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Non-Cooperative Distributed MPC with Iterative Learning

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Abstract: This paper presents a novel framework of distributed learning model predictive control (DLMPC) for multi-agent systems performing iterative tasks. The framework adopts a non-cooperative strategy in that each agent aims at optimizing its own objective. Local state and input trajectories from previous iterations are collected and used to recursively construct a time-varying safe set and terminal cost function. In this way, each subsystem is able to iteratively improve its control performance and ensure feasibility and stability in every iterations. No communication among subsystems is required during online control. Simulation on a benchmark example shows the efficacy of the proposed method.

Keywords: Distributed Systems, Model Predictive Control, Iterative Learning Control

1. INTRODUCTION

The emergence of spatially distributed systems with ever growing scales calls for distributed control methods in which the complexity of computation and communication does not increase with the size of the network. Distributed model predictive control (DMPC) has stood out as an effective way to cope with such systems due to its desired properties of providing safety guarantees and optimized control performance. Many DMPC methods have been proposed (Dunbar and Murray, 2006; Stewart et al., 2010; Farina and Scattolini, 2012; Conte et al., 2012) and devised for real-world applications (Negenborn et al., 2009; Ma et al., 2011; Hu et al., 2018). Among different DMPC architectures (Scattolini, 2009), non-cooperative schemes are of particular interest to the practitioners for its limited computation and transmission requirements.

One shared aspect in many large-scale distributed systems that cannot be overlooked is the repetitive execution of a same control task, or a routinely plan. Examples include load frequency control (Riverso and Ferrari-Trecate, 2012), water level control of irrigation canals (Negenborn et al., 2009) and building temperature control (Ma et al., 2011). In this setting, iterative learning control (ILC) is a powerful strategy since it exploits the fact that at each iteration, the system starts from the same initial condition pursuing the same control objectives. Information gathered from previous iterations is incorporated into the problem formulation at the next iteration to improve the closed loop control performance (Bristow et al., 2006). Many classical ILC approaches are used in conjunction with MPC to minimize the error in tracking a known reference signal (Lee et al., 1999; Cueli and Bordons, 2008). Recently, a novel reference-free MPC algorithm for ILC, known as Learning Model Predictive Control (LMPC), has been developed in Rosolia and Borrelli (2017a,b); Rosolia et al. (2017). It uses closed loop trajectories from past iterations to evaluate a terminal safe set and cost function, which in turn guarantee safety and improve the control performance. However, these methods are mainly designed for single-agent systems and may not scale well for large-scale multi-agent systems.

In this paper, we present distributed learning model predictive control (DLMPC), an ILC framework for large-scale, dynamically coupled linear systems. Within this framework, each subsystem maintains a reference trajectory, which encapsulates its "rough" plan, and transmits it to the neighbors. To ensure computational tractability, the subsystem dynamics are split into a nominal part and an error part which captures the coupling effect of the neighbors. We assume the subsystems are non-cooperative and seek to improve their own performance index given neighbor's reference trajectories. This is achieved by learning a time-varying terminal safe set and cost function that capture information regarding the reference trajectories. We show how to construct such sets and cost functions only from local trajectories collected during past iterations. Effect of neighbor's deviation from the reference is modeled as bounded disturbance. The controller is able to guarantee that: (i) the nominal iteration cost of each subsystem is non-increasing in the iterations; (ii) state and input constraints are satisfied at iteration l if they were satisfied at iteration $l-1$ and (iii) the closed loop global system converges asymptotically to the origin. Although the actual iteration cost cannot be guaranteed to improve

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due to neighbor's incompliance with the reference, yet we show on a benchmark of power network systems that the empirical control performance has improved significantly after a few iterations.

This paper is structured as follows. Section 2 introduces the problem setup. In Section 3, a time-varying LMPC for an individual agent is formulated. Section 4 presents the DLMPC framework for the global system. In Section 5 we test our algorithm on a benchmark example. Finally, Section 6 concludes the paper.

Notation: A set of integers ranging from a to b is denoted by $\mathcal{I}_{a:b}$. Concatenation of vectors $x_i \in \mathbb{R}^n$ is defined as $\mathrm{col}(x_a, \ldots, x_b) = \left[x_a^\top, \ldots, x_b^\top\right]^\top$. A sequence of vectors (or sets) $\{s_a, s_{a+1}, \ldots, s_b\}$ is denoted by $s_{[a:b]}$. The Minkowski sum is defined as $\mathbb{X} \oplus \mathbb{Y} = \{x + y | x \in \mathbb{X}, y \in \mathbb{Y}\}$. The Pontryagin difference is defined as $\mathbb{X} \ominus \mathbb{Y} = \{z | z + \mathbb{Y} \subseteq \mathbb{X}\}$.

2. PROBLEM SETUP

We consider a discrete-time linear dynamical system

$$\mathbf{\Sigma}: \; \mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t, \quad \mathbf{x}_0 = \mathbf{x}_S \tag{1}$$

with state vector $\mathbf{x}_t \in \mathbb{R}^n$, input vector $\mathbf{u}_t \in \mathbb{R}^m$, and a given initial state $\mathbf{x}_S \in \mathbb{R}^n$. We refer to (1) as the global system, with $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$ as the system and input matrix, respectively. We assume there exists a partition of (1) into M state-coupled but input-decoupled subsystems with the following dynamics

$$\Sigma_i: \; x_{t+1}^{[i]} = A_{ii} x_t^{[i]} + B_i u_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij} x_t^{[j]} \tag{2}$$

where $x_t^{[i]} \in \mathbb{R}^{n_i}$ and $u_t^{[i]} \in \mathbb{R}^{m_i}$ are state and input vector of Σ_i such that $\mathbf{x}_t = \mathrm{col}(x_t^{[1]}, \ldots, x_t^{[M]})$, $\sum_{i=1}^M n_i = n$ and $\mathbf{u}_t = \mathrm{col}(u_t^{[1]}, \ldots, u_t^{[M]})$, $\sum_{i=1}^M m_i = m$. The initial state of Σ_i is $x_S^{[i]} \in \mathbb{R}^{n_i}$. Matrices $A_{ij} \in \mathbb{R}^{n_i \times m_j}$ are the corresponding blocks in the system matrix $\mathbf{A}$ while the input matrix $\mathbf{B} = \mathrm{diag}(B_1, \ldots, B_M)$. Two sets defining the neighbors are introduced. The predecessor set $\mathcal{N}_i = \{j \in \mathcal{I}_{1:M} \backslash \{i\} | A_{ij} \neq 0\}$ contains indices of neighboring subsystems of Σ_i, whose action affects Σ_i. Likewise the follower set is $\bar{\mathcal{N}}_i = \{j \in \mathcal{I}_{1:M} \backslash \{i\} | A_{ji} \neq 0\}$. For each subsystem Σ_i a set of local state and input constraints are required to be satisfied

$$x_t^{[i]} \in \mathbb{X}^{[i]}, \; u_t^{[i]} \in \mathbb{U}^{[i]}, \quad \forall i \in \mathcal{I}_{1:M} \tag{3}$$

where $\mathbb{X}^{[i]}$ and $\mathbb{U}^{[i]}$ are polytopic sets containing the origin. This leads to the constraints $\mathbf{x}_t \in \mathbb{X} = \prod_{i=1}^M \mathbb{X}^{[i]}$ and $\mathbf{u}_t \in \mathbb{U} = \prod_{i=1}^M \mathbb{U}^{[i]}$ on the global system (1).

