partial t\partial x^2} = 0,\\ &q(t,x) \to 0 \text{ as } x \to -\infty, \quad q(t,a) = \frac{b-a}{2}\dot{h}_S(t),\\ &q(0,x) = \frac{\partial q}{\partial t}(0,x) = 0.\end{aligned} \tag{49}$$

For $f \in L^1[0,\infty]$, we denote by $\widehat{f}$ the Laplace transform of f. Applying the Laplace transform to both sides of (49), we obtain

$$\begin{aligned}&s^2\widehat{q} - (1 + s\mu)\frac{\partial^2\widehat{q}}{\partial x^2} = 0\\ &\widehat{q}(s,x) \to 0 \text{ as } x \to -\infty, \quad \widehat{q}(s,a) = \frac{b-a}{2}\widehat{h}_S, \ \mathrm{Re}(s) > 0.\end{aligned} \tag{50}$$

Hence we can conclude that

$$\widehat{q}(s,x) = \frac{b-a}{2}e^{\frac{-sa}{\sqrt{1+s\mu}}}e^{\frac{sx}{\sqrt{1+s\mu}}}\widehat{h}_S(s) \tag{51}$$

and

$$\begin{aligned}\widehat{h}\left(s,a^-\right) - \mu\frac{\partial\widehat{q}}{\partial x}\left(s,a^-\right) &= -\frac{l}{2}\left(\frac{1}{s} + \mu\right)\frac{s}{\sqrt{1+s\mu}}\widehat{h}_S(s)\\ &= -\frac{l}{2}(\sqrt{1+\mu s})\widehat{h}_S(s).\end{aligned} \tag{52}$$

In a similar way, we obtain

$$\widehat{h}\left(s,b^+\right) - \mu\frac{\partial\widehat{q}}{\partial x}\left(s,b^+\right) = -\frac{l}{2}\left(\sqrt{1+\mu s}\right)\widehat{h}_S(s). \tag{53}$$

Moreover, applying the Laplace transform to (48) we obtain

$$\widehat{q}(s,a) = \frac{l}{2}s\widehat{h}_S, \quad \widehat{q}(s,b) = -\frac{l}{2}s\widehat{h}_S. \tag{54}$$

Finally, applying Laplace transform to (25), we obtain

$$\begin{bmatrix} s\widehat{q}_a(s) \\ s\widehat{q}_b(s) \end{bmatrix} = \frac{M}{b-a} \begin{bmatrix} \widehat{u}(s) \\ -\widehat{u}(s) \end{bmatrix} + M \begin{bmatrix} \widehat{h}(s,a^-) - \mu \frac{\partial \widehat{q}}{\partial x}(s,a^-) - \widehat{h}_S(s) + \mu \frac{\widehat{q}_b - \widehat{q}_a}{b-a} \\ -\widehat{h}(s,b^+) + \mu \frac{\partial \widehat{q}}{\partial x}(s,b^+) + \widehat{h}_S(s) - \mu \frac{\widehat{q}_b - \widehat{q}_a}{b-a} \end{bmatrix}. \quad (55)$$

Hence, using (52) and (54) in (55), we conclude

$$\left(1 + \frac{l^3}{12}\right) s^2 \widehat{h}_S = l \left[-\frac{ls\sqrt{1+\mu s}}{2} - 1 - \mu s \right] \widehat{h}_S + \widehat{u}.$$

The above relation implies that

$$\widehat{h}_S(s) = G(s)\widehat{u}(s),$$

where G is given by (45), which ends the proof.

Lemma 7. Let F be the function defined by

$$F(s) = \left(1 + \frac{l^3}{12}\right) s^2 + \frac{l^2}{2} s\sqrt{1+\mu s} + \mu l s + l, \quad (56)$$

and let $\mathbb{C}_0$ be the open right-half plane, as defined in (29). Then there exists a neighborhood $\mathcal{O}$ of $\overline{\mathbb{C}_0}$ such that F is holomorphic on $\mathcal{O}$. Moreover, F does not vanish on $\overline{\mathbb{C}_0}$.

Proof. The fact that F is holomorphic on a neighborhood of $\overline{\mathbb{C}_0}$ follows from the corresponding property of each term is the right-hand side of (56) (including the one involving the square-root, for which we take the principal determination).

