Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On a generalized flat input definition and physical realizability

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Abstract: We generalize the definition of flat inputs to implicit nonlinear differential equations of order $\alpha > 1$. By allowing injections of input components and its time derivatives up to some finite order δ in both the dynamics and the output equations we show that physical realizability of generalized flat inputs can be achieved in cases that were shown to possess no physically realizable flat input. In addition, it can be shown that there always exists a physically realizable (generalized) flat input of order $\delta < \alpha$ in the linear case.

Keywords: Nonlinear systems, differential flatness, flat inputs, algorithm, physical realizability

1. INTRODUCTION

While linear systems have been well understood, the control of nonlinear systems faces many open problems. A typical approach in control theory is a linearization via (dynamic) feedback which then allows the usage of a rich set of linear methods. The property that characterizes the class of nonlinear systems which is feedback linearizable is differential flatness and has been introduced in the early 1990s by Fliess et al. (1992, 1995). Flat systems are in a certain sense equivalent to linear controllable systems (Fliess et al., 1999a). Informally, they possess a (possibly virtual) output called a flat output that (along with its derivatives) parametrizes the system variables, while the flat output itself is parametrized by the system variables (and derivatives thereof). Despite a lot of progress (see e.g. Lévine (2004, 2009, 2011); Schöberl and Schlacher (2007); Antritter and Lévine (2008); Franke and Röbenack (2013); Fritzsche et al. (2016b,a)) and the fact that many (industrial) systems are flat (see e.g. Fliess et al. (1999a) and the references therein), for arbitrary systems neither the existence of flat outputs nor the computation thereof have been answered sufficiently and remain open problems, see Fliess et al. (1999b); Martin et al. (2007).

A different approach to exploit flatness has been taken from a dual perspective: The question of where to influence a system such that a given output becomes flat motivates the definition of flat inputs, as defined by Waldherr and Zeitz (2008, 2010). For (locally) observable state space systems, there exist different approaches for the computation of flat inputs (Nicolau et al., 2018a; Fritzsche and Röbenack, 2018b). It has been shown that the computation of flat inputs is far easier (for this class of nonlinear systems) compared to flat outputs, and can even be carried out in a purely algebraic way, i.e., there is no integrability condition to be satisfied, see Fritzsche and Röbenack (2018b). Besides actuator placement, flat inputs have been shown to be useful as pure computational quantities for the control of non-flat systems (Stumper et al., 2009), for parameter identification (Schenkendorf and Mangold, 2014) and secure communication (Nicolau et al., 2018b).

While (local) controllability is a necessary condition for the existence of flat outputs, flat inputs exist for (certain) non-observable systems as well. For special non-observable two-output systems, the computation of flat inputs has been solved by Nicolau et al. (2018a), whereas the general non-observable case remains an open problem. However, as shown in Fritzsche et al. (2019) these difficulties can be circumvented and a flat input based tracking control can be achieved for non-observable systems with stable internal dynamics, too.

So far, flat inputs have been introduced for state space systems only, although most flatness based methods exist for implicit systems as well. When dealing with higher order differential equations, the computation of flat input vector fields is complicated by possible injections into definitional equations that arise when transforming these equations into state space representation. While Waldherr and Zeitz (2010) define the property of physical realizability of a flat input, we draw a different conclusion:

In this contribution we generalize the definition of flat inputs in different ways: (1) We allow injections of the input and its time derivatives up to some finite order δ, (2) we permit input injections (and derivatives up to some finite order) into the output equation, and (3) we define flat inputs for implicit nonlinear systems. None of these generalizations are a limitation regarding a physical realization. We describe how the algorithm proposed in Fritzsche and Röbenack (2018b) can be used to compute generalized (physically realizable) flat inputs for observable implicit nonlinear systems. Finally, while the order of input derivative injections δ can be arbitrary large (but finite), for linear systems there can always be found a generalized flat input with $\delta < \alpha$ where α is the order of our original differential equation.

2. PRELIMINARIES

2.1 Ore polynomial matrices

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers, $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ the set of positive natural numbers, t the time and $\mathfrak{K}$ the set of meromorphic functions. The set of polynomials in the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$ whose coefficients are elements in $\mathfrak{K}$ is a so-called Ore polynomial ring and will be denoted as $\mathfrak{K}[\frac{\mathrm{d}}{\mathrm{d}t}]$. The multiplication in this ring is non-commutative and determined by

$$\forall a \in \mathfrak{K} : \tfrac{\mathrm{d}}{\mathrm{d}t} a = \dot{a} + a \tfrac{\mathrm{d}}{\mathrm{d}t}. \tag{1}$$

The set of $m_1 \times m_2$ matrices whose coefficients are elements in $\mathfrak{K}[\frac{\mathrm{d}}{\mathrm{d}t}]$ will be denoted by $\mathfrak{K}^{m_1 \times m_2}[\frac{\mathrm{d}}{\mathrm{d}t}]$ (or $(\mathfrak{K}[\frac{\mathrm{d}}{\mathrm{d}t}])^{m_1 \times m_2}$ depending on the context). We also define

$$\mathrm{row}(\mathbf{A}_1(\tfrac{\mathrm{d}}{\mathrm{d}t}), \ldots, \mathbf{A}_p(\tfrac{\mathrm{d}}{\mathrm{d}t})) \coloneqq \begin{pmatrix} \mathbf{A}_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \vdots \\ \mathbf{A}_p(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix}$$

and

$$\mathrm{col}(\mathbf{B}_1(\tfrac{\mathrm{d}}{\mathrm{d}t}), \ldots, \mathbf{B}_p(\tfrac{\mathrm{d}}{\mathrm{d}t})) \coloneqq (\mathbf{B}_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) \ \ldots \ \mathbf{B}_p(\tfrac{\mathrm{d}}{\mathrm{d}t}))$$

for $\mathbf{A}_i(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{\bullet \times m_2}[\frac{\mathrm{d}}{\mathrm{d}t}]$, $i \in \{1, \ldots, p\}$ and $\mathbf{B}_j(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{m_1 \times \bullet}[\frac{\mathrm{d}}{\mathrm{d}t}]$, $j \in \{1, \ldots, p\}$.