2.1 Non-Cooperative Control

In this paper, we seek to achieve distributed control of the global system (1) in terms of the subsystems (2) with efficient computation and communication. Thus we adopt a non-cooperative control framework in which, as illustrated in Figure 1, each subsystem Σ_i optimizes over its own state and input variables $x^{[i]}$ and $u^{[i]}$ while taking into account neighbor's effects based on their future intentions. This is done by requiring that each subsystem Σ_i pre-communicates a reference trajectory $\tilde{x}_{[0:\infty]}^{[i]}$ encapsulating its "rough" plan with the followers Σ_j, $\forall j \in \bar{\mathcal{N}}_i$. Now we can rewrite the subsystem dynamics (2) as

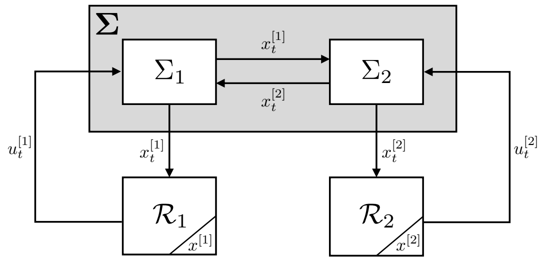

Fig. 1. An illustration of non-cooperative control. Solid arrows between Σ_1 and Σ_2 denote physical interactions between those two subsystems. A local controller $\mathcal{R}_i$ only optimizes over local states $x^{[i]}$.

$$x_{t+1}^{[i]} = A_{ii} x_t^{[i]} + B_i u_t^{[i]} + c_t^{[i]} + w_t^{[i]} \tag{4}$$

where the offset term $c_t^{[i]}$ is defined as the summation over neighbor's reference trajectories

$$c_t^{[i]} = \sum_{j \in \mathcal{N}_i} A_{ij} \tilde{x}_t^{[j]} \tag{5}$$

The discrepancy between the actual and reference trajectories of the neighbors can be written as

$$w_t^{[i]} = \sum_{j \in \mathcal{N}_i} A_{ij} (x_t^{[j]} - \tilde{x}_t^{[j]}) \tag{6}$$

which is considered as a disturbance term in (4). Apparently, this term is bounded only when the actual state $x_t^{[i]}$ of each subsystem does not deviate too far away from its communicated reference state $\tilde{x}_t^{[i]}$. We formalize this intuition by requiring the existence of a time-invariant and bounded set $\mathcal{E}^{[i]}$ such that $x_t^{[i]} - \tilde{x}_t^{[i]} \in \mathcal{E}^{[i]}$ for all $t \geq 0$. Subsequently, the disturbance term $w_t^{[i]}$ in (6) satisfies

$$w_t^{[i]} \in \mathbb{W}^{[i]} = \bigoplus_{j \in \mathcal{N}_i} A_{ij} \mathcal{E}^{[j]} \tag{7}$$

where the disturbance set $\mathbb{W}^{[i]}$ is bounded as well.

Each subsystem Σ_i would then solve the following infinite horizon robust optimal control problem

$$\begin{aligned}
\min_{\bar{u}_0^{[i]}, \bar{u}_1^{[i]}(\cdot), \ldots} \quad & \sum_{k=0}^{\infty} h_i\left(\bar{x}_k^{[i]}(\mathbf{0}), \bar{u}_k^{[i]}\left(\bar{x}_k^{[i]}(\mathbf{0})\right)\right) && \text{(8a)}\\
\text{s.t.} \quad & \bar{x}_{k+1}^{[i]} = A_{ii} \bar{x}_k^{[i]} + B_i \bar{u}_k^{[i]} + c_k^{[i]} + w_k^{[i]} && \text{(8b)}\\
& \bar{x}_k^{[i]} \in \mathbb{X}^{[i]}, \; \bar{u}_k^{[i]} \in \mathbb{U}^{[i]}, \; \forall w_k^{[i]} \in \mathbb{W}^{[i]} && \text{(8c)}\\
& \bar{x}_k^{[i]} - \tilde{x}_k^{[i]} \in \mathcal{E}^{[i]} && \text{(8d)}\\
& \bar{x}_0^{[i]} = x_S^{[i]}, \; k = 0, 1, \ldots, && \text{(8e)}
\end{aligned}$$

where $\bar{(\cdot)}$ denotes decision variables. Sequences $\tilde{x}_{[0:\infty]}$ and $c_{[0:\infty]}^{[i]}$ are given. The stage cost function $h_i(\cdot, \cdot)$ is continuous, jointly convex and satisfies

$$h_i(0, 0) = 0 \text{ and } h_i(x_t^{[i]}, u_t^{[i]}) > 0, \; \forall x_t^{[i]}, u_t^{[i]} \neq 0. \tag{9}$$

The cost function (8a) aims at minimizing the nominal cost which assumes the disturbance sequence that affects the state $x_t^{[i]}$ is $w_{[0:t-1]}^{[i]} = \mathbf{0}$ for all $t > 0$, leading to the notation $x_t^{[i]}(\mathbf{0})$. Other types of cost functions such as the worst-case or expected-value cost can also be used.

Nonetheless in this paper we choose the nominal cost due to its conceptual simplicity and practical usefulness. See Rosolia et al. (2017) for additional discussions on this topic. The collective solution to (8) of all subsystems

$$\bar{\mathbf{x}}_t^\star = \mathrm{col}(\bar{x}_t^{[1],\star}, \ldots, \bar{x}_t^{[M],\star}), \quad t = 0, 1, \ldots,$$
$$\bar{\mathbf{u}}_t^\star = \mathrm{col}(\bar{u}_t^{[1],\star}, \ldots, \bar{u}_t^{[M],\star}), \quad t = 0, 1, \ldots,$$

constitutes a feasible trajectory for the global system (1) subject to constraints $\mathbf{x}_t \in \mathbb{X}$ and $\mathbf{u}_t \in \mathbb{U}$.

2.2 The Nominal Dynamics

In general, problem (8) is computationally intractable because (i) the decision variable $u_t^{[i]}(\cdot) : \mathbb{X}^{[i]} \mapsto \mathbb{U}^{[i]}$ lies in an infinite-dimensional space of state-feedback policies and (ii) the control horizon is infinite. This section focuses on approximating control policies such that the decision variables are in finite dimensions. Subsequent sections in the paper will discuss how to approximately solve problem (8) with finite horizons using the idea of ILC.

