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A terminal state contractive nonlinear MPC with output zones and input targets ⋆

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Abstract: Terminal equality and inequality constraints along with terminal costs are known to be ingredients that grant stability in many Nonlinear Model Predictive Control (NMPC) approaches. Despite the availability of different methods for computing a suitable terminal set and cost, they usually rely on the linearization of the system and the design of terminal stabilizing control laws. Thus, approaches based on contracting constraints represent an alternative to circumvent the calculation of terminal sets and penalties. The present work proposes an NMPC based on a terminal state contracting constraint. This approach also avoids the need of large prediction horizon, helping to alleviate the computational burden usually associated with NMPC. Another contribution of this proposal is a formulation in terms of output zone control and input targets, designed for the common situation in the process industry where the number of degrees of freedom is not enough to independently track the setpoint of all controlled variables. A simulated case study is presented with the application of the proposed controller to the well-known quadruple-tank process.

Keywords: Nonlinear predictive control, Constrained control, Tracking

1. INTRODUCTION

The design of model predictive controllers based on rigorous nonlinear models with guaranteed stability becomes challenging in some applications when the usual ingredients for stability such as infinite prediction horizon or terminal equality constraints (Mayne et al., 2000) may not be included in the optimization problem due to prohibitive computational times. Moreover, terminal equality constraints may reduce the feasibility domain (Chen and Allgöwer, 1998), which is undesirable in industrial applications where feasible solutions are required at every time step.

In order to address the drawbacks associated with terminal equality constraints, Michalska and Mayne (1993) first developed a dual-mode controller based on a terminal inequality constraint, calculated from a linearization of the system model near the origin. The two modes of operation consist of a receding horizon controller applied outside the terminal region and a linear state feedback controller for the case when the states lie inside this region. Chen and Allgöwer (1998) proposed a quasi-infinite horizon formulation that is based on a terminal inequality and cost to avoid the need to switch between controllers. Ferramosca et al. (2009) developed a similar stabilizing controller based on the same stability ingredients but suitable for output tracking. A generalized terminal state constraint was developed by Fagiano and Teel (2013) which consists of a terminal equality constraint applied at any fixed point lying between the initial state and the desired reference, enlarging the controller feasibility set in comparison with a conventional terminal equality constraint.

Some authors choose to avoid terminal equality or inequality constraints either due to the need of using large prediction horizons or the difficulty of computing a controlled-invariant set to be used in a terminal inequality constraint. As an alternative, an NMPC based on a contracting constraint was proposed by Yang and Polak (1993), in which the authors failed to realize that this approach conferred close-loop exponential stability, as pointed out in (Kothare and Morari, 2000). In their work, Kothare and Morari (2000) impose a state contraction between the initial and terminal states, but their controller does not follow the receding horizon principle since it is based on the application of the whole control sequence to the system before the control problem is recomputed. A first-state contractive strategy that is implemented in the usual moving horizon fashion was proposed by Xie and Fierro (2008), which represents a very restrictive approach despite the straightforward stability proof. Other approaches featuring a cost contracting constraint (Mejía and Stipanović, 2009) and a terminal constraint-free contractive formulation (Alamir, 2017) were also proposed.

In this study, a new NMPC approach based on a contracting constraint is presented. As most contractive strategies, this formulation is independent of system linearization and also

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avoids the *a priori* computation of a terminal set that is controlled-invariant with respect to a stabilizing control law. Another aspect of practical relevance considered in the present paper is derived from the observation that in many chemical processes there are more controlled variables than manipulated ones, being impossible to lead all controlled variables to independent setpoints. Therefore, this work considers zone tracking instead of reference tracking (González and Odloak, 2009). In addition, the proposed controller also features constraints on input increments, which are not commonly addressed in the NMPC literature but play an important role regarding actuators preservation and equipment damage prevention.

The subsequent sections are organized as follows. Section 2 provides a first approach for the zone control NMPC based on a terminal equality constraint, whereas in Section 3 this controller is extended using a terminal contracting constraint in order to enlarge the feasibility domain of the control problem. In Section 4, the application of the proposed controller to the well-known quadruple-tank process is presented alongside some simulation results and Section 5 draws some final remarks.