Definition 1. Let $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) = \sum_{i=0}^{d} \mathbf{A}_i \cdot (\frac{\mathrm{d}}{\mathrm{d}t})^i \in \mathfrak{K}^{p \times q}[\frac{\mathrm{d}}{\mathrm{d}t}]$ where $\mathbf{A}_i \in \mathfrak{K}^{p \times q}$. We define the *degree* of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$ w.r.t. $\frac{\mathrm{d}}{\mathrm{d}t}$ as $\deg \mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) \coloneqq d$.

Since in general the inverse of a polynomial is not a polynomial, the following definition characterizes an important set.

Definition 2. A matrix $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{p \times p}[\frac{\mathrm{d}}{\mathrm{d}t}]$ is called *unimodular* if there exists a matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{p \times p}[\frac{\mathrm{d}}{\mathrm{d}t}]$ such that $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) = \mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) = \mathbf{I}_p$. The set of unimodular $p \times p$ matrices is denoted by $\mathcal{U}_p[\frac{\mathrm{d}}{\mathrm{d}t}]$.

This definition now motivates an interesting property for non-square matrices, which is typically introduced using the so-called *Smith normal form*. However, to keep it simple we will use an equivalent, more appropriate definition here (cf. Antritter and Middeke (2011); Fritzsche and Röbenack (2018a)).

Definition 3. A matrix $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{p \times q}[\frac{\mathrm{d}}{\mathrm{d}t}]$ is called *hyper-regular* if it can be completed to a unimodular matrix, i.e., if there exists a matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})$ (of suitable size) such that

$$\begin{cases} \mathrm{row}(\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}), \mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})) \in \mathcal{U}_q[\frac{\mathrm{d}}{\mathrm{d}t}], & p < q \\ \mathrm{col}(\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}), \mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})) \in \mathcal{U}_p[\frac{\mathrm{d}}{\mathrm{d}t}], & p > q. \end{cases}$$

The matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})$ is then referred to as a *unimodular completion* of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$.

Obviously, a hyper-regular square matrix is unimodular, and vice versa.

Lemma and Definition 4. A matrix $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{p \times q}[\frac{\mathrm{d}}{\mathrm{d}t}]$ with $p \neq q$ is hyper-regular, iff there exists a matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{q \times p}[\frac{\mathrm{d}}{\mathrm{d}t}]$ such that

$$\begin{cases} \mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) = \mathbf{I}_p, & p < q \\ \mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) = \mathbf{I}_q, & p > q. \end{cases}$$

The matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})$ is then called a *right pseudo inverse* of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$ denoted by $\mathbf{A}^{+\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t})$ if $p < q$, and a *left pseudo inverse* of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$ symbolized by $\mathbf{A}^{+\mathsf{L}}(\frac{\mathrm{d}}{\mathrm{d}t})$ if $p > q$.

Proof. We show the case $p < q$: Hyper-regularity of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$ implies the existence of a unimodular completion $\mathbf{C}(\frac{\mathrm{d}}{\mathrm{d}t})$, i.e., $\mathbf{D}(\frac{\mathrm{d}}{\mathrm{d}t}) \coloneqq \mathrm{row}(\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}), \mathbf{C}(\frac{\mathrm{d}}{\mathrm{d}t})) \in \mathcal{U}_q[\frac{\mathrm{d}}{\mathrm{d}t}]$. The matrix $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t})$ consists of the first p columns of the inverse of $\mathbf{D}(\frac{\mathrm{d}}{\mathrm{d}t})$. The case $p > q$ is similar. ■

For a practical computation of right and left pseudo inverses we refer to Fritzsche and Röbenack (2018a). Evidently, the inverse of a unimodular matrix is unique, i.e., it is left and right invertible. A useful fact about hyper-regular matrices can be stated as follows:

Lemma 5. Let $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{p \times q}[\frac{\mathrm{d}}{\mathrm{d}t}]$ be hyper-regular, $p < q$, and $r \in \mathbb{N}$. Then the matrix

$$\mathbf{W}(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \begin{pmatrix} \mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{0} \\ \mathbf{T}_1(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \in \mathfrak{K}^{(p+r) \times (q+r)}[\tfrac{\mathrm{d}}{\mathrm{d}t}]$$

is hyper-regular for arbitrary matrices $\mathbf{T}_1(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{r \times q}[\frac{\mathrm{d}}{\mathrm{d}t}]$ and $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathcal{U}_r[\frac{\mathrm{d}}{\mathrm{d}t}]$.