Now we review the approach in Farina and Scattolini (2012) which is able to handle (i). Consider the disturbance-free *nominal dynamics* associated with the actual subsystem (4) defined as

$$z_{t+1}^{[i]} = A_{ii} z_t^{[i]} + B_i v_t^{[i]} + c_t^{[i]} \tag{10}$$

in which the state and input are $z_t^{[i]}$ and $v_t^{[i]}$, respectively. The control law for each subsystem Σ_i, for all $t \geq 0$, is parameterized by

$$u_t^{[i]} = v_t^{[i]} + K_i(x_t^{[i]} - z_t^{[i]}) \tag{11}$$

which, along with (4) and (10), defines the *error dynamics*

$$e_{t+1}^{[i]} = \Phi_i e_t^{[i]} + w_t^{[i]} \tag{12}$$

where

$$e_t^{[i]} = x_t^{[i]} - z_t^{[i]} \tag{13}$$

Assumption 2.1. There exists matrices $K_i \in \mathbb{R}^{m_i \times n_i}$ and $\mathbf{K} = \mathrm{diag}(K_1, \ldots, K_M)$ such that: (i). $\mathbf{\Phi} = \mathbf{A} + \mathbf{BK}$ is Schur, (ii). $\Phi_i = A_{ii} + B_i K_i$ is Schur for all $i \in \mathcal{I}_{1:M}$.

Remark 2.1. A necessary and sufficient condition for the existence of decentralized feedback gain $\mathbf{K}$ satisfying Assumption 2.1 can be found in Wang and Davison (1973). In Betti et al. (2014) an LMI-based approach is proposed to effectively design the block-diagonal matrix $\mathbf{K}$.

By Assumption 2.1 and that $\mathbb{W}^{[i]}$ is bounded, it is shown in Rakovic et al. (2005) that there exists a minimal robust positive invariant (mRPI) set $\mathbb{Z}^{[i]}$ for the error dynamics (12) such that

$$e_t^{[i]} \in \mathbb{Z}^{[i]}, \ \forall w_t^{[i]} \in \mathbb{W}^{[i]}, \ \forall t \geq 0 \tag{14}$$

In addition, we define the deviation set $\mathbb{E}^{[i]}$ which satisfies

$$0 \in \mathbb{E}^{[i]} \text{ and } \mathbb{E}^{[i]} \oplus \mathbb{Z}^{[i]} \subseteq \mathcal{E}^{[i]} \tag{15}$$

and enforce that

$$z_t^{[i]} - \tilde{x}_t^{[i]} \in \mathbb{E}^{[i]} \tag{16}$$

for all $t \geq 0$. In other words, the deviation between the nominal and the reference trajectories is required to be bounded. We refer interested readers to Farina and Scattolini (2012); Betti et al. (2014) for details on synthesizing sets $\mathcal{E}^{[i]}$, $\mathbb{Z}^{[i]}$ and $\mathbb{E}^{[i]}$.

Now we are in place to introduce for the nominal dynamics (10) of each subsystem Σ_i the following infinite horizon optimal control problem

$$\min_{\bar{v}_{[0:\infty]}^{[i]}} \quad \sum_{k=0}^{\infty} h_i(\bar{z}_k^{[i]}, \bar{v}_k^{[i]}) \tag{17a}$$
$$\text{s.t.} \quad \bar{z}_{k+1}^{[i]} = A_{ii}\bar{z}_k^{[i]} + B_i \bar{v}_k^{[i]} + c_k^{[i]} \tag{17b}$$
$$\bar{z}_k^{[i]} \in \bar{\mathbb{X}}^{[i]}, \ \bar{v}_k^{[i]} \in \bar{\mathbb{U}}^{[i]} \tag{17c}$$
$$\bar{z}_k^{[i]} - \tilde{x}_k^{[i]} \in \mathbb{E}^{[i]} \tag{17d}$$
$$\bar{z}_0^{[i]} = x_S^{[i]}, \ k = 0, 1, \ldots, \tag{17e}$$

where sequences $\tilde{x}_{[0:\infty]}^{[i]}$, $c_{[0:\infty]}^{[i]}$ are given. Local state and input constraints (3) are tightened in (17c) according to

$$\bar{\mathbb{X}}^{[i]} = \mathbb{X}^{[i]} \ominus \mathbb{Z}^{[i]}, \ \bar{\mathbb{U}}^{[i]} = \mathbb{U}^{[i]} \ominus K_i \mathbb{Z}^{[i]} \tag{18}$$

From (11) we have that the optimal cost of (17) is an upper bound on that of (8), i.e.

$$\sum_{k=0}^{\infty} h_i(\bar{z}_k^{[i],\star}, \bar{v}_k^{[i],\star}) \geq \sum_{k=0}^{\infty} h_i\left(\bar{x}_k^{[i],\star}(\mathbf{0}), \bar{u}_k^{[i],\star}\left(\bar{x}_k^{[i],\star}(\mathbf{0})\right)\right)$$

Remark 2.2. The DMPC scheme in Farina and Scattolini (2012) produces for each subsystem a closed loop (nominal) trajectory which is a feasible but sub-optimal solution to (17). In this paper, we take a step further to search for the optimal solution to (17).

3. LMPC WITH REFERENCE TRAJECTORY

This section discusses an iterative learning control scheme that exploits previous data to optimally solve problem (17) for an *individual* subsystem Σ_i. The next section extends the method to deal with the global system (1).

In this paper, we assume that problem (17) is to be solved over and over again for the same initial condition $x_0^{l,[i]} = z_0^{l,[i]} = x_S^{[i]}$ for each subsystem Σ_i and all iterations $l \geq 0$. At iteration l, let the sequences of vectors

$$z_{[0:\infty]}^{l,[i]} = \{z_0^{l,[i]}, z_1^{l,[i]}, \ldots, z_t^{l,[i]}, \ldots\} \tag{19a}$$
$$v_{[0:\infty]}^{l,[i]} = \{v_0^{l,[i]}, v_1^{l,[i]}, \ldots, v_t^{l,[i]}, \ldots\} \tag{19b}$$

denote the realized state and input trajectories of the nominal dynamics (10) associated with subsystem Σ_i. Likewise, define $x_{[0:\infty]}^{l,[i]}$ and $u_{[0:\infty]}^{l,[i]}$ for actual model (2). It is assumed that the reference trajectory $\tilde{x}_{[0:\infty]}^{[i]}$ is given and remains fixed for all $l \geq 1$. Moreover, Σ_i has access to the reference trajectories of its neighbors $\tilde{x}_{[0:\infty]}^{[j]}$, $j \in \mathcal{N}_i$ to compute the offset term $c_{[0:\infty]}^{[i]}$ using (5). Finally, we make the following assumption on the reference trajectories.

Assumption 3.1. There exists a finite time $t' \geq 0$ such that

$$\tilde{x}_t^{[i]} = 0, \ c_t^{[i]} = \sum_{j \in \mathcal{N}_i} \tilde{x}_t^{[j]} = 0, \quad \forall t \geq t'. \tag{20}$$

Remark 3.1. The above assumption facilitates the development of our ILC algorithms in subsequent sections. Such reference trajectories can be generated via DMPC proposed in Farina and Scattolini (2012) by setting the origin as the terminal constraint, at the cost of a smaller region of attraction. Better methods for generating reference trajectories are left for future research and are beyond the scope of this paper.

