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Lyapunov-based Singular Perturbation Results in the Framework of Hybrid Systems ⋆

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Abstract: Stability properties of singularly perturbed hybrid systems are investigated via Lyapunov functions with assistance from the invariance principle. Both continuously differentiable Lyapunov functions and non-smooth Lyapunov functions are considered. In each case, under appropriate assumptions, uniform asymptotic stability and uniform global asymptotic stability are established. An estimate of the basin of attraction is given for the former property. Two examples are given to illustrate the proposed theoretical results based on continuously differentiable Lyapunov functions. In addition, one example for switched learning inclusions with unstable modes is given to show the effectiveness of the results obtained based on non-smooth Lyapunov functions.

Keywords: Singular perturbation, hybrid dynamical systems, Lyapunov function methods, uniform asymptotic stability, uniform global asymptotic stability.

1. INTRODUCTION

The analysis and design of singularly perturbed systems have a rich history in the control literature; see Kokotovic et al. (1976) and Saksena et al. (1984) for early surveys. Singularly perturbed systems can be analyzed with various tools; one very efficient and constructive approach is through Lyapunov functions, as employed early on by Chow and Kokotovic (1981), Grujić (1981) and Saberi and Khalil (1984) for example.

With the development of hybrid systems in the control literature over the last several decades, similar singular perturbation results are desirable in the hybrid systems setting as well. This paper provides another step in that direction. We develop results in the hybrid systems framework of Goebel et al. (2012). Some singular perturbation results for these hybrid systems already can be found in Sanfelice et al. (2006), Sanfelice and Teel (2011) and Wang et al. (2012), for example. These works establish semi-global, practical asymptotic stability for singularly perturbed systems, exploiting robustness of stability in hybrid systems that satisfy certain basic conditions. To the best of our knowledge, there are no results in the hybrid systems literature that characterize when actual asymptotic stability (as opposed to practical asymptotic stability) is achieved. To produce such results, we combine the singular perturbation for hybrid systems ideas found in Sanfelice and Teel (2011) with the Lyapunov function ideas of Saberi and Khalil (1984), where quadratic-like Lyapunov functions are employed. Like in Saberi and Khalil (1984), we are able to give an estimate of the basin of attraction in terms of the singular perturbation parameter and bounds on the Lyapunov function. We consider both continuously differentiable and non-smooth Lyapunov functions. Using these Lyapunov functions, uniform asymptotic stability (UAS) and uniform global asymptotic stability (UGAS) results are guaranteed under some mild assumptions. Moreover, the proposed methods can be applied to some practical problems such as robot source seeking Cochran and Krstić (2009), Nash seeking with adversarial agents Frihauf et al. (2012), and distributed optimization with attacks Wang et al. (2020). In addition, compared with Sanfelice and Teel (2011) and Wang et al. (2012), the assumptions we make are stronger and the conclusions we draw are also stronger.

The rest of the paper is organized as follows. In Section 2, some preliminaries are given. In Sections 3, a class of singularly perturbed hybrid systems is considered and the main results are presented. In Section 4, three examples are given to illustrate the main results. Section 5 contains the conclusions.

2. PRELIMINARIES

$\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real and integer numbers, respectively; moreover, $\mathbb{R}_{>0}:=(0,\infty)$ and $\mathbb{R}_{\geq 0}:=[0,\infty)$. $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. $\mathbf{0}$ stands for zero matrix/vector with appropriate dimension. $\mathbf{1}_n := (1,\ldots,1)^T \in \mathbb{R}^n$. $I_n \in \mathbb{R}^{n\times n}$ stands for the identity matrix. Given $x,y \in \mathbb{R}^n$, $\langle x,y\rangle := x^T y$ and $(x,y) := (x^T, y^T)^T$. $\otimes$ is the Kronecker product. $P > 0$ denotes a symmetric positive definite matrix. $\mathbb{B}$ is open unit ball and $\overline{\mathbb{B}}$ is closed unit ball. Given a compact set $\mathbb{W} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_{\mathbb{W}} := \inf_{y\in\mathbb{W}} |x-y|$; also, given $\rho \in \mathbb{R}_{>0} \cup \{\infty\}$, we use $\mathbb{B}^{\rho}_{\mathbb{W}} := \{x \in \mathbb{R}^n : |x|_{\mathbb{W}} < \rho\}$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semi-continuous (OSC) at $x \in \mathbb{R}^n$ if, for all convergent sequences $\{(x_i,y_i)\}_{i=1}^{\infty}$ satisfying $y_i \in M(x_i)$ for all i, the limit $(x,y) = \lim_{i\to\infty}(x_i,y_i)$ sat-