2. NOTATION AND PROBLEM FORMULATION

The following notation is used in this study. For a given positive-definite symmetric matrix $P \in \mathbb{R}^{n_x \times n_x}$, the weighted Euclidean norm of $x \in \mathbb{R}^{n_x}$ is denoted as $\|x\|_P := \sqrt{x^T P x}$. The set of real numbers is denoted as $\mathbb{R}$, while $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{[a,b]}$ correspond to the set of integers, non-negative integers and integers between a and b, respectively. For a given initial state $x(k)$, the n-steps ahead predicted state is denoted by $x(n|k)$ and $u(n|k)$ represents the n-th control of the input sequence computed at time step k. Other definitions will be presented throughout the article as needed.

The goal of this paper is to propose an NMPC for zone control and tracking of input targets without excessively tight terminal constraints. However, before achieving such formulation, it is instructive to start with an optimization problem that explicitly forces the system to achieve a steady state at the end of the prediction horizon, similarly to the proposal of Fagiano and Teel (2013) but including a zone constraint on the steady state. Later in Section 3 the problem is rewritten with a terminal state contracting constraint.

Consider a discrete time-invariant nonlinear system represented by the following model:

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

in which $x \in \mathcal{X}$ and $u \in \mathcal{U}$ are the system states and inputs, respectively, and $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is closed and $\mathcal{U} \subset \mathbb{R}^{n_u}$ is compact. The successor state is computed for a given pair (x, u), in which $f(\cdot)$ is implicitly defined by difference equations with $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. Also, since we aim to treat the case in which the input moves are constrained, let us define $\Delta u(k) = u(k) - u(k-1)$ such that $\Delta u \in \mathcal{U}_\Delta \subset \mathbb{R}^{n_u}$.

Assumption 1. The set $\mathcal{X}_{ss}$ of equilibrium points $x_{ss} \in \mathcal{X}_{ss} \subset \mathbb{R}^{n_x}$ with $\mathcal{X}_{ss} := \{x_{ss} \in \mathbb{R}^{n_x} : x_{ss} = f(x_{ss}, u_{ss}),$ $x_{ss,min} \leq x_{ss} \leq x_{ss,max}, u_{ss} \in \mathcal{U}\}$ is non empty.

Then, consider the following infinite horizon control cost function:

$$\begin{aligned} V(k) = &\sum_{j=0}^{\infty} \|x(j|k) - x_{ss,k}\|^2_{Q_x} + \sum_{j=0}^{\infty} \|\Delta u(j|k)\|^2_R \\ &+ \|u(N_c - 1|k) - u_{des}\|^2_{Q_u} \end{aligned} \tag{2}$$

in which u_{des} denotes an optimizing input target, assumed known. Also, consider after a given control horizon of length N_c we have that

$$\Delta u(j|k) = 0, \qquad j \geq N_c \tag{3}$$

Thus, a sequence of control actions can be defined as follows:

$$\mathbf{u}_k = \left[u(0|k)^T \ u(1|k)^T \ \cdots \ u(N_c - 1|k)^T\right]^T \tag{4}$$

Considering a prediction horizon of length $N_p \geq N_c$, then the following terminal constraint is enforced:

$$x(N_p|k) - x_{ss,k} = 0 \tag{5}$$

in which $x_{ss,k}$ is defined as the steady state corresponding to the last control action of $\mathbf{u}_k$, that is, $x_{ss,k} = f(x_{ss,k}, u(N_c - 1|k))$. Notice that such $u(N_c - 1|k)$ exists due to Assumption 1.

Therefore, assuming (3) and (5) hold, then the summations in the control cost function can be reduced and the nonlinear MPC results from the solution to the following finite horizon problem:

Problem 1. $\mathcal{P}_1(x(k))$:

$$\begin{aligned} \min_{\mathbf{u}_k, x_{ss,k}, u_{ss,k}} \quad V(k) = &\sum_{j=0}^{N_p-1} \|x(j|k) - x_{ss,k}\|^2_{Q_x} \\ &+ \sum_{j=0}^{N_c-1} \|\Delta u(j|k)\|^2_R \\ &+ \|u(N_c - 1|k) - u_{des}\|^2_{Q_u} \end{aligned} \tag{6}$$

subject to

$$x(0|k) = x(k) \tag{7}$$

$$x(j+1|k) = f(x(j|k), u(j|k)), \quad \forall j \in \mathbb{Z}_{[0, N_p - 1]} \tag{8}$$

$$x_{ss,k} = f(x_{ss,k}, u(N_c - 1|k)) \tag{9}$$

$$x(N_p|k) - x_{ss,k} = 0 \tag{10}$$

$$x_{ss,k} \in \mathcal{X}_{ss} \tag{11}$$

$$u(j|k) \in \mathcal{U}, \quad \forall j \in \mathbb{Z}_{[0, N_c - 1]} \tag{12}$$

$$\Delta u(j|k) \in \mathcal{U}_\Delta, \quad \forall j \in \mathbb{Z}_{[0, N_c - 1]} \tag{13}$$

The following lemma ensures the recursive feasibility of Problem 1.