3.1 Time-varying Safe Sets

Recall in Rosolia and Borrelli (2017b) the definition of sampled safe set

$$\mathcal{SS}^{l,[i]} = \left\{ \bigcup_{h=0}^{l} \bigcup_{t=0}^{\infty} z_t^{h,[i]} \right\} \tag{21}$$

which collects all nominal state trajectories of Σ_i up to iteration l. The convex hull of its elements

$$\mathcal{CS}^{l,[i]} = \mathrm{conv}(\mathcal{SS}^{l,[i]}) \tag{22}$$

also known as the convex safe set, is a control invariant set for constrained linear systems (Borrelli et al., 2017). It is the usage of convex safe sets that leads to a computationally efficient LMPC problem formulated as a quadratic program (Rosolia and Borrelli, 2017a). However, in our case, the convex safe set $\mathcal{CS}^{l,[i]}$ is no longer control invariant for (10). The reason is twofold:

(1) System (10) is nonlinear due to the time-varying offset $c_t^{[i]}$, results in Rosolia and Borrelli (2017a) which build upon linearity cannot be directly used,
(2) Constraint (16) is time-varying and at each time only a subset of realized states satisfies it.

In the next we show how to split $\mathcal{SS}^{l,[i]}$ into subsets indexed by time and use them to construct a time-varying, yet convex LMPC problem.

Definition 3.1. (Time-varying sampled safe set).

$$\mathcal{SS}_t^{l,[i]} := \begin{cases} \left\{ \bigcup_{h=0}^{l} z_t^{h,[i]} \right\} & \text{if } t < t' \\ \mathcal{SS}_\infty^{l,[i]} = \left\{ \bigcup_{h=0}^{l} \bigcup_{t \geq t'} z_t^{h,[i]} \right\} & \text{otherwise} \end{cases} \tag{23}$$

Definition 3.2. (Time-varying convex safe set).

$$\mathcal{CS}_t^{l,[i]} = \mathrm{conv}(\mathcal{SS}_t^{l,[i]}), \quad \mathcal{CS}_\infty^{l,[i]} = \mathrm{conv}(\mathcal{SS}_\infty^{l,[i]}) \tag{24}$$

Remark 3.2. For all $t \geq t'$, the set $\mathcal{CS}_\infty^{l,[i]}$ is a control invariant set for system (10) since, by Assumption 3.1, the system is now a linear system with $c_t^{[i]} = 0$ and time-invariant constraints $z_t^{[i]} \in \mathbb{E}^{[i]}$ and $v_t^{[i]} \in \bar{\mathbb{U}}^{[i]}$.

3.2 Terminal Cost

At time t of the l-th iteration, the empirical (realized) cost-to-go associated with the closed loop state and input trajectories (19) is defined as

$$J_{t\to\infty}^{l,[i]}(z_t^{l,[i]}) = \sum_{k=t}^{\infty} h_i(z_k^{l,[i]}, v_k^{l,[i]}) \tag{25}$$

Subsequently we define the l-th nominal iteration cost as the cost (25) along the l-th trajectory with $t = 0$, i.e.

$$J_{0\to\infty}^{l,[i]}(z_0^{l,[i]}) = \sum_{k=0}^{\infty} h_i(z_k^{l,[i]}, v_k^{l,[i]}) \tag{26}$$

The iteration cost is a quantification of the control performance at iteration l and, obviously, a lower cost represents a better control performance.

Given a state z_t of system (10), its associated (optimal) cost-to-go is approximated by the barycentric function (Jones and Morari, 2010).

Definition 3.3. (Time-varying barycentric function). For all $t < t'$,

$$P_t^{l,[i]}(z_t) := \begin{cases} p_t^{l,\star}(z_t) & \text{if } z_t \in \mathcal{CS}_t^{l,[i]} \\ +\infty & \text{if } z_t \notin \mathcal{CS}_t^{l,[i]} \end{cases} \tag{27}$$

and

$$p_t^{l,\star}(z_t) := \min_{\lambda^h \geq 0} \sum_{h=0}^{l} \lambda^h J_{t\to\infty}^{h,[i]}(z_t^{h,[i]}) \tag{28a}$$

$$\text{s.t.} \ \sum_{h=0}^{l} \lambda^h = 1, \ \sum_{h=0}^{l} \lambda^h z_t^{h,[i]} = z_t \tag{28b}$$

For all $t \geq t'$,

$$P_\infty^{l,[i]}(z_t) := \begin{cases} p_\infty^{l,\star}(z_t) & \text{if } z_t \in \mathcal{CS}_\infty^{l,[i]} \\ +\infty & \text{if } z_t \notin \mathcal{CS}_\infty^{l,[i]} \end{cases} \tag{29}$$

and

$$p_\infty^{l,\star}(z_t) := \min_{\lambda_k^h \geq 0} \sum_{h=0}^{l} \sum_{k=t'}^{\infty} \lambda_k^h J_{k\to\infty}^{h,[i]}(z_k^{h,[i]}) \tag{30a}$$

$$\text{s.t.} \ \sum_{h=0}^{l} \sum_{k=t'}^{\infty} \lambda_k^h = 1, \ \sum_{h=0}^{l} \sum_{k=t'}^{\infty} \lambda_k^h z_k^{h,[i]} = z_t \tag{30b}$$

Remark 3.3. For all $t \geq 0$, $P_\infty^{l,[i]}(\cdot)$ is a time-invariant function defined on the set $\mathcal{CS}_\infty^{l,[i]}$.

Intuitively, the function $P_t^{l,[i]}(\cdot)$ assigns to each point z_t in $\mathcal{CS}_t^{l,[i]}$ the minimum cost-to-go along the trajectories contained in the tube $\{\mathcal{CS}_t^{l,[i]}, \mathcal{CS}_{t+1}^{l,[i]}, \ldots, \mathcal{CS}_{t'-1}^{l,[i]}, \mathcal{CS}_\infty^{l,[i]}\}$.

3.3 Learning MPC for Subsystem Σ_i

During the l-th iteration, at each time t, subsystem Σ_i solves the following finite horizon optimal problem

$$\begin{aligned} & J_{t\to t+N}^{l,[i],\star}(z_t^{l,[i]}) = \\ & \min_{\bar{v}_{[t:t+N-1]}^{l,[i]}} \quad \sum_{k=t}^{t+N-1} h_i(\bar{z}_k^{[i]}, \bar{v}_k^{[i]}) + P_{t+N}^{l-1,[i]}(\bar{z}_{t+N}^{[i]}) && (31a) \\ & \text{s.t.} \quad \bar{z}_{k+1}^{[i]} = A_{ii}\bar{z}_k^{[i]} + B_i \bar{v}_k^{[i]} + c_k^{[i]} && (31b) \\ & \qquad \bar{z}_k^{[i]} \in \bar{\mathbb{X}}^{[i]}, \ \bar{v}_k^{[i]} \in \bar{\mathbb{U}}^{[i]} && (31c) \\ & \qquad \bar{z}_k^{[i]} - \tilde{x}_k^{[i]} \in \mathbb{E}^{[i]} && (31d) \\ & \qquad \bar{z}_t^{[i]} = z_t^{l,[i]}, \ \bar{z}_{t+N}^{[i]} \in \mathcal{CS}_{t+N}^{l-1,[i]} && (31e) \\ & \qquad k = t, \ldots, t+N-1 \end{aligned}$$

Upon solving (31), the controller applies

$$v_t^{l,[i]} = \bar{v}_t^{l,[i],\star} \tag{32}$$

to the nominal system (10) and

$$u_t^{l,[i]} = \bar{v}_t^{l,[i],\star} + K_i(x_t^{l,[i]} - z_t^{l,[i]}) \tag{33}$$

to the actual system (2). We make the following assumptions on the initial feasibility of (31).