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isfies $y \in M(x)$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded (LB) at $x \in \mathbb{R}^n$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^m$ is bounded. Given a set $\Omega \subset \mathbb{R}^n$, the mapping M is said to be OSC and LB relative to Ω if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $M(x)$ for $x \in \Omega$ and $\emptyset$ for $x \notin \Omega$ is OSC and LB at each $x \in \Omega$. Given a set $\Omega \subset \mathbb{R}^n$, $\overline{\text{co}}\Omega$ stands for the closed convex hull of Ω. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{L}$, i.e., $\alpha \in \mathscr{L}$, if: (i) it is continuous, (ii) non-increasing, and (iii) converging to zero as its argument grows unbounded. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$, i.e., $\alpha \in \mathscr{K}$, if: (i) it is continuous, (ii) zero at zero, and (iii) strictly increasing. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}_\infty$, i.e., $\alpha \in \mathscr{K}_\infty$, if $\alpha \in \mathscr{K}$ and α grows unbounded as its argument grows unbounded. A function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathscr{KL}$, i.e., $\alpha \in \mathscr{KL}$ if: (i) it is of class $\mathscr{K}$ in its first argument; (ii) it is of class $\mathscr{L}$ in its second argument. Given a continuously differentiable function $W : \mathbb{R}^{n_1+n_2} \to \mathbb{R}_{\geq 0}$, define $\nabla_{x_2} W(x_1,x_2) := \frac{\partial W(x_1,x_2)}{\partial x_2}, \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$.

In this paper, we consider the hybrid systems framework that appears in Goebel et al. (2012). These hybrid models have state $x \in \mathbb{R}^n$ and are written formally as

$$\begin{aligned} \dot{x} &\in F(x),\ x \in C \\ x^+ &\in G(x),\ x \in D, \end{aligned} \tag{1}$$

where $C \subset \mathbb{R}^n$ is *flow set*, $D \subset \mathbb{R}^n$ is *jump set*, the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map* and the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map*. System (1) is represented by the notation $\mathscr{H} := \{C,F,D,G\}$. Solutions are defined on hybrid time domains. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1}([s_j,s_{j+1}],j)$ for some finite sequence of times $0 = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_J$. It is a *hybrid time domain* if for all $(T,J) \in E$, $E \cap ([0,T] \times \{0,\ldots,J\})$ is a compact hybrid domain.

Definition 1. A function $x : \text{dom}\, x \mapsto \mathbb{R}^n$ is a *hybrid arc* if dom x is a hybrid time domain and $t \mapsto x(t,j)$ is locally absolutely continuous for each j such that the interval $I_j := \{t : (t,j) \in \text{dom}\, x\}$ has nonempty interior. A hybrid arc is *complete* if its domain is unbounded. A hybrid arc x is a *solution to system (1)* if $x(0,0) \in \overline{C} \cup D$, and the following two conditions hold:
1) for all $j \in \mathbb{Z}_{\geq 0}$ such that I_j has nonempty interior

$$\begin{aligned} x(t,j) &\in C && \text{for all } t \in \text{int}(I_j), \\ \dot{x}(t,j) &\in F(x(t,j)) && \text{for almost all } t \in I_j; \end{aligned}$$

2) for all $(t,j) \in \text{dom}\, x$ such that $(t,j+1) \in \text{dom}\, x$,

$$\begin{aligned} x(t,j) &\in D, \\ x(t,j+1) &\in G(x(t,j)). \end{aligned}$$

■

Definition 2. Consider a hybrid system $\mathscr{H}$ on $\mathbb{R}^n$. Let $\mathscr{W} \subset \mathbb{R}^n$ be closed. The set $\mathscr{W}$ is said to be:

- *uniformly asymptotically stable* (UAS) if there exist a function $\bar{\beta} \in \mathscr{KL}$ and a positive constant c, such that for any solution x to $\mathscr{H}$ with $|x(0,0)|_{\mathscr{W}} < c$,
$$|x(t,j)|_{\mathscr{W}} \leq \bar{\beta}(|x(0,0)|_{\mathscr{W}}, t+j),\ \forall (t,j) \in \text{dom}\, x; \tag{2}$$
- *uniformly globally asymptotically stable* (UGAS) if inequality (2) is satisfied for any initial state. ■