Proof. Due to hyper-regularity of $\mathbf{A}(\frac{\mathrm{d}}{\mathrm{d}t})$ there exists a right pseudo inverse $\mathbf{A}^{+\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathfrak{K}^{q \times p}[\frac{\mathrm{d}}{\mathrm{d}t}]$. It is easy to verify that

$$\mathbf{W}^{+\mathsf{R}}(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \begin{pmatrix} \mathbf{A}^{+\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{0} \\ -\mathbf{B}^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{T}_1(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{A}^{+\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{B}^{-1}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix}$$

is a right pseudo inverse of $\mathbf{W}(\frac{\mathrm{d}}{\mathrm{d}t})$, thus $\mathbf{W}(\frac{\mathrm{d}}{\mathrm{d}t})$ is hyper-regular. ■

We now bridge the preceding definitions with nonlinear systems.

2.2 Differential flatness and flat outputs

Let

$$\mathbf{0}_m = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\alpha)}), \quad \mathbf{x}(t) \in \mathbb{R}^n, \alpha \in \mathbb{N}_+ \tag{3}$$

be an implicit system, where $\mathbf{F}$ is assumed meromorphic.

Definition 6. The matrix

$$\mathcal{J}_{\mathbf{x}}(\mathbf{F}) \coloneqq \left(\sum_{i=0}^{\alpha} \frac{\partial \mathbf{F}}{\partial \mathbf{x}^{(i)}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^i \right) \in \mathfrak{K}^{m \times n}[\tfrac{\mathrm{d}}{\mathrm{d}t}]$$

is called *generalized Jacobian* of $\mathbf{F}$.

Definition 7. (Fliess et al. (1992, 1995)). The system described by (3) with $m < n$ is called *(differentially) flat* if there exists an $n - m$ tuple $\mathbf{y}$ such that

(1) $\mathbf{y} = \mathbf{h}_{\mathbf{y}}(\mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}), \beta \in \mathbb{N}$,
(2) $\mathbf{x} = \mathbf{g}_{\mathbf{x}}(\mathbf{y}, \dot{\mathbf{y}}, \ldots, \mathbf{y}^{(\gamma)}), \gamma \in \mathbb{N}$, and
(3) the components of $\mathbf{y}$ are differentially independent, i.e., there exists no differential equation of the form $\mathbf{R}(\mathbf{y}, \dot{\mathbf{y}}, \ldots, \mathbf{y}^{(\varphi)}) = \mathbf{0}, \varphi \in \mathbb{N}$.

The tuple $\mathbf{y}$ is then called *flat output* of (3).

Note that flatness is a local property, i.e., flat outputs may not be globally defined. A necessary condition for the existence of flat outputs is (local) controllability, which can be linked to hyper-regularity (Lévine, 2011):

Proposition 8. The system (3) with $m < n$ is (locally) controllable iff $\mathcal{J}_{\mathbf{x}}(\mathbf{F}) \in \mathfrak{K}^{m \times n}[\frac{\mathrm{d}}{\mathrm{d}t}]$ is hyper-regular.

Proof. See Lévine (2011). ■

Dual to Proposition 8, system (3) with $m > n$ is (locally) observable iff $\mathcal{J}_{\mathbf{x}}(\mathbf{F}) \in \mathfrak{K}^{m \times n}[\frac{\mathrm{d}}{\mathrm{d}t}]$ is hyper-regular. However, (local) observability is not a necessary condition for the existence of flat inputs (Waldherr and Zeitz, 2010) as mentioned in the introduction.

A more constructive proposition for flat output computation which can be utilized for the computation of flat inputs as well is the following sufficient condition for differential flatness (Lévine, 2011).

Proposition 9. The system (3) with $m < n$ is flat, if there exists a (virtual) output

$$\mathbf{y} = \mathbf{h_x}(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(\beta)}), \quad \mathbf{y}(t) \in \mathbb{R}^{n-m}, \beta \in \mathbb{N}$$

such that

$$\mathcal{J}_{\mathbf{x}}\left(\begin{smallmatrix} \mathbf{F} \\ \mathbf{h_x} \end{smallmatrix} \right) \in \mathcal{U}_n[\tfrac{\mathrm{d}}{\mathrm{d}t}].$$

Proof. See (Lévine, 2011, Thm. 3). ∎

3. GENERALIZED FLAT INPUTS FOR IMPLICIT SYSTEMS

3.1 Flat inputs for state space systems and physical realizability

In Waldherr and Zeitz (2008, 2010) flat inputs have been introduced for state space systems as follows:

Definition 10. Given a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(t) \in \mathbb{R}^n$ with an output $\mathbf{y} = \mathbf{h}(\mathbf{x})$, $\mathbf{y}(t) \in \mathbb{R}^m$, the (independent) components of any $\mathbf{g}_\mathrm{f} = (\mathbf{g}_{\mathrm{f},1}, \dots, \mathbf{g}_{\mathrm{f},m})$ that render

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{g}_{\mathrm{f},i}(\mathbf{x}) u_{\mathrm{f},i} \tag{4}$$

flat such that $\mathbf{y}$ is a flat output are called *flat input vector fields*. The corresponding $\mathbf{u}_\mathrm{f}(t) := (u_{\mathrm{f},1}(t), \dots, u_{\mathrm{f},m}(t))^\mathsf{T}$ is then called a *flat input*.