Assumption 3.2. The initial trajectory $z_{[0:\infty]}^{0,[i]}$, the reference trajectory $\tilde{x}_{[0:\infty]}^{[i]}$ and the offset sequence $c_{[0:\infty]}^{[i]}$ are given and satisfy (31b)-(31c)-(31d). Moreover, $\tilde{x}_{[0:\infty]}^{[i]}$ satisfies Assumption 3.1 and $z_{[0:\infty]}^{0,[i]}$ is convergent to the origin.

3.4 Feasibility, Stability and Convergence Properties

Theorem 3.1. Consider subsystem Σ_i with the nominal dynamics (10) in closed loop with LMPC (31), (32). Let Assumption 3.2 hold. Then problem (31) is feasible for all times $t \geq 0$ and all iterations $l \geq 1$. Moreover, the origin is asymptotically stable.

Proof. See Appendix A.

Theorem 3.2. Consider subsystem Σ_i with the nominal dynamics (10) in closed loop with (31), (32). Let Assumption 3.2 hold. Then the iteration cost does not increase with iteration l: $J^{l,[i]}_{0\to\infty}(z^{l,[i]}_0) \leq J^{l-1,[i]}_{0\to\infty}(z^{l-1,[i]}_0)$.

Proof. See Appendix B.

Theorem 3.3. Consider subsystem Σ_i with the nominal dynamics (10) in closed loop with (31), (32). Let Assumption 3.2 hold. If (31) converges to a steady state trajectory $z^{\infty,[i]}_{[0:\infty]}$ as $l \to \infty$, then $(z^{\infty,[i]}_{[0:\infty]}, v^{\infty,[i]}_{[0:\infty]})$ is the global optimal solution to the infinite horizon control problem (17).

Proof. Follows Theorem 3 in Rosolia and Borrelli (2017b), convexity of (17), (31) and decreasing of $J^{l,\star}_{t\to t+N}(z^l_t)$ as proven in (A.3).

4. DISTRIBUTED LEARNING MPC

In this section we extend results in Section 3 and propose an ILC framework: distributed learning model predictive control (DLMPC) for the the global system (1).

4.1 Implementation of DLMPC

Similar to (26), we define the actual iteration cost

$$\bar{J}^{l,[i]}_{0\to\infty}(x^{[i]}_0) = \sum_{k=0}^{\infty} h(x^{l,[i]}_k, u^{l,[i]}_k) \tag{34}$$

associated with $(x^{l,[i]}_{[0:\infty]}, u^{l,[i]}_{[0:\infty]})$, the state and input trajectories of Σ_i with dynamics (2) at iteration l. To evaluate the control performance of the proposed algorithm on the global system (1), we look at two metrics:

(1) **Metric 1**: Iteration cost $\bar{J}^{l,[i]}_{0\to\infty}(x^{[i]}_0)$ of a particular subsystem Σ_i
(2) **Metric 2**: Plant-wide performance defined by the summation of iteration costs over all subsystems, i.e.

$$\bar{J}^{l,\mathrm{sum}}_{0\to\infty}(\mathbf{x}_0) = \sum_{i\in\mathcal{I}_{1:M}} \bar{J}^{l,[i]}_{0\to\infty}(x^{[i]}_0) \tag{35}$$

The procedure of implementing DLMPC is provided in Algorithm 1. Note that during online control (Line 9-24), no communication is required among subsystems.

4.2 Properties of the DLMPC Controller

Theorem 4.1. Let Assumption 2.1, 3.1 and 3.2 hold. The global system (1) in closed loop with DLMPC (31) and (33) satisfies state and input constraints $\mathbf{x}_t \in \mathbb{X}$, $\mathbf{u}_t \in \mathbb{U}$ for all $t \geq 0$ and all $l \geq 1$. The state trajectory $\mathbf{x}_t$ converges asymptotically to the origin for all iterations $l \geq 1$.

Proof. See Appendix C.

Algorithm 1 DLMPC for the global system $\mathbf{\Sigma}$

1: **Given** (for all subsystems $\Sigma_i,\ i \in \mathcal{I}_{1:M}$):
2: Initial state $x^{[i]}_0$ and desired iterations l_d
3: Initial trajectories $(z^{0,[i]}_{[0:\infty]}, v^{0,[i]}_{[0:\infty]})$
4: Ego reference trajectory $\tilde{x}^{[i]}_{[0:\infty]}$ and offsets $c^{[i]}_{[0:\infty]}$
5: Sets computed offline: $\bar{\mathbb{X}}^{[i]}$, $\bar{\mathbb{U}}^{[i]}$ and $\mathbb{E}^{[i]}$
6: **Initialization**:
7: Find t' following (20)
8: Construct sets $\mathcal{CS}^{l,[i]}_{[0:\infty]}$ using (23), (24) for all $i \in \mathcal{I}_{1:M}$
9: Set the iteration counter: $l \leftarrow 1$
10: **Learning Iterations**:
11: **while** $l \leq l_d$ **do**
12: Reset initial state $\mathbf{x}^l_0 \leftarrow \mathbf{x}_0$ and time index $t \leftarrow 0$
13: **while** $\mathbf{x}^l_t \neq 0$ **do**
14: **for** subsystem index $i \leftarrow 1$ to M **do**
15: Solve LMPC (31) and obtain $v^{l,[i],\star}_{[t:t+N-1]}$
16: Update state: $z^{l,[i]}_{t+1} \leftarrow A_{ii} z^{l,[i]}_t + B_i v^{l,[i],\star}_t + c^{[i]}_t$
17: Compute $u^{l,[i]}_t$ using (33)
18: **end for**
19: Set current input: $\mathbf{u}^l_t \leftarrow \mathrm{col}(u^{l,[1]}_t, \ldots, u^{l,[M]}_t)$
20: Update state: $\mathbf{x}^l_{t+1} \leftarrow \mathbf{A}\mathbf{x}^l_t + \mathbf{B}\mathbf{u}^l_t$
21: $t \leftarrow t+1$
22: **end while**
23: Update sets $\mathcal{CS}^{l-1,[i]}_{[0:\infty]}$ for all $i \in \mathcal{I}_{1:M}$
24: $l \leftarrow l+1$
25: **end while**
26: **Determining Output**:
27: **if** Using Metric 1 for Σ_i **then**
28: Choose $l^\star = \mathrm{argmin}_{l=0,\ldots,l_d}\ \bar{J}^{l,[i]}_{0\to\infty}(x^{[i]}_0)$
29: **else**
30: Choose $l^\star = \mathrm{argmin}_{l=0,\ldots,l_d}\ \bar{J}^{l,\mathrm{sum}}_{0\to\infty}(\mathbf{x}_0)$
31: **end if**
32: Output the trajectory $\mathbf{x}^{l^\star}_{[0:\infty]}$ and $\mathbf{u}^{l^\star}_{[0:\infty]}$

Remark 4.1. The properties of non-increasing iteration cost and convergence to the optimal solution as we proved for $(z^{\infty,[i]}_{[0:\infty]}, v^{\infty,[i]}_{[0:\infty]})$, in general, do not hold for $(x^{\infty,[i]}_{[0:\infty]}, u^{\infty,[i]}_{[0:\infty]})$ due to neighbor's incompliance with their reference trajectories. Nevertheless, we show in the numerical example that the control performance of the actual subsystems (2) appears to improve dramatically as the DLMPC is applied for the nominal subsystems (10).