3. MODELING AND STABILITY ANALYSIS

Consider the following hybrid system:

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} \dot{x}_1 \in F_1(x_1,x_2) \\ \dot{x}_2 \in \dfrac{1}{\varepsilon} F_2(x_1,x_2,\varepsilon) \end{array} \right\} & (x_1,x_2) \in C \times X_1 \\ x^+ \in G(x) \quad \} & (x_1,x_2) \in D \times X_2, \end{array} \right. \tag{3}$$

where $x := (x_1,x_2) \in \mathbb{R}^n$ with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $\varepsilon > 0$ is small, $C \times X_1$ is the flow set, $D \times X_2$ is the jump set, the set-valued mappings F_1, F_2 comprise the flow map, and the set-valued mapping G is the jump map. Define $F_\varepsilon(x) := (F_1(x_1,x_2), F_2(x_1,x_2,\varepsilon))$ for all $x \in C \times X_1$ and $F_\varepsilon(x) = \varnothing$ otherwise.

Assumption 1. $C \times X_1$ and $D \times X_2$ are closed sets, F_ε is OSC and LB with convex nonempty values on C for each $\varepsilon > 0$, and G is OSC and LB and $G(x)$ is nonempty for each $x \in D$. ■

By setting $\tau = \frac{t}{\varepsilon}$, the boundary-layer system of system (3) is:

$$\frac{dx_2}{d\tau} \in F_2(x_1,x_2,0),\ (x_1,x_2) \in C \times X_1. \tag{4}$$

Treating x_1 as a constant, the boundary layer system (4) is supposed to have (via a subsequent assumption) a quasi-steady-state equilibrium manifold, like in classical singular perturbation theory, which is expressed in terms of a set-valued mapping H satisfying the following assumption.

Assumption 2. $H : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ is OSC and LB. For each $x_1 \in C$, $H(x_1)$ is a nonempty subset of X_1. ■

By constraining x_2 to the set $H(x_1)$ during flows, the reduced system for the singularly perturbed system (3) is obtained as follows:

$$\begin{aligned} \dot{x}_1 &\in F_r(x_1),\ \ x_1 \in C \\ x_1^+ &\in G_r(x_1),\ \ x_1 \in D, \end{aligned} \tag{5}$$

where

$$\begin{aligned} F_r(x_1) &:= \overline{\text{co}}\{v \in \mathbb{R}^{n_1} : v \in F_1(x_1,x_2), x_2 \in H(x_1)\} \\ G_r(x_1) &:= \{v_1 \in \mathbb{R}^{n_1} : (v_1,v_2) \in G(x_1,x_2), x_2 \in X_2, v_2 \in X_1 \cup X_2\}. \end{aligned}$$

The objective of this paper is to establish stability conditions for system (3) based on Lyapunov functions for the reduced system (5) and the boundary layer system (4). We will first assume that the reduced system has a compact set $\mathbb{W}_1$ UAS with a Lyapunov characterization. The following definition quantifies the property that we will use.

Definition 3. The function $V : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ is said to be *a Lyapunov function for the reduced system (5)* parametrized by $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, the continuous, positive definite function α_3, the continuous, positive semidefinite function α_4, the positive real numbers L_1, d_1 and $\rho \in \mathbb{R}_{>0} \cup \{\infty\}$ if:

(1) for all $x_1 \in C \cup D \cup G_r(D)$, it holds that
$$\alpha_1(|x_1|_{\mathbb{W}_1}) \leq V(x_1) \leq \alpha_2(|x_1|_{\mathbb{W}_1}); \tag{6}$$
(2) for all $x_1 \in C \cap \mathbb{B}^{\rho}_{\mathbb{W}_1}$ and $f_r \in F_r(x_1)$, it holds that
$$\langle \nabla V(x_1), f_r \rangle \leq -L_1 \alpha_3^2(|x_1|_{\mathbb{W}_1});$$
(3) and for all $x_1 \in D \cap \mathbb{B}^{\rho}_{\mathbb{W}_1}, g_1 \in G_r(x_1)$, it holds that
$$V(g_1) - V(x_1) \leq -d_1 \alpha_4(|x_1|_{\mathbb{W}_1}). \tag{7}$$
■

Next, we assume that the boundary layer, with x_1 constant, has the set $x_2 \in H(x_1)$ UAS with a Lyapunov function. The following definition quantifies the property that we will use.