Lemma 1. For an undisturbed nominal system, if Problem 1 is feasible at time step k, it will remain feasible for any subsequent time step.

Proof. Suppose Problem 1 has a feasible solution at time step k, which results in the following optimal control sequence:

$$\mathbf{u}^*_k = \left[u^*(0|k)^T \ u^*(1|k)^T \ \cdots \ u^*(N_c - 1|k)^T\right]^T \tag{14}$$

Also, consider a candidate solution to Problem 1 at time step $k+1$ given as follows:

$$\tilde{\mathbf{u}}_{k+1} = \left[u^*(1|k)^T \ \cdots \ u^*(N_c - 1|k)^T \ u^*(N_c - 1|k)^T\right]^T \tag{15}$$

It is easy to see that $\tilde{\mathbf{u}}_{k+1}$ satisfies input constraints (12) and (13). Also, since $\tilde{u}(N_c-1|k+1) = u^*(N_c-1|k)$, then $\tilde{x}_{ss,k+1} = x_{ss,k}$ and $\tilde{x}(N_p|k+1) = x^*(N_p|k)$, which implies that constraints (9) and (11) are satisfied as well.

Now, observe that, at time step k, $x^*(N_p|k) = x_{ss,k}$, which means that $x^*(N_p|k)$ is a steady state. Therefore, since $\tilde{x}_{ss,k+1} = x_{ss,k}$, it results that $\tilde{x}(N_p|k+1) = \tilde{x}_{ss,k+1}$, which means that the candidate solution $\tilde{\mathbf{u}}_{k+1}$ also satisfies the terminal constraint (10). Therefore, by induction, Problem 1 remains feasible at any time step $k+j > k$, $j \in \mathbb{Z}_+$. □

After showing the recursive feasibility of Problem 1, the following theorem addresses the convergence and stability of the controller.

Theorem 1. Consider the nominal system in the absence of disturbances and let Assumption 1 hold. Also, suppose that Problem 1 has a feasible solution at time step k and that u_{des} is an admissible input target. The controller drives the system into the control zone in which the pair (x_{ss}, u_{des}), $x_{ss} \in \mathcal{X}_{ss}$, $u_{des} \in \mathcal{U}$ is an asymptotically stable solution to Problem 1.

Proof. The proof follows the standard steps of stability guarantees in MPC literature. Let $\mathbf{u}_k^*$ denote the optimal control sequence computed as solution of Problem 1 at time step k that produces an optimal cost value $V^*(k)$. The sequential solution of Problem 1 is guaranteed by Lemma 1. Now, at time step $k+1$, consider a sub-optimal solution resulting from the control sequence $\tilde{\mathbf{u}}_{k+1}$ defined in (15). Moreover, as already shown in the proof of Lemma 1, $\tilde{x}_{ss,k+1} = x^*_{ss,k}$ because $\tilde{u}(N_c-1|k+1) = u^*(N_c-1|k)$. Also, observe that $\tilde{x}(j|k+1) = x^*(j+1|k)$ and that $\Delta\tilde{u}(N_c-1|k+1) = 0$. The corresponding cost value of the shifted solution is given by $\tilde{V}(k+1)$. Comparing the cost at the consecutive steps k and $k+1$, it follows that

$$V^*(k) - \tilde{V}(k+1) = \|x^*(0|k) - x^*_{ss,k}\|^2_{Q_x} + \|\Delta u^*(0|k)\|^2_R \tag{16}$$

Thus, since Q_x and R are positive-definite matrices, the left-hand side of (16) is non-negative, which means that $\tilde{V}(k+1) \leq V^*(k)$ and it also implies that $V^*(k+1) \leq V^*(k)$. Consequently, the sequence of control cost at subsequent time steps is non-increasing and, since the control cost is built such that is bounded below by zero, it converges to zero provided u_{des} is reachable. Convergence of the control cost to the origin implies that $x(k) \to x_{ss}$ and that $u(k) \to u_{ss} = u_{des}$ as $k \to \infty$. □