5. NUMERICAL EXAMPLE

In this section, we apply the proposed DLMPC scheme to a benchmark example of power network systems (Saadat et al., 1999; Riverso and Ferrari-Trecate, 2012). The system is composed of four power generation areas coupled through tie-lines. The network is configured according to Scenario 1 in Riverso and Ferrari-Trecate (2012) and sketched in Figure 2. The control objective is to keep the frequency at a nominal level when the load of power changes. In addition, we seek to iteratively improve the control performance of the considered system.

Following Zeilinger et al. (2013) we consider for each subsystem (area) $\Sigma_i, i \in \mathcal{I}_{1:4}$ the LTI dynamics in form of (2), which is linearized around equilibrium and discretized with a sampling time of 1 second. The state $x^{[i]} \in \mathbb{R}^4$ is defined as the deviation from the desired state $x_r^{[i]} = [0, 0, \Delta P_L^{[i]}, \Delta P_L^{[i]}]^\top$ and similarly the desired input $u_r^{[i]} = \Delta P_L^{[i]}$, where $\Delta P_L^{[i]}$ is the change in local power load. For simplicity we consider a step change of $\Delta P_L^{[1]} = 0.15$, $\Delta P_L^{[2]} = -0.15$, $\Delta P_L^{[3]} = 0.12$ and $\Delta P_L^{[4]} = 0.28$ at $t = 0$ of the simulation. To solve a regulation problem in form of (31), we define the initial state of both the nominal and actual subsystem to be $z_0^{[i]} = x_0^{[i]} = -x_r^{[i]}$. For each subsystem Σ_i, sets $\mathcal{E}^{[i]}$, $\mathbb{Z}^{[i]}$ and the decentralized state-feedback gain matrix K_i are synthesized following Betti et al. (2014). DMPC in Farina and Scattolini (2012) with zero terminal constraint is designed and served as the baseline. It is also used to generate a reference trajectory $\tilde{x}_{[0:\infty]}^{[i]}$ and a feasible closed loop trajectory to initialize $P_{[0:\infty]}^{0,[i]}(\cdot)$ and $\mathcal{CS}_{[0:\infty]}^{0,[i]}$. We use a prediction horizon $N = 5$ for (31) and $N = 6$ for the baseline. Subsequently, t' in (20) is found to be $t' = 6$. The rest of model and controller parameters are chosen identical to Scenario 1 in Riverso and Ferrari-Trecate (2012). Quadratic programs (31) are solved in MATLAB using the solver quadprog and the YALMIP interface Löfberg (2004). The termination criterion is chosen to be $\|\mathbf{x}_t\|_2 \leq 10^{-8}$.

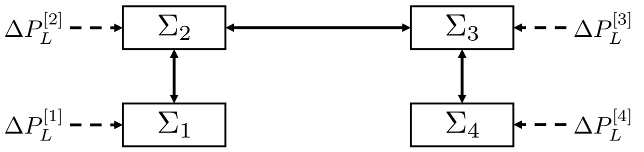

Fig. 2. Power network with four coupled generation areas

5.1 Learning Nominal Trajectories

Table 1 summarizes the nominal iteration costs $J_{0\to\infty}^{l,[i]}(z_0^{[i]})$ of Σ_i in closed loop with (31), (32) over learning iterations $l = 0, \ldots, 28$. As suggested by Theorem 3.2, the iteration cost is non-increasing for all $\Sigma_i, i \in \mathcal{I}_{1:4}$ as the iteration index increments. In addition, we obtain the solution to the infinite horizon control problem (17) by solving it using a finite horizon $T = 50$, long enough to ensure that the terminal state reaches the origin. The associated iteration costs $J_{0\to\infty}^{\star,[i]}(z_0^{[i]})$ are listed on the last line of Table 1. One may observe that $J_{0\to\infty}^{l,[i]}(z_0^{[i]})$ converges to $J_{0\to\infty}^{\star,[i]}(z_0^{[i]})$ after $l = 28$ iterations for all four subsystems.

5.2 Control Performance of the Actual Systems

Now we examine the control performance of DLMPC for the actual subsystems (2) and the global system (1) through the two metrics defined in Section 4.1. Table 2 shows Metric 1: $\bar{J}_{0\to\infty}^{l,[i]}(x_0^{[i]})$ and Metric 2: $\bar{J}_{0\to\infty}^{l,\text{sum}}(\mathbf{x}_0)$ over learning iterations $l = 0, \ldots, 11$. For all iterations $l \geq 1$ the DLMPC successfully steers $\mathbf{x}_t$ to the origin, which empirically validates Theorem 4.1. As pointed out by Remark 4.1, non-increasing property of the actual iteration cost $\bar{J}_{0\to\infty}^{l,[i]}(x_0^{[i]})$ does not hold due to the coupling effect from the neighbors. The costs selected by Metric 1 are highlighted in boldface in column 2-5 of Table 2. For

Table 1. Nominal iteration cost

Iteration	Iteration cost $J_{0\to\infty}^{l,[i]}(z_0^{[i]})$			
	Σ_1	Σ_2	Σ_3	Σ_4
$l = 0$	0.48156	0.32609	0.18844	1.52845
$l = 1$	0.40011	0.29701	0.17509	1.36560
$l = 2$	0.35381	0.28352	0.17219	1.26562
$l = 3$	0.32999	0.27668	0.17080	1.21154
…		…		
$l = 25$	0.30121	0.26827	0.16923	1.15885
$l = 26$	**0.30120**	**0.26826**	0.16923	1.15884
$l = 27$	0.30120	0.26826	**0.16922**	1.15884
$l = 28$	0.30120	0.26826	0.16922	**1.15883**
$J_{0\to\infty}^{\star,[i]}(z_0^{[i]})$	0.30119	0.26824	0.16922	1.15881

example, if the DLMPC aims at improving the control performance of Σ_2, then Algorithm 1 would output the trajectories $\mathbf{x}_{[0:\infty]}^6$ and $\mathbf{u}_{[0:\infty]}^6$ from iteration $l = 6$. On the other hand, if Metric 2 is chosen, then trajectories from iteration $l = 10$ would be selected since it yields the lowest plant-wide cost $\bar{J}_{0\to\infty}^{10,\text{sum}}(\mathbf{x}_0) = 1.90806$. Comparing with the cost from DMPC $\bar{J}_{0\to\infty}^{0,\text{sum}}(\mathbf{x}_0) = 2.51864$, the cost $\bar{J}_{0\to\infty}^{10,\text{sum}}(\mathbf{x}_0)$ achieves a percentage decrease of 24.24%.