Definition 4. A function $W:\mathbb{R}^{n_1+n_2}\to\mathbb{R}_{\geq 0}$ is said to be a *Lyapunov function for the boundary layer system (4)* parameterized by $\beta_1,\beta_2\in\mathscr{K}_\infty$, the continuous, positive definite function α_5, the continuous, positive semidefinite function α_4, the positive real numbers L_2,d_2 and $\rho\in\mathbb{R}_{>0}\cup\{\infty\}$ if:

(1) for all $x\in(C\times X_1)\cup(D\times X_2)\cup G(D\times X_2)$, it holds that
$$\beta_1(|x_2|_{H(x_1)})\leq W(x_1,x_2)\leq\beta_2(|x_2|_{H(x_1)}); \quad (8)$$
(2) for all $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$, and $\tilde{f}_2\in F_2(x_1,x_2,0)$, it holds that
$$\langle\nabla_{x_2}W(x_1,x_2),\tilde{f}_2\rangle\leq -L_2\alpha_5^2(|x_2|_{H(x_1)});$$
(3) and for all $x\in(D\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_2$, it holds that
$$W(g)\leq W(x_1,x_2)+d_2\alpha_4(|x_1|_{\mathbb{W}_1}),\ g\in G(x). \quad (9)$$
■

Following Saberi and Khalil (1984), for the full, singularly perturbed system (3), we use a Lyapunov function of the form
$$U(x):=(1-d)V(x_1)+dW(x) \quad (10)$$
with $d\in(0,1)$ to establish UAS of the set
$$\mathbb{W}:=\{(x_1,x_2)|x_1\in\mathbb{W}_1,x_2\in H(x_1)\} \quad (11)$$
when $\varepsilon>0$ is sufficiently small. The following holds for U:

Proposition 1. If V is *a Lyapunov function for the reduced system (5)* and W is *a Lyapunov function for the boundary layer system (4)*, then there exist functions $\alpha_{U,1,d},\alpha_{U,2}\in\mathscr{K}_\infty$ ($\alpha_{U,2}$ can be taken independent of d) such that for all $x\in(C\times X_1)\cup(D\times X_2)\cup G(D\times X_2)$, it holds that
$$\alpha_{U,1,d}(|x|_{\mathbb{W}})\leq U(x)\leq\alpha_{U,2}(|x|_{\mathbb{W}}). \quad (12)$$

UAS of $\mathbb{W}_1$ for the reduced system (5) and UAS of $x_2\in H(x_1)$ for the boundary layer (4) are already enough for semi-global (for the UGAS case), practical asymptotic stability of $\mathbb{W}$ for (3), as established in Sanfelice and Teel (2011) and Wang et al. (2012). However, in the present paper we are interested in conditions that give a global, asymptotic (not just practical) result, and we are interested in giving an estimate for the basin of attraction in the local case. For such results, we require coupling conditions between Lyapunov functions, like in the continuous-time results of Saberi and Khalil (1984). The following definition quantifies the coupling conditions.

Definition 5. A Lyapunov function V for the reduced system, a Lyapunov function W for the boundary layer system and the continuous, positive definite functions α_3 and α_5 are said to satisfy the *coupling conditions* with the coupling parameters $M_i>0(i=1,2,3,4,5)$ and $\rho\in\mathbb{R}_{>0}\cup\{\infty\}$ if the following conditions hold:

(1) for all $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$, and for $\forall f_1\in F_1(x_1,x_2),\exists f_r\in F_r(x_1)$ such that
$$\langle\nabla V(x_1),f_1-f_r\rangle\leq M_1\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)});$$
(2) for all $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$, it holds that
$$\langle\nabla_{x_1}W(x_1,x_2),f_1\rangle\leq M_2\alpha_5^2(|x_2|_{H(x_1)})+M_3\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)});$$
(3) for all $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$ and $f_2\in F_2(x_1,x_2,\varepsilon),\exists\tilde{f}_2\in F_2(x_1,x_2,0)$ such that
$$\langle\nabla_{x_2}W(x_1,x_2),f_2-\tilde{f}_2\rangle\leq\ \varepsilon M_4\alpha_5^2(|x_2|_{H(x_1)})+\varepsilon M_5\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)}).$$
■

We codify our main assumptions as follows.