3. NONLINEAR MPC WITH TERMINAL CONTRACTION CONSTRAINT

Since Problem 1 considers constraints on the input increments, this may lead to a smaller feasibility set in comparison to the majority of NMPC approaches available in the literature that do not consider this type of input constraint. For instance, consider a given initial $u(k-1|k) \in \mathcal{U}$, then notice that Problem 1 may turn infeasible if the prediction horizon N_p is not large enough to allow the computation of a $u(N_c-1|k) \in \mathcal{U}$ such that $x_{ss,k} \in \mathcal{X}_{ss}$. In other words, constraints (9) and (11)-(13) may not be simultaneously satisfied. Moreover, even if a $x_{ss,k} \in \mathcal{X}_{ss}$ with $u(N_c-1|k) \in \mathcal{U}$ could be computed, infeasibilities may arise when the terminal state $x(N_p|k)$ is not an equilibrium point of the system, failing to satisfy constraint (10). In fact, this is the main problem of terminal equality-based NMPC approaches, requiring large prediction horizons, which also increases computational cost (Chen and Allgöwer, 1998).

For this reason, we propose some modifications intended to enlarge the feasibility set of the controller. Thus, instead of forcing the equilibrium point to be defined as a function of the last control action $u(N_c-1|k)$ of the control horizon as in (9), we now let the optimizer to chose any $u_{ss,k} \in \mathcal{U}$ such that $x_{ss,k} \in \mathcal{X}_{ss}$. Therefore, constraint (9) is substituted by $x_{ss,k} = f(x_{ss,k}, u_{ss,k})$. Besides, constraint (10) can be relaxed by including a slack variable $\delta_k \in \mathbb{R}^{n_x}$ as follows:

$$x(N_p|k) - x_{ss,k} - \delta_k = 0 \tag{17}$$

Although the standard approach would be to penalize this slack variable in the controller cost function, it would be no different from the well-known terminal penalty, which has been shown to suffice in guaranteeing close-loop stability for open-loop stable systems (Chen and Allgöwer, 1998). However, besides the fact that this procedure alone guarantees stability only for stable systems, it still requires a suitable terminal penalty to be computed offline.

Thus, in order to escape this step of computing terminal weights, we want to ensure that the distance between $x(N_p|k)$ and a feasible steady state $x_{ss,k}$ decreases as $k \to \infty$. For this purpose, the following terminal contraction constraint is appended to the control problem:

$$\|\delta_k\|^2_S \leq \alpha_k \|\delta_{k-1}\|^2_S \tag{18}$$

in which S is a positive-definite matrix, δ_{k-1} is computed at previous time step and α_k is a contraction parameter, with $\alpha_k \in [\alpha_{min}, 1)$.

Observe that, unlike some existing contractive NMPC strategies (Kothare and Morari, 2000; Xie and Fierro, 2008), the contraction is imposed only at the end of the prediction horizon and the contraction parameter is considered as a decision variable of the control problem. This is done because choosing a fixed contraction parameter is not intuitive and small values of α_k may compromise system performance due to the multi-objective characteristic of the problem. Besides, a lower bound α_{min} is considered in order to avoid the computation of small values of α_k due to its overpenalization in the cost function.

Then, the extended nonlinear MPC is based on the solution of the following optimization problem:

Problem 2. $\mathcal{P}_2(x(k), \delta_{k-1})$:

$$\min_{\mathbf{u}_k, x_{ss,k}, u_{ss,k}, \delta_k, \alpha_k} \quad V_2(k) = \sum_{j=0}^{N_p} \|x(j|k) - x_{ss,k}\|^2_{Q_x} + \sum_{j=0}^{N_c-1} \|\Delta u(j|k)\|^2_R + \|u_{ss,k} - u_{des}\|^2_{Q_u} + \|\alpha_k - \alpha_{min}\|^2_W \tag{19}$$

subject to

$$x(0|k) = x(k) \tag{20}$$

$$x(j+1|k) = f(x(j|k), u(j|k)), \quad \forall j \in \mathbb{Z}_{[0,N_p-1]} \tag{21}$$