Oftentimes, the state space description results from introducing equations of the form $\dot{z}_i = z_{i+1}$ which we will refer to as *definitional equations*. These are used to replace higher order derivatives in the original differential equation and thus reducing the order of the system. Injecting flat inputs into these definitional equations however prevents us from reversing this process, so this is to be avoided. Waldherr and Zeitz (2010) give the following

Definition 11. A given flat input vector field of a state space system is called *physically realizable* if the flat input does not act on the definitional equations.

This definition is then motivated using the

Example 12. Consider the linear second-order system

$$\ddot{q} + d\dot{q} + kq = 0 \tag{5a}$$

where d, k are constants, and the output equation

$$y = h(\mathbf{x}) = -kx_1 - dx_2. \tag{5b}$$

Introducing the state vector $\mathbf{x} := (q, \dot{q})^\mathsf{T}$ the system (5a) is rewritten in state space form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 \\ -kx_1 - dx_2 \end{pmatrix} \tag{6}$$

which introduces a definitional equation $\dot{x}_1 = x_2$. In Waldherr and Zeitz (2010), the suggested flat input vector field reads $\boldsymbol{\gamma} = \varepsilon(\mathbf{x})(d, -k)^\mathsf{T}$ where $\varepsilon(\mathbf{x}) \neq 0$ as an arbitrary function is a degree of freedom, i.e., the flat input system yields

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \boldsymbol{\gamma} u = \begin{pmatrix} x_2 \\ -kx_1 - dx_2 \end{pmatrix} + \varepsilon(\mathbf{x}) \begin{pmatrix} d \\ -k \end{pmatrix} u \tag{7}$$

and (5b) is a flat output. Obviously this flat input vector field injects into the definitional equation $\dot{x}_1 = x_2$. The conversion back into a second order system is therefore not possible, i.e., this flat input is not physically realizable – regardless of the choice of $\varepsilon(\mathbf{x})$.

Waldherr and Zeitz (2010) then provide a condition for physical realizability of flat inputs. In the following subsection we will show that a generalized flat input definition allows more freedom with respect to a physical realization.

3.2 A generalized flat inputs definition

Flat outputs have been defined and analyzed for implicit systems which justifies the urge to define flat inputs for implicit systems, too. Another motivation to generalize Definition 10 is the somewhat artificial restriction of not allowing a feedthrough of the flat input, i.e., by Definition 10 there are no input injections to be made in the output equation. Lastly, we allow affine injections of input derivatives in both, the dynamics and the output equation. Note that none of these generalizations limits a physical realizability.

We examine implicit systems of the form

$$\mathbf{0}_n = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(\alpha)}), \quad \mathbf{x}(t) \in \mathbb{R}^n, \alpha \in \mathbb{N}_+ \tag{8a}$$

with an output equation

$$\mathbf{y} = \mathbf{G}(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(\alpha)}), \quad \mathbf{y}(t) \in \mathbb{R}^m, m < n. \tag{8b}$$

Definition 13. Consider systems of the form (8). If there exist matrices

$$\mathbf{K}_i(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(\gamma)}) \in \mathfrak{K}^{(n+m)\times m}, i \in \{0, \dots, \delta\}, \tag{9}$$

with $\gamma, \delta \in \mathbb{N}$ that render

$$\mathbf{0}_n = \mathbf{F} + (\mathbf{I}_n \ \mathbf{0}_{n,m}) \cdot \sum_{i=0}^{\delta} \mathbf{K}_i \mathbf{u}^{(i)}, \quad \mathbf{u}(t) \in \mathbb{R}^m \tag{10}$$

differentially flat such that

$$\tilde{\mathbf{y}} = \mathbf{G} + (\mathbf{0}_{m,n} \ \mathbf{I}_m) \cdot \sum_{i=0}^{\delta} \mathbf{K}_i \mathbf{u}^{(i)} \tag{11}$$

is a flat output of (10), then the input $\mathbf{u}$ is called a *(generalized) flat input* of (8).

Note that Definition 13 is in accordance with Definition 10 for $\delta = 0$ and $(\mathbf{0}_{m,n}, \mathbf{I}_m)\mathbf{K}_0 = \mathbf{0}_{m\times m}$. We may therefore drop the specification *generalized* if this is clear from the context.

3.3 On the computation of generalized flat inputs for (locally) observable nonlinear systems

For the generalized flat input computation we will utilize the algorithm described in Fritzsche and Röbenack (2018b) which assumes (8) to be a first order (locally) observable system. Therefore, we first carry out an order reduction of (8): Introducing $\mathbf{z} := \mathrm{row}(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(\alpha-1)})$ with definitional equations

$$\mathbf{0}_{(\alpha-1)n} = (\dot{z}_i - z_{i+n})_{i\in\{1,\dots,(\alpha-1)n\}} =: \mathbf{E}(\mathbf{z}, \dot{\mathbf{z}}) \tag{12}$$

allows the substitution of $\mathbf{x}^{(i)}$ for $i \in \{0, 1, \ldots, \alpha\}$ in (8), i.e., we get the reduced first order system

$$\begin{pmatrix} \mathbf{0}_{\alpha n} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\mathbf{z}, \dot{\mathbf{z}}) \\ \mathbf{F}(\mathbf{z}, \dot{\mathbf{z}}) \\ \mathbf{G}(\mathbf{z}, \dot{\mathbf{z}}) \end{pmatrix} =: \mathbf{H}(\mathbf{z}, \dot{\mathbf{z}}) \tag{13}$$