We also compare our result with the solution obtained by a centralized MPC (CMPC) controller designed for the global system (1) with terminal set and cost synthesized following Borrelli et al. (2017). Surprisingly, the cost $\bar{J}_{0\to\infty}^{10,\text{sum}}(\mathbf{x}_0) = 1.90806$ of DLMPC chosen by Metric 2 is only 0.2675% higher than the CMPC solution $J_{0\to\infty}^{\text{CMPC}}(\mathbf{x}_0) = 1.90297$. Interestingly, we observe that the iteration cost $J_{0\to\infty}^{\text{CMPC},[3]}(x_0^{[3]}) = 0.17231$ of subsystem Σ_3 from CMPC is higher than $\bar{J}_{0\to\infty}^{3,[3]}(x_0^{[3]}) = 0.16923$, the cost chosen by Metric 1 for Σ_3. This explains the non-cooperative nature of the proposed DLMPC algorithm: each subsystem is acting selfishly to optimize their own performance, but in CMPC there must be some subsystems (in this case Σ_3) that sacrifice their own performance to achieve the optimality for the global system.

Table 2. Actual iteration cost of subsystem Σ_i and the global system

Iter.	Iteration cost $\bar{J}_{0\to\infty}^{l,[i]}(x_0^{[i]})$				
l	Σ_1	Σ_2	Σ_3	Σ_4	$\bar{J}_{0\to\infty}^{l,\text{sum}}(\cdot)$
0	0.47871	0.31987	0.18944	1.53063	2.51864
1	0.39814	0.29101	0.17355	1.36619	2.22888
2	0.35275	0.27864	0.17005	1.26567	2.06712
3	0.32970	0.27332	**0.16923**	1.21172	1.98397
4	0.31728	0.27118	0.16953	1.18564	1.94364
5	0.31056	0.27051	0.17006	1.17283	1.92396
6	0.30701	**0.27050**	0.17055	1.16675	1.91480
7	0.30512	0.27073	0.17097	1.16381	1.91063
8	0.30414	0.27105	0.17130	1.16235	1.90885
9	0.30364	0.27138	0.17156	1.16163	1.90820
10	0.30337	0.27168	0.17176	1.16126	**1.90806**
11	**0.30324**	0.27193	0.17191	**1.16108**	1.90815
CMPC	0.30261	0.26915	0.17231	1.15890	1.90297

Figure 3 compares the inputs obtained by DMPC with the ones given by DLMPC using Metric 2 ($l = 10$) for all four subsystems. One may observe that DLMPC uses much less control efforts to reach the origin than DMPC does.

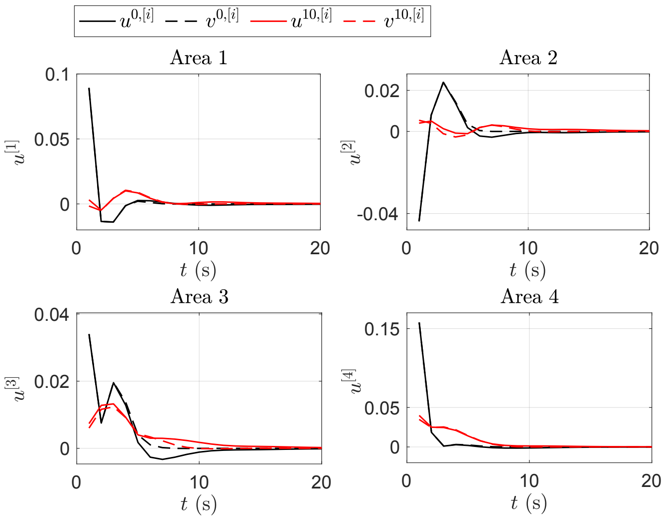

Fig. 3. The actual input (solid lines) and nominal input (dashed lines) produced by DMPC (black lines) and DLMPC (red lines) with Metric 2 ($l = 10$) in all four generation areas.

6. CONCLUSIONS

In this paper, distributed learning model predictive control (DLMPC) is introduced for large-scale dynamically coupled linear systems performing iterative tasks. Each agent makes decisions locally and improves its control performance by repetitively solving a learning MPC problem of which the terminal set and cost function are learned from past iterations. Stability and feasibility are guaranteed. Numerical simulation on a benchmark example demonstrates the usefulness of the proposed control framework. Future work includes co-learning the reference trajectories and extending the approach to the setting of cooperative DMPC methods such as Stewart et al. (2010).

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Appendix A. PROOF OF THEOREM 3.1

In the following (and subsequent) proofs we drop the superscript $(\cdot)^{[i]}$ for brevity of notation and refer every quantities to subsystem Σ_i unless otherwise specified.

We start by showing recursive feasibility. By Assumption 3.2, the initial state sequence $z^0_{[0:N]}$ and input sequence $v^0_{[0:N-1]}$ is a feasible solution to the LMPC (31) at $t=0$ of the l-th iteration. Assume that LMPC (31) is feasible at time t of the l-th iteration. Denote $z^{l,\star}_{[t:t+N]|t}$ and $v^{l,\star}_{[t:t+N-1]|t}$ as the optimal solution. In the next, we consider two cases.

Case 1: $t+N<t'$. Note that (31e) enforces that $z^{l,\star}_{t+N|t} \in \mathcal{CS}^{l-1}_{t+N}$. Define the input $\hat{v}_{t+N} = \sum_{h=0}^{l-1} \lambda^{h,\star} v^h_{t+N} \in \bar{\mathbb{U}}$ and the state $\hat{z}_{t+N+1} = \sum_{h=0}^{l-1} \lambda^{h,\star} z^h_{t+N+1} \in \mathcal{CS}^{l-1}_{t+N+1}$. Note that $\hat{z}_{t+N+1} - \tilde{x}_{t+N+1} \in \mathbb{E}$. Consider the following pair of state and input trajectory

$$[z^{l,\star}_{t+1|t},\ z^{l,\star}_{t+2|t},\ \ldots,\ z^{l,\star}_{t+N-1|t},\ z^{l,\star}_{t+N|t},\ \hat{z}_{t+N+1}] \quad \text{(A.1a)}$$
$$[v^{l,\star}_{t+1|t},\ v^{l,\star}_{t+2|t},\ \ldots,\ v^{l,\star}_{t+N-1|t},\ \hat{v}_{t+N}] \quad \text{(A.1b)}$$

which satisfies constraints (31b)-(31c)-(31d)-(31e) at $t+1$ and therefore it is a feasible solution to the MPC problem (31) of Σ_i to be solved at $t+1$.

Case 2: $t+N \geq t'$. From Remark 3.2 problem (31) is now time-invariant as considered in Rosolia and Borrelli (2017a). Therefore the same argument as in Case 1 is provided by Theorem 1 in Rosolia and Borrelli (2017a). The reminder of the proof is completed by induction.