Let $s \in \overline{\mathbb{C}_0}$.We set $z := \sqrt{1+\mu s}$. Since $z^2 = 1 + \mu s$ we have

$$\mathrm{Re}(z^2 - 1) \geq 0. \quad (57)$$

In particular $\mathrm{Re}(z^2) > 0$, which implies that $\arg(z^2) \in (-\pi/2, \pi/2)$. Consequently, we have

$$\arg(z) \in (-\pi/4, \pi/4). \quad (58)$$

As $s = \frac{z^2-1}{\mu}$, $F(s) = 0$ is equivalent to

$$\left(1 + \frac{l^3}{12}\right) \left(\frac{z^2-1}{\mu}\right)^2 + \frac{l^2}{2} \left(\frac{z^2-1}{\mu}\right) z + l z^2 = 0. \quad (59)$$

Multiplying (59) by $\frac{\mu^2}{1+\frac{l^3}{12}}$, we obtain

$$(z^2-1) + 2\frac{l^2\mu}{4\left(1+\frac{l^3}{12}\right)}(z^2-1)z + \frac{l^4\mu^2 z^2}{16\left(1+\frac{l^3}{12}\right)^2} + \frac{l\mu^2 z^2}{16\left(1+\frac{l^3}{12}\right)^2}\left(16 + \frac{l^3}{3}\right) = 0,$$

and then

$$\left(z^2 - 1 + \frac{l^2\mu z}{4\left(1+\frac{l^3}{12}\right)}\right)^2 + \frac{l\mu^2 z^2}{16\left(1+\frac{l^3}{12}\right)^2}\left(16 + \frac{l^3}{3}\right) = 0. \quad (60)$$

Hence equation (59) is equivalent to $P(z)Q(z) = 0$, where

$$P(z) := z^2 - 1 + \frac{l^2\mu z}{4\left(1+\frac{l^3}{12}\right)} + iz\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}} \quad (61)$$

$$Q(z) := z^2 - 1 + \frac{l^2\mu z}{4\left(1+\frac{l^3}{12}\right)} - iz\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}}. \quad (62)$$

Let us prove that $P(z) \neq 0$ and $Q(z) \neq 0$. To this aim we write $z = x + iy$ with $x > 0$ (due to (57) and (58)) and $y \in \mathbb{R}$. We note that:

$$\mathrm{Re}(P(z)) = \mu\mathrm{Re}(s) + \frac{l^2\mu x}{4\left(1+\frac{l^3}{12}\right)} - y\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}},$$

$$\mathrm{Im}(P(z)) = y\left(2x + \frac{l^2\mu}{4\left(1+\frac{l^3}{12}\right)}\right) + x\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}},$$

$$\mathrm{Re}(Q(z)) = \mu\mathrm{Re}(s) + \frac{l^2\mu x}{4\left(1+\frac{l^3}{12}\right)} + y\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}},$$

$$\mathrm{Im}(Q(z)) = y\left(2x + \frac{l^2\mu}{4\left(1+\frac{l^3}{12}\right)}\right) - x\frac{\mu\sqrt{l}}{4\left(1+\frac{l^3}{12}\right)}\sqrt{16 + \frac{l^3}{3}}.$$

If $\mathrm{Im}(P(z)) = 0$, since l, μ, x are positive, then $y < 0$. But since $\mathrm{Re}(z^2 - 1) \geq 0$, this implies that $\mathrm{Re}(P(z)) > 0$. Therefore $P(z) \neq 0$.

If $\mathrm{Re}(Q(z)) = 0$, since $\mathrm{Re}(z^2 - 1) \geq 0$ and $x > 0$ are positive we conclude that $y < 0$. This implies that $\mathrm{Im}(Q(z)) < 0$. Therefore $Q(z) \neq 0$.

Thus $F(s) \neq 0$ on $\overline{\mathbb{C}_0}$, which concludes the proof of the Lemma.

Note that this result can be extended to the case of a more function defined by $\tilde{F}(s) := s^2 + a_1 s\sqrt{s+\epsilon} + a_2 s + a_3$, only requiring the positivity of the coefficients ϵ, a_i but not relying at all on the specific values of the physical parameters and the algebraic relations between them, see the companion paper Vergara-Hermosilla et al. (2020).

3.2 Proof of Theorem 4

By Lemma 7 we know that the function F defined in (56) is not vanishing on $\overline{\mathbb{C}_0}$. Moreover, since

$$\lim_{|s| \to \infty} |F(s)| = +\infty,$$

we have that the map $s \mapsto |F(s)|$ is bounded from below on $\overline{\mathbb{C}_0}$. We conclude that the transfer function G defined in (45) is such that

$$\sup_{s \in \mathbb{C}_0} |G(s)| < \infty.$$

By the Paley-Wiener theorem (see, for instance, (Rudin, 1987, Section 19.2)), this implies that (1)-(8) is input-output stable, so that the proof of the theorem is complete.

4. FURTHER WORK

Other issues that have not been mentionned in this work, but that are interesting for our future research are the following ones:

- Study of the sectorial properties of the resolvent of the operator A defined in eq. (27), in order to establish results related to the existence and uniqueness of strong solutions and strong stability of the system.
- Study of the new system with vertical velocity as output, instead of the height: would it be a well-posed system in some suitable state space? Moreover, the stability properties of this new system and its applications to energy wave extractors will be investigated.

- Time-domain analyis of both these infinite-dimensional systems, together with a careful study of the asymptotic behaviour as a function of the physical parameters, following Vergara-Hermosilla et al. (2020).

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