Assumption 3. For $\rho\in\mathbb{R}_{>0}\cup\{\infty\}$, the following conditions hold:

- The function V is *a Lyapunov function for the reduced system (5)* parameterized by $\alpha_1,\alpha_2\in\mathscr{K}_\infty$, the continuous, positive definite function α_3, the continuous, positive semidefinite function α_4, the positive real numbers L_1,d_1 and the given ρ;
- The function W is *a Lyapunov function for the boundary layer system (4)* parameterized by $\beta_1,\beta_2\in\mathscr{K}_\infty$, the continuous, positive definite function α_5, the positive real numbers L_2,d_2 and the given ρ;
- The functions V and W and the continuous, positive definite functions α_3 and α_5 satisfy the *coupling conditions* with the coupling parameters $M_i>0(i=1,2,3,4,5)$ and the given ρ. ■

Theorem 1. Consider system (3) and the set $\mathbb{W}$ defined in (11). Suppose Assumptions 1-3 hold and let
$$\varepsilon^*(d):=\frac{L_1L_2}{L_1\gamma_1+[M_1(1-d)+\gamma_2d]^2/4d(1-d)},$$
$$d\leq\frac{d_1}{d_1+d_2}, \quad (13)$$
where $\gamma_1:=M_2+M_4,\gamma_2:=M_3+M_5$. If $\varepsilon\in(0,\varepsilon^*(d))$ and any complete solution x of the discrete-time dynamics in (3) with $x(0,0)\in(D\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_2$ converges to $\mathbb{W}$, then $\mathbb{W}$ is UAS for system (3) with basin of attraction containing $\mathbb{B}_{\mathbb{W}}^{\alpha_{U,2}^{-1}\circ\alpha_1(\rho)}$ where $\alpha_{U,2}\in\mathscr{K}_\infty$ comes from (12) in Proposition 1.

Proof. Choose the Lyapunov candidate function U given in (10). Then, based on conditions of Theorem 1, for all $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$ and $f\in F_\varepsilon(x)$ we have that
$$\begin{aligned}&\langle\nabla U(x),f\rangle\\&=(1-d)\langle\nabla V(x_1),f_1\rangle+d\langle\nabla_{x_2}W(x_1,x_2),\frac{1}{\varepsilon}f_2\rangle\\&\qquad+d\langle\nabla_{x_1}W(x_1,x_2),f_1\rangle\\&=(1-d)\langle\nabla V(x_1),f_r\rangle+(1-d)\langle\nabla V(x_1),f_1-f_r\rangle\\&\quad+\frac{d}{\varepsilon}\langle\nabla_{x_2}W(x_1,x_2),\tilde{f}_2\rangle+\frac{d}{\varepsilon}\langle\nabla_{x_2}W(x_1,x_2),f_2-\tilde{f}_2\rangle\\&\quad+d\langle\nabla_{x_1}W(x_1,x_2),f_1\rangle.\end{aligned}$$
It follows from the definitions of the Lyapunov functions that
$$\begin{aligned}&\langle\nabla U(x),f\rangle\\&\leq-L_1(1-d)\alpha_3^2(|x_1|_{\mathbb{W}_1})+M_1(1-d)\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)})\\&\quad-L_2\frac{d}{\varepsilon}\alpha_5^2(|x_2|_{H(x_1)})+dM_4\alpha_5^2(|x_2|_{H(x_1)})\\&\quad+dM_5\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)})+dM_2\alpha_5^2(|x_2|_{H(x_1)})\\&\quad+dM_3\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)})\\&=-L_1(1-d)\alpha_3^2(|x_1|_{\mathbb{W}_1})-d\Big(\frac{L_2}{\varepsilon}-M_2-M_4\Big)\alpha_5^2(|x_2|_{H(x_1)})\\&\quad+((1-d)M_1+dM_3+dM_5)\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)}).\end{aligned}$$
Therefore, from (12) and (13), there exists a continuous, positive definite function α which is independent of ε such that for $x\in(C\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_1$ and for all $f\in F_\varepsilon(x)$, we have
$$\langle\nabla U(x),f\rangle\leq-\alpha(U(x)). \quad (14)$$
In addition, for all $x\in(D\cap\mathbb{B}^{\rho}_{\mathbb{W}_1})\times X_2$ and all $g\in G(x)$, from (7), (9) and (13) we obtain

$$\begin{aligned}&U(g)-U(x_1,x_2)\\ =&(1-d)V(g_1)+dW(g)-(1-d)V(x_1)-dW(x_1,x_2)\\ &\le -((1-d)d_1-dd_2)\alpha_4(|x_1|_{\mathbb{W}_1})\\ &\le 0. \end{aligned} \tag{15}$$