where $\mathbf{z}(t) \in \mathbb{R}^{\alpha n}$, $\mathbf{y}(t) \in \mathbb{R}^m$. The generalized Jacobian of (13) is then computed by

$$\mathcal{J}_{\mathbf{z}}(\mathbf{H}) = \underbrace{\frac{\partial \mathbf{H}}{\partial \dot{\mathbf{z}}}}_{=:\mathbf{P}_1} \frac{\mathrm{d}}{\mathrm{d}t} + \underbrace{\frac{\partial \mathbf{H}}{\partial \mathbf{z}}}_{=:\mathbf{P}_0} \in \mathfrak{K}^{(\alpha n+m)\times \alpha n}[\tfrac{\mathrm{d}}{\mathrm{d}t}] \tag{14}$$

and has the structure

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{I}_{(\alpha-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_\alpha \end{pmatrix} =: \begin{pmatrix} \mathbf{P}_{1,1} \\ \mathbf{P}_{1,2} \end{pmatrix}, \tag{15a}$$

$$\mathbf{P}_0 = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_{(\alpha-1)n} \\ \mathbf{Q}_0 & \mathbf{\Gamma} \end{pmatrix} =: \begin{pmatrix} \mathbf{P}_{0,1} \\ \mathbf{P}_{0,2} \end{pmatrix} \tag{15b}$$

where

$$\mathbf{\Gamma} := \mathrm{col}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{\alpha-1}) \in \mathfrak{K}^{(n+m)\times(\alpha-1)n}. \tag{16}$$

The matrices $\mathbf{Q}_i$ are computed as

$$\mathbf{Q}_i := \frac{\partial}{\partial \mathbf{x}^{(i)}} \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} \in \mathfrak{K}^{(n+m)\times n}, \quad i \in \{0, \ldots, \alpha\} \tag{17}$$

where $x_i^{(j)}$ is replaced by z_{i+jn} for $i \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, \alpha-1\}$ and $x_i^{(\alpha)}$ is replaced by $\dot{z}_{i+(\alpha-1)n}$ for $i \in \{0, 1, \ldots, n\}$.

Next, we compute a hyper-regular completion $\mathbf{C}(\frac{\mathrm{d}}{\mathrm{d}t})$ of (14). If we assume

$$\mathrm{rank}\, \frac{\partial}{\partial \mathbf{x}^{(\alpha)}} \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = n \tag{18}$$

or equivalently $\mathrm{rank}\, \mathbf{P}_1 = \alpha n$ then we can apply the algorithm proposed in Fritzsche and Röbenack (2018b) to compute a unimodular completion [1] [2] [3]. We get

$$(\mathcal{J}_{\mathbf{z}}(\mathbf{H})\ \mathbf{C}(\tfrac{\mathrm{d}}{\mathrm{d}t})) =: \begin{pmatrix} \mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1} & \mathbf{C}_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2} & \mathbf{C}_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \in \mathcal{U}_{\alpha n+m}[\tfrac{\mathrm{d}}{\mathrm{d}t}]$$

The goal is to modify this unimodular completion in order to avoid input injections in the definitional equations. The following lemma will help to achieve that.

Lemma 14. The generalized Jacobian of definitional equations $\mathcal{J}_{\mathbf{z}}(\mathbf{E}) = \mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1}$ is hyper-regular.

Proof. We need to show hyper-regularity of

$$\mathcal{J}_{\mathbf{z}}(\mathbf{E}) = \left(\mathbf{I}_{(\alpha-1)n}\tfrac{\mathrm{d}}{\mathrm{d}t}\ \mathbf{0}_{(\alpha-1)n,n}\right) + \left(\mathbf{0}_{(\alpha-1)n,n}\ -\mathbf{I}_{(\alpha-1)n}\right)$$

$$= \begin{pmatrix} \mathbf{I}_n\frac{\mathrm{d}}{\mathrm{d}t} & -\mathbf{I}_n & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n\frac{\mathrm{d}}{\mathrm{d}t} & -\mathbf{I}_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I}_n\frac{\mathrm{d}}{\mathrm{d}t} & -\mathbf{I}_n \end{pmatrix} \in \mathfrak{K}^{(\alpha-1)n\times\alpha n}[\tfrac{\mathrm{d}}{\mathrm{d}t}]$$

(cf. also the similarities to singular matrix pencils in (Gantmacher, 1990, §3)). Obviously, hyper-regularity of $\mathcal{J}_{\mathbf{z}}(\mathbf{E})$ is independent of $\mathbf{n}$. For a fixed n we prove hyper-regularity of $\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)$ by induction over α: Let $\alpha = 2$, then $\mathcal{J}_{\mathbf{z}}(\mathbf{E})(2) = (\mathbf{I}_n\frac{\mathrm{d}}{\mathrm{d}t}, -\mathbf{I}_n)$ is right invertible by $(\mathcal{J}_{\mathbf{z}}(\mathbf{E})(2))^{+\mathsf{R}} = (\mathbf{0}, -\mathbf{I}_n)^{\mathsf{T}}$ and thus hyper-regular. Now, assume $\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)$ to be hyper-regular for some arbitrary $\alpha > 2$, then

$$\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha+1) = \begin{pmatrix} \mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha) & \mathbf{0} \\ \mathbf{T}_1(\frac{\mathrm{d}}{\mathrm{d}t}) & -\mathbf{I}_n \end{pmatrix}$$

where $\mathbf{T}_1(\frac{\mathrm{d}}{\mathrm{d}t}) := (\mathbf{0}, \mathbf{I}_n\frac{\mathrm{d}}{\mathrm{d}t})$. Setting $\mathbf{B}(\frac{\mathrm{d}}{\mathrm{d}t}) = -\mathbf{I}_n$ we can apply Lemma 5 which completes the proof. ■