$$x_{ss,k} = f(x_{ss,k}, u_{ss,k}) \tag{22}$$

$$x(N_p|k) - x_{ss,k} = \delta_k \tag{23}$$
$$x_{ss,k} \in \mathcal{X}_{ss} \tag{24}$$
$$u_{ss,k} \in \mathcal{U} \tag{25}$$
$$x(k+j|k) \in \mathcal{X}, \quad \forall j \in \mathbb{Z}_{[0,N_p-1]} \tag{26}$$
$$u(k+j|k) \in \mathcal{U}, \quad \forall j \in \mathbb{Z}_{[0,N_c-1]} \tag{27}$$
$$\Delta u(k+j|k) \in \mathcal{U}_\Delta, \quad \forall j \in \mathbb{Z}_{[0,N_c-1]} \tag{28}$$
$$\|\delta_k\|_S^2 \leq \alpha_k \|\delta_{k-1}\|_S^2, \quad \alpha_k \in [\alpha_{min}, 1) \tag{29}$$

Remark 1. Notice that δ_{k-1} must be fed back from previous sample time. However, in practical applications one often has $x(0|k) \neq x(1|k-1)$ due to disturbances. In this case, δ_{k-1} should be replaced by $\tilde{\delta}_k$ given as follows:

$$\tilde{\delta}_k = x(N_p - 1|k) - x^*_{ss,k-1} \tag{30}$$

in which $x(N_p-1|k)$ is the state prediction at N_p-1 starting from $x(0|k) = x(k)$ and applying the control sequence $\tilde{\mathbf{u}}_k = \left[u^*(1|k-1)^T \ \cdots \ u^*(N_c-1|k-1)^T\right]^T$ computed at time step $k-1$. This procedure results in $\tilde{\delta}_k = \delta_{k-1}$ when $x(k) = x(1|k-1)$, i.e. when the model is nominal and in the absence of disturbances .

Remark 2. In practice, we should allow α_k to be also 1 in the case where the terminal contraction can not be satisfied. However, the weight W must be chosen properly so that the contraction is guaranteed whenever it is feasible.

Since the feasibility set of Problem 2 is not necessarily the same as the controlled-invariant set under the implicit control law generated from the sequential application of the proposed controller, recursive feasibility may not be assured. In other words, since the feasibility set is not a controlled-invariant set *per se*, Problem 2 can generate inputs that steer the system to outside the feasibility set. In this regard, we shall make the following assumption:

Assumption 2. The set $\mathcal{X}$ is a $\mathcal{U}$-controlled invariant set that contains a neighborhood of $x_{ss} \in \mathcal{X}_{ss}$.

Remark 3. In general, the above defined Assumption 2 is not satisfied for any $\mathcal{X}$ simply based on physical system limits, but state constraints can be tightened so that this assumption is satisfied (Alamir, 2017).

Assumption 3. There exists a constant $\kappa \in (0, \infty)$ such that $\|x(k+j) - x_{ss}\| \leq \kappa \|x(k) - x_{ss}\|$, $\forall j \in \mathbb{Z}_+$.

Remark 4. Notice that by Assumption 3 finite escape time systems are out of the scope of this study.

The following theorem ensures the convergence of δ_{k+j} to the origin.

Theorem 2. Let Assumptions 1-3 hold. Assume Problem 2 is feasible at time step k, then δ_{k+j} converges exponentially to the origin as $j \to \infty$.

Proof. Assumption 1 is necessary to guarantee that there is a steady state satisfying constraints (22), (24)-(25), avoiding the situation where Problem 2 is trivially infeasible. Provided Problem 2 is feasible at time step k, it will remain feasible for every subsequent time step $k+j$, $j \in \mathbb{Z}_+$ by virtue of Assumption 2. Therefore, the sequential solution of Problem 2 leads to the following relationship:

$$\|\delta_{k+j}\|_S^2 \leq \alpha_{k+j} \|\delta_{k+j-1}\|_S^2 \leq \cdots \leq \prod_{i=1}^{j} \alpha_{k+i} \|\delta_k\|_S^2 \tag{31}$$

Define $\bar{\alpha}$ as follows:

$$\bar{\alpha} := \max\{\alpha_{k+i} : i \in \mathbb{Z}_{1:j} : j \in \mathbb{Z}_+\} \tag{32}$$