Also, using (6), (12), (14), (15) and

$$\alpha_1(|x_1(t,j)|_{\mathbb{W}_1})\le V(x_1(t,j))\le U(x(t,j))\\ \le U(x(0,0))\le \alpha_{U,2}(|x(0,0)|_{\mathbb{W}})$$

for any initial value satisfying

$$|x(0,0)|_{\mathbb{W}}<\alpha_{U,2}^{-1}\circ\alpha_1(\rho),$$

we have $|x_1(t,j)|_{\mathbb{W}_1}<\rho$ for all $(t,j)\in$ dom x. Then $\mathbb{W}$ is uniformly stable. Since system (3) satisfies Assumption 1 and $\mathbb{W}$ is compact, combining the conditions of Theorem 1 and the invariance principle for hybrid systems in Sanfelice et al. (2007), we conclude that $\mathbb{W}$ is UAS for (3) with basin of attraction containing $\mathbb{B}_{\mathbb{W}}^{\alpha_{U,2}^{-1}\circ\alpha_1(\rho)}$. ■

Theorem 2. Suppose all the conditions in Theorem 1 hold globally, i.e., $\rho=\infty$. Then $\mathbb{W}$ is UGAS.

In Theorems 1 and 2, we consider only continuously differentiable functions V. In practical applications, however, a non-smooth function V is sometimes needed. Thus, it is useful to establish the following results.

Definition 6. A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be regular if for all v, the usual one-sided directional derivative $f^{'}(z;v)$ exists and $f^{'}(z;v)=f^{\circ}(z;v)$. ■

Theorem 3. Suppose all the conditions in Theorem 1 hold except that V is replaced by a locally Lipschitz, regular function, condition (2) in the definition of a Lyapunov function for the reduced system is replaced by:
for all $x_1\in C\cap\mathbb{B}_{\mathbb{W}_1}^{\rho}$ and all $f_r\in F_r(x_1)$, we have that

$$\min_{p\in\partial_{x_1}V(x_1)}\langle p,f_r\rangle\le -L_1\alpha_3^2(|x_1|_{\mathbb{W}_1})$$

and coupling condition (1) in Definition 5 is replace by:
for all $x\in(C\cap\mathbb{B}_{\mathbb{W}_1}^{\rho})\times X_1$, and for $\forall f_1\in F_1(x_1,x_2)$, $\exists f_r\in F_r(x_1)$ such that

$$\max_{p\in\partial_{x_1}V(x_1)}\langle p,f_1-f_r\rangle\le M_1\alpha_3(|x_1|_{\mathbb{W}_1})\alpha_5(|x_2|_{H(x_1)}).$$

Then $\mathbb{W}$ is UAS for (3) with basin of attraction containing $\mathbb{B}_{\mathbb{W}}^{\alpha_{U,2}^{-1}\circ\alpha_1(\rho)}$ where $\alpha_{U,2}\in\mathcal{K}_\infty$ is from (12) in Proposition 1.

Proof. From Bacciotti and Ceragioli (1999), Teel (2000) and the proof of Theorem 1, the conclusion follows. ■

Theorem 4. If all of the conditions in Theorem 3 hold globally, i.e. $\rho=\infty$, then the set $\mathbb{W}$ defined in (11) is UGAS for (3).

4. EXAMPLES

We present three examples to illustrate the meaning of the proposed results in Theorems 1, 2 and 4, respectively. The first example is a linear system used to show the proposed continuously differentiable Lyapunov analysis may not have any conservatism for certain examples and UGAS result is obtained. The second example is a nonlinear system used to present a local result by considering continuously differentiable Lyapunov functions. The third example is the switched learning inclusions with unstable modes used to obtain UGAS result based on non-smooth Lyapunov functions.