To avoid input injections into the definitional equations we now right multiply $\mathrm{col}(\mathcal{J}_{\mathbf{z}}(\mathbf{H}), \mathbf{C}(\frac{\mathrm{d}}{\mathrm{d}t}))$ by a matrix

$$\mathbf{T}(\tfrac{\mathrm{d}}{\mathrm{d}t}) := \begin{pmatrix} \mathbf{I}_{\alpha n} & \mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \mathbf{0} & \mathbf{I}_m \end{pmatrix} \in \mathcal{U}_{\alpha n+m}[\tfrac{\mathrm{d}}{\mathrm{d}t}],$$

which results in

$$\begin{aligned} &(\mathcal{J}_{\mathbf{z}}(\mathbf{H})\ \mathbf{C}(\tfrac{\mathrm{d}}{\mathrm{d}t}))\,\mathbf{T}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \\ &= \begin{pmatrix} \mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1} & \mathbf{C}_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2} & \mathbf{C}_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \begin{pmatrix} \mathbf{I}_{\alpha n} & \mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \mathbf{0} & \mathbf{I}_m \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1} & (\mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1})\mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{C}_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2} & (\mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2})\mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{C}_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \\ &=: \left(\mathcal{J}_{\mathbf{z}}(\mathbf{H})\ \tilde{\mathbf{C}}(\tfrac{\mathrm{d}}{\mathrm{d}t})\right). \end{aligned}$$

From the requirement

$$(\mathbf{P}_{1,1}\tfrac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1})\mathbf{L}(\tfrac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{C}_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) \stackrel{!}{=} \mathbf{0}$$

follows

$$\mathbf{L}(\tfrac{\mathrm{d}}{\mathrm{d}t}) = -(\mathbf{P}_{1,1}\tfrac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1})^{+\mathsf{R}}\mathbf{C}_1(\tfrac{\mathrm{d}}{\mathrm{d}t}),$$

i.e.,

$$\begin{pmatrix} \mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1} & \mathbf{0} \\ \mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2} & (\mathbf{P}_{1,2}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2})\mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{C}_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix}$$

Note that $(\mathbf{P}_{1,1}\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,1})^{+\mathsf{R}}$ exists because of Lemma 14. Due to invertibility of $\mathbf{T}(\frac{\mathrm{d}}{\mathrm{d}t})$, we now have a unimodular completion of the generalized Jacobian of system (13) without input injections in the definitional equations. Integration of the corresponding (vector-valued) 1-forms results in

$$\begin{aligned} \begin{pmatrix} \mathbf{0}_{\alpha n} \\ \tilde{\mathbf{y}} \end{pmatrix} &= \int \left(\mathcal{J}_{\mathbf{z}}(\mathbf{H})\ \tilde{\mathbf{C}}(\tfrac{\mathrm{d}}{\mathrm{d}t})\right) \begin{pmatrix} \mathrm{d}\mathbf{z} \\ \mathrm{d}\mathbf{u} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{E}(\mathbf{z}, \dot{\mathbf{z}}) \\ \mathbf{F}(\mathbf{z}, \dot{\mathbf{z}}) \\ \mathbf{G}(\mathbf{z}, \dot{\mathbf{z}}) \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{(\alpha-1)n} \\ \int \mathbf{M}(\frac{\mathrm{d}}{\mathrm{d}t})\mathrm{d}\mathbf{u} \end{pmatrix} \end{aligned} \tag{19}$$

where

$$\mathbf{M}(\tfrac{\mathrm{d}}{\mathrm{d}t}) := (\mathbf{P}_{1,2}\tfrac{\mathrm{d}}{\mathrm{d}t} + \mathbf{P}_{0,2})\mathbf{L}(\tfrac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{C}_2(\tfrac{\mathrm{d}}{\mathrm{d}t}).$$

If the generalized Jacobian of (19) is unimodular, then equation (19) is a physically realizable generalized flat input system. Resubstitution finally yields the higher order generalized flat input system

$$\mathbf{0}_n = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\alpha)}) + (\mathbf{I}_n\ \mathbf{0}) \int \mathbf{M}(\tfrac{\mathrm{d}}{\mathrm{d}t})\mathrm{d}\mathbf{u} \tag{20a}$$

where

$$\tilde{\mathbf{y}} = \mathbf{G}(\mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\alpha)}) + (\mathbf{0}\ \mathbf{I}_m) \int \mathbf{M}(\tfrac{\mathrm{d}}{\mathrm{d}t})\mathrm{d}\mathbf{u} \tag{20b}$$

is a flat output.

A general condition for the existence of generalized flat inputs is subject to further investigation. On the one

[1] In this case we can omit the last step in Fritzsche and Röbenack (2018b), i.e., the completion $\mathbf{C}(\frac{\mathrm{d}}{\mathrm{d}t})$ is an element in $\mathfrak{K}^{(\alpha n+m)\times m}$, not in $\mathfrak{K}^{(\alpha n+m)\times m}[\frac{\mathrm{d}}{\mathrm{d}t}]$.