Now, observing that $\prod_{i=1}^{j} \alpha_{k+i} \leq \bar{\alpha}^j$, it follows that:

$$\|\delta_{k+j}\|_S^2 \leq \bar{\alpha}^j \|\delta_k\|_S^2 \tag{33}$$

From Assumption 3, there exists a finite number $\kappa > 0$ such that:

$$\|x(N_p|k) - x_{ss,k}\|_S^2 \leq \kappa \|x(k) - x_{ss,k}\|_S^2 \tag{34}$$

Since $\delta_k = x(N_p|k) - x_{ss,k}$, the multiplication of (34) by $\bar{\alpha}^j$ results in the following inequality:

$$\bar{\alpha}^j \|\delta_k\|_S^2 \leq \bar{\alpha}^j \kappa \|x(k) - x_{ss,k}\|_S^2 \tag{35}$$

Then, combining (33) and (35) yields:

$$\|\delta_{k+j}\|_S^2 \leq \bar{\alpha}^j \kappa \|x(k) - x_{ss,k}\|_S^2 \tag{36}$$

Recognizing that, since $\bar{\alpha} \leq e^{(\bar{\alpha}-1)}$, then $\bar{\alpha}^j \leq e^{-(1-\bar{\alpha})j}$, we have the following relationship:

$$\|\delta_{k+j}\|_S^2 \leq \kappa e^{-(1-\bar{\alpha})j} \|x(k) - x_{ss,k}\|_S^2 \tag{37}$$

Therefore, $\|\delta_{k+j}\|_S^2$ goes exponentially to zero as $j \to \infty$, which implies that δ_{k+j} also converges to zero. □

Remark 5. Notice that once δ_{k+j} is zero, we have that $\alpha_{k+j} = \alpha_{min}$, $x(N_p|k+j) = x_{ss,k+j}$ and $u(N_c|k+j) = u_{ss,k+j}$. Therefore, Problem 2 reduces to Problem 1, whose asymptotic stability is guaranteed by Theorem 1.

4. APPLICATION EXAMPLE

In this section, we describe the quadruple-tank process and provide a first-principles model that is used here as a simulated case study. Next, we define two simulation scenarios with changes in the control zones and input targets and provide some results.

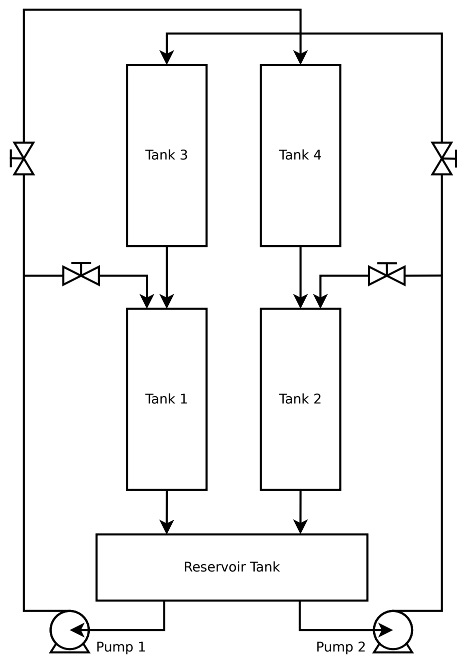

Fig. 1. Schematic diagram of the quadruple-tank process.

Table 1. Variables of the four-tank system.

Symbol	Meaning
x_i	water level of tank i
a_i	cross-section of the outlet hole of tank i
A_i	cross-section of tank i
γ_i	percentage of flow directed to the tank i
g	acceleration due to gravity
u_j	flow rate after pump j

4.1 The quadruple-tank process

The quadruple-tank process (Johansson, 2000) is a well-known benchmark that was developed for testing and designing controllers for nonlinear multivariable processes. This plant consists of four tanks, two pumps and some valves. A schematic diagram is depicted in Fig. 1. This system has adjustable zeros, which allows one to set up the process to operate either in minimum or non-minimum phase conditions by only changing the position of the valves that adjust the feed of the tanks.

The goal is to control the level of the lower tanks 1 and 2 by manipulating the feed flow through the two pumps. Hence, this system has two inputs $u = [u_1\ u_2]^T$ related to the pumps and four states $x = [x_1\ x_2\ x_3\ x_4]^T$ corresponding to the level of each of the four tanks, which are assumed to be measured in this work.