Asymptotic stability is established by showing that the optimal cost $J^{l,\star}_{t\to t+N}(\cdot)$ from LMPC (31) is a Lyapunov function. Note that if $t+N \geq t'$, the proof directly follows Rosolia and Borrelli (2017a). Therefore, we focus on the case when $t+N<t'$. From (9) we have that $J^{l,\star}_{t\to t+N}(z) \succ 0,\ \forall z \in \mathbb{R}^{n_i}\setminus\{0\}$ and $J^{l,\star}_{t\to t+N}(0)=0$. The optimal cost at time t is

$$\begin{aligned}
&J^{l,\star}_{t\to t+N}(z^l_t) = h(z^{l,\star}_{t|t}, v^{l,\star}_{t|t}) + \sum_{k=t+1}^{t+N-1} h(z^{l,\star}_{k|t}, v^{l,\star}_{k|t}) \\
&+ \sum_{h=0}^{l-1} \lambda^{h,\star} h(z^h_{t+N}, v^h_{t+N}) + \sum_{h=0}^{l-1} \lambda^{h,\star} \sum_{k=t+N+1}^{\infty} h(z^h_k, v^h_k) \\
&\geq h(z^{l,\star}_{t|t}, v^{l,\star}_{t|t}) + \sum_{k=t+1}^{t+N-1} h(z^{l,\star}_{k|t}, v^{l,\star}_{k|t}) \\
&+ h(\sum_{h=0}^{l-1} \lambda^{h,\star} z^h_{t+N}, \sum_{h=0}^{l-1} \lambda^{h,\star} v^h_{t+N}) + \sum_{h=0}^{l-1} \lambda^{h,\star} J^h_{t+N+1\to\infty}(\cdot) \\
&\geq h(z^{l,\star}_{t|t}, v^{l,\star}_{t|t}) + J^{l,\star}_{t+1\to t+N+1}(z^{l,\star}_{t+1|t})
\end{aligned} \quad \text{(A.2)}$$

We conclude from (A.2) that for all $z^l_t \in \mathbb{R}^{n_i}\setminus\{0\}$

$$J^{l,\star}_{t+1\to t+N+1}(z^l_{t+1}) - J^{l,\star}_{t\to t+N}(z^l_t) \leq -h(z^l_t, v^l_t) < 0 \quad \text{(A.3)}$$

Appendix B. PROOF OF THEOREM 3.2

By (26) the iteration cost of the $l-1$-th iteration is

$$\begin{aligned}
J^{l-1}_{0\to\infty}(z^{l-1}_0) &\geq \min_{\bar{v}_{[0:N-1]}} \left[\sum_{k=0}^{N-1} h(\bar{z}_k, \bar{v}_k) + P^{l-1}_N(\bar{z}_N) \right] \\
&= J^{l,\star}_{0\to N}(z^l_0)
\end{aligned} \quad \text{(B.1)}$$

By (A.3) we can show that $J^{l,\star}_{0\to N}(z^l_0)$ is lower bounded by

$$J^{l,\star}_{0\to N}(z^l_0) \geq \lim_{t\to\infty} \left[\sum_{k=0}^{t-1} h(z^l_k, v^l_k) + J^{l,\star}_{t\to N+t}(z^l_t) \right] \quad \text{(B.2)}$$

From Theorem 3.1 we have that $\lim_{t\to\infty} z^l_t = 0$. By continuity of $h(\cdot,\cdot)$ we have that

$$J^{l,\star}_{0\to N}(z^l_0) \geq \sum_{k=0}^{\infty} h(z^l_k, v^l_k) = J^l_{0\to\infty}(z^l_0) \quad \text{(B.3)}$$

Finally from (B.1) and (B.3) we conclude that

$$J^l_{0\to\infty}(z^l_0) \leq J^{l,\star}_{0\to N}(z^l_0) \leq J^{l-1}_{0\to\infty}(z^{l-1}_0) \quad \text{(B.4)}$$

Appendix C. PROOF OF THEOREM 4.1

The proof relies on Theorem 3.1. Since it holds over all iteration $l \geq 1$, we drop the iteration index l for simplicity of notation. Define a set of collective vectors: $\mathbf{z}_t = \mathrm{col}(z^{[1]}_t, \ldots, z^{[M]}_t)$, similarly for $\mathbf{w}_t$ and $\mathbf{e}_t$. In addition, let $\mathbf{A}_D = \mathrm{diag}(A_{11}, \ldots, A_{MM})$ and $\mathbf{A}_C = \mathbf{A} - \mathbf{A}_D$.

By feasibility of (31) and boundedness of $e^{[i]}_t$ (14) we have

$$x^{[i]}_t \in \bar{\mathbb{X}}^{[i]} \oplus \mathbb{Z}^{[i]} \subseteq \mathbb{X}^{[i]},\ u^{[i]}_t \in \bar{\mathbb{U}}^{[i]} \oplus K_i \mathbb{Z}^{[i]} \subseteq \mathbb{U}^{[i]} \quad \text{(C.1)}$$

for all $t \geq 0$. It follows from (C.1) and (3) that

$$\mathbf{x}_t \in \mathbb{X}^{[i]},\ \mathbf{u}_t \in \mathbb{U}^{[i]}, \quad \forall t \geq 0 \quad \text{(C.2)}$$

Note that for all $t \geq t'$ we have that $w^{[i]}_t = \sum_{j\in\mathcal{N}_i} A_{ij} x^{[j]}_t$ since $\tilde{x}^{[j]}_t = 0,\ t \geq t'$ as defined (20). Consequently,

$$\begin{aligned}
\mathbf{w}_t &= \mathrm{col}(\sum_{j_1\in\mathcal{N}_1} A_{1j_1} x^{[j_1]}_t, \ldots, \sum_{j_M\in\mathcal{N}_M} A_{Mj_M} x^{[j_M]}_t) \\
&= \mathbf{A}_C \mathbf{x}_t, \quad \forall t \geq t'
\end{aligned} \quad \text{(C.3)}$$

Recall that the global state $\mathbf{x}_t$ is decomposed as

$$\mathbf{x}_t = \mathbf{z}_t + \mathbf{e}_t \quad \text{(C.4)}$$

From (12), (C.3) and (C.4) we have that

$$\begin{aligned}
\mathbf{e}_{t+1} &= (\mathbf{A}_D + \mathbf{BK})\mathbf{e}_t + \mathbf{A}_C \mathbf{x}_t \\
&= (\mathbf{A}_D + \mathbf{BK})\mathbf{e}_t + \mathbf{A}_C \mathbf{e}_t + \mathbf{A}_C \mathbf{z}_t \\
&= \mathbf{\Phi}\mathbf{e}_t + \mathbf{A}_C \mathbf{z}_t
\end{aligned} \quad \text{(C.5)}$$

By Theorem 3.1 the collective nominal state $\mathbf{z}_t$ satisfies

$$\lim_{t\to\infty} \mathbf{z}_t = 0 \text{ and } \lim_{t\to\infty} \mathbf{A}_C \mathbf{z}_t = 0 \quad \text{(C.6)}$$

Therefore (C.5) is a linear system with diminishing input and initial condition $\mathbf{e}_{t'} \in \prod_{i=1}^{M} \mathbb{Z}^{[i]}$. The state equation is

$$\mathbf{e}_t = \Phi^{t-t'} \mathbf{e}_{t'} + \sum_{k=t'}^{t-1} \Phi^{t-1-k} \mathbf{A}_C \mathbf{z}_k, \quad t > t' \quad \text{(C.7)}$$

By Assumption 2.1 and (C.6) we have that

$$\lim_{t\to\infty} \Phi^{t-t'} \mathbf{e}_{t'} = 0 \text{ and } \lim_{t\to\infty} \Phi^{t-1-k} \mathbf{A}_C \mathbf{z}_k = 0 \quad \text{(C.8)}$$

for all $k \geq t'$. It then follows that

$$\lim_{t\to\infty} \mathbf{e}_t = 0 \quad \text{(C.9)}$$

Combining (C.4), (C.6) and (C.9) we conclude with

$$\lim_{t\to\infty} \mathbf{x}_t = 0 \quad \text{(C.10)}$$

This completes the proof.