[2] Different than the approach in Nicolau et al. (2018a) the algorithm presented in Fritzsche and Röbenack (2018b) does not rely on Lie-derivatives, thus $\frac{\partial \mathbf{F}}{\partial \mathbf{x}^{(\alpha)}}$ is not assumed to be a unit matrix.

[3] If condition (18) is not satisfied, the algorithm in Fritzsche and Röbenack (2018b) can not directly be applied. However, (local) observability ensures the existence of unimodular completion.

hand, flat inputs have been shown to exist for (certain) non-observable systems as well, where our approach does not apply. On the other hand, the main difficulty of our method is to ensure unimodularity of the generalized Jacobian of (19). However, we can state the following proposition which also solves the motivational problem for the definition of physical realizability as given in Waldherr and Zeitz (2010).

Proposition 15. For every observable linear system there exists a physically realizable (generalized) flat input.

The proof of this proposition follows from the preceding paragraph where the last unimodularity condition is always satisfied. Furthermore, we can state the following Lemma about the order of input derivatives in a physically realizable generalized flat input system:

Lemma 16. For an observable system (8) which is linear with the restriction (18) there exists a (generalized) flat input where the order of derivatives is $\delta < \alpha$.

Proof. Since the algorithm for the computation of unimodular completions in Fritzsche and Röbenack (2018b) always results in a $\frac{\mathrm{d}}{\mathrm{d}t}$-free completion, the order of input derivatives is specified by $[\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)]^{+\mathsf{R}}$, only. Due to the structure of the definitional equations $\mathbf{0} = \mathbf{E}(\mathbf{z}, \dot{\mathbf{z}})$ a right pseudo inverse of $\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)$ is given by

$$[\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)]^{+\mathsf{R}} = -\sum_{i=0}^{\alpha-2} \begin{pmatrix} \mathbf{0}_{(1+i)n,(\alpha-1-i)n} & \mathbf{0}_{(1+i)n,in} \\ \mathbf{I}_{(\alpha-1-i)n} & \mathbf{0}_{(\alpha-1-i)n,in} \end{pmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i,$$

i.e., due to $\deg \mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha) = 1$ and $\deg [\mathcal{J}_{\mathbf{z}}(\mathbf{E})(\alpha)]^{+\mathsf{R}} = \alpha - 2$ we have $\deg \mathbf{M}(\frac{\mathrm{d}}{\mathrm{d}t}) = \alpha - 1$ thus $\delta \leq \alpha - 1$. ∎

Remark 17. Right pseudo inverses are not unique, but allow arbitrary powers of $\frac{\mathrm{d}}{\mathrm{d}t}$ since

$$\mathbf{I}_{(\alpha-1)n} = \mathbf{P}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \left[\mathbf{P}^{+\mathsf{R}}(\tfrac{\mathrm{d}}{\mathrm{d}t}) + \mathbf{P}^{\perp\mathsf{R}}(\tfrac{\mathrm{d}}{\mathrm{d}t})\mathbf{M}(\tfrac{\mathrm{d}}{\mathrm{d}t})\right]$$

where $\mathbf{P}^{\perp\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t})$ is a right orthogonal complement of $\mathbf{P}(\frac{\mathrm{d}}{\mathrm{d}t})$ and $\mathbf{M}(\frac{\mathrm{d}}{\mathrm{d}t})$ is an arbitrary matrix of suitable size (see e.g. Fritzsche and Röbenack (2018a)). Thus, it should be possible to find a generalized flat input for $\delta \geq \alpha$.

Remark 18. For control purposes, the injection of time derivatives of $\mathbf{u}$ suggests the usage of dynamic controllers, and quasi-static feedbacks (Delaleau and Rudolph, 1998).

4. EXAMPLE

We consider Example 12. Assuming $\varepsilon(\mathbf{x}) \equiv 1$, the generalized Jacobian of the implicit version of (7) with $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) := \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}) - \boldsymbol{\gamma} u$ and $\mathbf{z} := (x_1, x_2, u)^\mathsf{T}$ reads

$$\mathcal{J}_{\mathbf{z}}\left(\begin{smallmatrix}\mathbf{F}\\ h\end{smallmatrix}\right) = \left(\begin{array}{cc|c} \frac{\mathrm{d}}{\mathrm{d}t} & -1 & -d \\ \hline k & \frac{\mathrm{d}}{\mathrm{d}t} + d & k \\ -k & -d & 0 \end{array}\right) =: \begin{pmatrix} \mathbf{V}_1(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{H}_1 \\ \mathbf{V}_2(\frac{\mathrm{d}}{\mathrm{d}t}) & \mathbf{H}_2 \end{pmatrix} \tag{21}$$

and is unimodular, so this will be our starting point. Here, $\mathbf{V}_1(\frac{\mathrm{d}}{\mathrm{d}t})$ is the generalized Jacobian of the definitional equations w.r.t. $\mathbf{x}$. According to Lemma 14 this matrix is always hyper-regular, i.e., there exists a pseudo inverse, e.g. $\mathbf{V}_1^{+\mathsf{R}} = (0, -1)^\mathsf{T}$, while $\mathbf{H}_1 = -d$. Right multiplication of (21) by the matrix

$$\mathbf{T}(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \begin{pmatrix} \mathbf{I}_2 & -\mathbf{V}_1^{+\mathsf{R}}(\frac{\mathrm{d}}{\mathrm{d}t})\mathbf{H}_1 \\ \mathbf{0} & \mathbf{I}_1 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -d \\ \hline 0 & 0 & 1 \end{array}\right) \in \mathfrak{U}_n[\tfrac{\mathrm{d}}{\mathrm{d}t}]$$