The first-principles model of this system can be obtained by using mass balances and Bernoulli's law, resulting in the following system of ordinary differential equations (ODE):

$$\frac{dx_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gx_1} + \frac{a_3}{A_1}\sqrt{2gx_3} + \frac{\gamma_1}{A_1}u_1 \qquad (38)$$

$$\frac{dx_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gx_2} + \frac{a_4}{A_2}\sqrt{2gx_4} + \frac{\gamma_2}{A_2}u_2 \qquad (39)$$

$$\frac{dx_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gx_3} + \frac{(1-\gamma_2)}{A_3}u_2 \qquad (40)$$

$$\frac{dx_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gx_4} + \frac{(1-\gamma_1)}{A_4}u_1 \qquad (41)$$

in which the variables are described in Table 1.

We used the same values of model parameters provided in Raff et al. (2006), which are also given in Table 2.

Table 2. Model parameters (Raff et al., 2006).

Parameter	Value	Units
A_1, A_2	50.27	cm^2
A_3, A_4	28.27	cm^2
a_1	0.233	cm^2
a_2	0.242	cm^2
a_3, a_4	0.127	cm^2
g	981	cm/s^2
γ_1, γ_2	0.4	-

4.2 Simulation results

Simulation scenarios were carried out with two different controllers, in which controller I consists in the NMPC defined by Problem 2 and controller II corresponds to the same control problem, but without the terminal contraction constraint (29).

Table 3. System constraints.

Variable	Min. value	Max. value	Unit
x	$[7.5\ 7.5\ 3.5\ 4.5]^T$	$[28\ 28\ 28\ 28]^T$	cm
u	$[0\ 0]^T$	$[60\ 60]^T$	mL/s
Δu	$[-5\ \ -5]^T$	$[5\ 5]^T$	mL/s

For performing the simulation, the sampling time was chosen as $T_s = 2\,\text{s}$ and the NMPC tuning parameters are $N_p = 8$, $N_c = 4$, $Q_x = diag([1\ 1\ 10^{-2}\ 10^{-2}])$, $Q_u = diag([1\ 0])$, $R = 10^{-2}diag([1\ 1])$, $S = diag([1\ 1\ 1\ 1])$ and $W = 10^2$. Observe that Q_u was chosen such that only the first input has an optimizing target, which is assumed to be known. We considered the minimum contraction parameter as $\alpha_{min} = 0.5$. The simulation starts with $x(k) = [8.4495\ 7.8327\ 10.2386\ 10.2386]^T$, $u(k-1) = [30\ 30]^T$, $u_{1,des} = 39$ and control zones $x_{ss,min} = [12\ 11\ 3.5\ 4.5]^T$ and $x_{ss,max} = [14\ 13\ 28\ 28]^T$. Then, at $t = 6\,\text{min}$, the input target and the control zones change to $u_{1,des} = 41$, $x_{ss,min} = [14\ 13\ 3.5\ 4.5]^T$ and $x_{ss,max} = [15\ 15\ 28\ 28]^T$. Finally, at $t = 12\,\text{min}$, the input target returns to $u_{1,des} = 39$. In addition, the controller considers the system constraints given in Table 3.

Since only the levels of tanks 1 and 2 are controlled inside their zones, we set the zones for the other tanks to be their physical limits. The evolution of system states, denoted as x for controller I and x_{nc} for controller II (no contraction), are depicted in Fig. 2, while the inputs, represented as u for controller I and u_{nc} for controller II, are shown in Fig. 3. In the plots, the steady states x_{ss} and inputs u_{ss} computed with controller II are omitted for clarity.

In the scenario simulated with controller I, the outputs start outside their control zones and were steered into them, while the predicted steady states were updated to steer the input u_1 to its target as well. As shown in Fig. 4, the contraction parameter decreases until it reaches its minimum value and the quantity $\|\delta\|_S^2$ converges to zero, as expected. After zones and target changes at $t = 6$ min, the system is driven to the new control zones and input target. This change in operating point required the terminal contraction to be attenuated as can be observed from the increase of α, which also results in $\|\delta\|_S^2$ to increase, although it converged to zero again. Finally, the system was driven to a new operating point after the input target changed at $t = 12$ min. However, this last change did not perturb the system enough to produce a deviation between the terminal predicted state and the predicted steady state, which is a particular case in which the controller defined by Problem 1 could be applied.

Regarding the simulation with controller II, it failed to maintain the levels of tanks 1 and 2 inside their control zones and did not steer input u_1 to its target. In fact, the system only stabilized because the upper limit of the level of tank 3 was reached at about $t = 7$ min, which forced the controller to compute appropriate inputs to prevent state constraint violation. The quantity $\|\delta\|_S^2$ corresponding to controller II is absent in Fig. 4 because it was much larger than the one of controller I. This scenario shows the importance of the terminal contraction constraint as an ingredient that confers closed-loop stability and guarantees the convergence of states to inside the control zones.

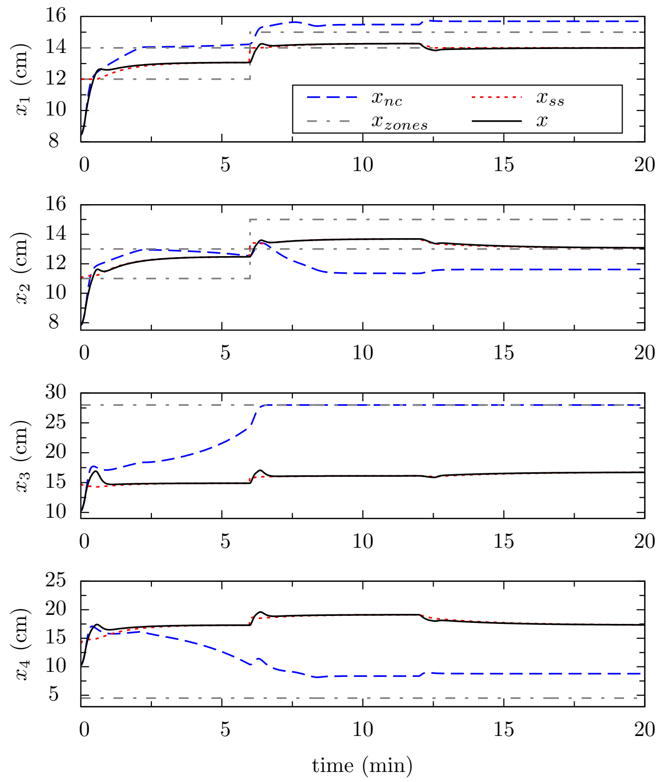

Fig. 2. System states, steady states and control zones.

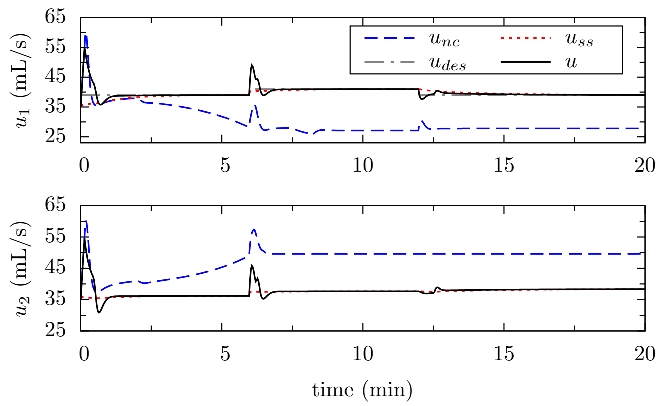

Fig. 3. System inputs, steady inputs and input targets.

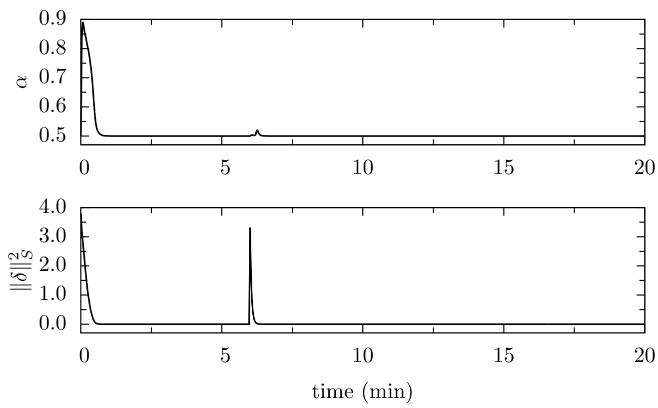

Fig. 4. Contraction parameter and $\|\delta\|_S^2$.

5. CONCLUSION

This paper presented a formulation of a contractive NMPC to control systems in terms of output zones and input targets. The proposed controller does not depend on predefined stabilizing control laws nor on terminal sets. The simulated scenarios showed the role of the contracting constraint in driving the outputs inside their control zones and also grating closed-loop stability to the system, which was not the case for the controller without this stability ingredient. Our ongoing research is devoted to the development of other contractive NMPC approaches that are suitable for industrial applications.

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