results in

$$\begin{aligned} \mathcal{J}_{\mathbf{z}}\left(\begin{smallmatrix}\mathbf{F}\\ h\end{smallmatrix}\right)\mathbf{T}(\tfrac{\mathrm{d}}{\mathrm{d}t}) &= \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} & -1 & -d \\ k & \frac{\mathrm{d}}{\mathrm{d}t} + d & k \\ -k & -d & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} & -1 & 0 \\ k & \frac{\mathrm{d}}{\mathrm{d}t} + d & -d\frac{\mathrm{d}}{\mathrm{d}t} - d^2 + k \\ -k & -d & d^2 \end{pmatrix}. \end{aligned}$$

The flat input injected (first order) system can be computed from

$$\begin{pmatrix} \mathbf{0}_n \\ \tilde{y} \end{pmatrix} = \int \mathcal{J}_{\mathbf{z}}\left(\begin{smallmatrix}\mathbf{F}\\ h\end{smallmatrix}\right)\mathbf{T}(\tfrac{\mathrm{d}}{\mathrm{d}t})\mathrm{d}\mathbf{z} \tag{22}$$

and yields

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \begin{pmatrix} 0 \\ d^2 - k \end{pmatrix} u + \begin{pmatrix} 0 \\ d \end{pmatrix} \dot{u} \tag{23a}$$

$$\tilde{y} = h(\mathbf{x}) + d^2 u. \tag{23b}$$

Note that instead of injecting into the definitional equation, we now deal with derivatives of u and additionally inject into the output equation. This allows the transformation back into a second-order system which results in

$$\ddot{q} + d\dot{q} + kq = (d^2 - k)u + d\dot{u} \tag{24a}$$

where

$$\tilde{y} = -kq - d\dot{q} + d^2 u \tag{24b}$$

is a flat output, i.e., according to Definition (13) we get

$$\mathbf{K}_0 = \begin{pmatrix} k - d^2 \\ d^2 \end{pmatrix}, \quad \mathbf{K}_1 = \begin{pmatrix} -d \\ 0 \end{pmatrix}. \tag{25}$$

We can prove this result using Proposition 9 e.g. with methods from Fritzsche and Röbenack (2018a) (or using linear methods as we are dealing with constant coefficients here) by verifying that

$$\mathcal{J}_{\tilde{\mathbf{z}}}\left(\begin{smallmatrix}\tilde{\mathbf{F}}\\ \tilde{y}\end{smallmatrix}\right) = \begin{pmatrix} (\frac{\mathrm{d}}{\mathrm{d}t})^2 + d\frac{\mathrm{d}}{\mathrm{d}t} + k & -d\frac{\mathrm{d}}{\mathrm{d}t} - (d^2 - k) \\ -d\frac{\mathrm{d}}{\mathrm{d}t} - k & d^2 \end{pmatrix} \in \mathfrak{U}_2[\tfrac{\mathrm{d}}{\mathrm{d}t}] \tag{26}$$

where $\tilde{\mathbf{F}}$ is the implicit version of (24a), and $\tilde{\mathbf{z}} := (q, u)^\mathsf{T}$. Since

$$\begin{pmatrix} \frac{d^2}{k^2} & \frac{d}{k^2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{d^2 - k}{k^2} \\ \frac{d}{k^2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{k} & \frac{1}{k^2}(\frac{\mathrm{d}}{\mathrm{d}t})^2 + \frac{d}{k^2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{k} \end{pmatrix}$$

is the inverse of (26) the input injection is in fact a physically realizable (generalized) flat input. Obviously, we have $\delta = \alpha - 1 = 1$.

5. CONCLUSION

Differential flatness is the key property to understand feedback linearization and is characterized by a (possibly virtual) output called flat output. For the class of flat systems, many control problems are solved, however essential questions have not been answered, in particular the questions whether or not an arbitrary system is flat, and if so, how to compute a flat output. In order to incorporate the flatness property in the design process of (industrial) systems, a dual perspective is given by the actuator placement problem which motivates the definition of flat inputs: Given an uncontrolled system, we are interested in an input injected system such that a given output becomes flat. So far, the definition of flat inputs has only be given for state space systems (Waldherr and Zeitz, 2008, 2010).

Many state space systems originate from differential equations of order $\alpha > 1$ through the introduction of definitional equations. However, the injection into definitional equations prevents the conversion back to the higher order differential equation, and thus hinders a physical realization.

Therefore, in this contribution we have generalized the existing flat input definition in multiple ways:

- We allow the injection of time derivatives of $\mathbf{u}(t)$ up to a finite order δ,
- we permit input injections in the output equation, and
- we define flat inputs for implicit systems.

None of these generalizations limit the physical realizability of the input injection. Instead, this liberates degrees of freedom for a conversion of the flat input state space system back to the higher order system and thus renders it physically realizable. We have shown, that for linear observable systems there always exists a physically realizable generalized flat input of order $\delta < \alpha$. Furthermore, we have presented how the algorithm for the computation of flat inputs for state space systems as presented in Fritzsche and Röbenack (2018b) can be used to compute physically realizable generalized flat inputs for implicit systems. We have illustrated these results using a simple example system from Waldherr and Zeitz (2010).

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