Example 1. Consider a linear system with states $x_1:=(z,\tau)$ and $x_2:=y$:

$$\begin{cases}\left.\begin{aligned}\dot z&=z+y\\ \dot\tau&=1\\ \varepsilon\dot y&=-(y+2z)\end{aligned}\right\} & \tau\in[0,T]\\ \left.\begin{aligned}z^+&=2z\\ \tau^+&=0\\ y^+&=ay+bz\end{aligned}\right\} & \tau=T,\end{cases}$$

where $a\in(-1,1)$, $b\in\mathbb{R}$ and $e^T>2$. The *flow set* is $C\times X_1:=\Big(\mathbb{R}^{n_1}\times[0,T]\Big)\times\mathbb{R}^{n_1}$ and the *jump set* is $D\times X_2:=\Big(\mathbb{R}^{n_1}\times\{T\}\Big)\times\mathbb{R}^{n_1}$.

Setting $\varepsilon=0$, we obtain the quasi-steady-state equilibrium manifold $H(x_1)=-2z$ and the reduced system is:

$$\begin{bmatrix}\dot z\\ \dot\tau\end{bmatrix}=F_r(x_1):=\begin{cases}-z\\ 1\end{cases}\quad x_1\in C,$$
$$\begin{bmatrix}z^+\\ \tau^+\end{bmatrix}=G_r(x_1):=\begin{cases}2z\\ 0\end{cases}\quad x_1\in D.$$

Given $\mathbb{W}_1:=\{0\}\times[0,T]$ and $\mathbb{W}_2:=\{\vartheta|\vartheta=-2z,x_1\in\mathbb{W}_1\}$. Choose $V(x_1):=z^2\exp(c\tau)$ and $W(x_1,x_2):=(y+2z)^2$, then by computing we have that conditions (1) of Definitions 3 and 4 hold obviously. Moreover, we obtain the conditions (2) and (3) of Definition 3 as follows: $\langle\nabla V(x_1),f_r\rangle\le-(2-c)|x_1|_{\mathbb{W}_1}^2$, $V(g_1)-V(x_1)\le(4-e^{cT})|x_1|_{\mathbb{W}_1}^2$. Since $e^T>2$, there exists $c<2$ such that $4-e^{cT}<0$.

The conditions (2) and (3) of Definition 4 are as follows:

$$\begin{aligned}&\langle\nabla_{x_2}W(x_1,x_2),\tilde f_2\rangle\le-2|x_2|_{H(x_1)}^2,\\ &W(g)=[a(y+2z)+(b+4-2a)z]^2\\ &=a^2(y+2z)^2+(b+4-2a)^2z^2+2a(y+2z)(b+4-2a)z\\ &\le(a^2+c_1a^2)W(x_1,x_2)+\frac{1+c_1}{c_1}(b+4-2a)^2|x_1|_{\mathbb{W}_1}^2,\end{aligned}$$

where $c_1:=\frac{1-a^2}{a^2}$.

The *coupling conditions* of Definition 5 are as follows:

(1) $\langle\nabla V(x_1),f_1-f_r\rangle\le 2e^{cT}|x_1|_{\mathbb{W}_1}|x_2|_{H(x_1)}$;
(2) $\langle\nabla_{x_1}W(x_1,x_2),f_1\rangle\le4|x_2|_{H(x_1)}^2+4|x_2|_{H(x_1)}|x_1|_{\mathbb{W}_1}$;
(3) $\langle\nabla_{x_2}W(x_1,x_2),f_2-\tilde f_2\rangle=0$.

Then using Theorem 2, we can conclude that $\mathbb{W}$ is UGAS for all $\varepsilon<\varepsilon^*$ where ε^* is given by $\varepsilon^*:=\frac{2d(2-c)(1-d)}{4d(2-c)(1-d)+(e^{cT}(1-d)+2d)^2}$ with $d\le\frac{(e^{cT}-4)(1-a^2)}{(e^{cT}-4)(1-a^2)+(b+4-2a)^2}$. ■

Example 1 shows that the proposed Lyapunov analysis may not have any conservatism for certain examples, since the reduced system is a sampled-data system with flow dynamics $\dot z=-z$ and jump dynamics $z^+=2z$, which is asymptotically stable if $2e^{-T}<1$, i.e., $e^T>2$.

Next we will give a nonlinear example to present a local result.

Example 2. Consider the system given in Example 1 with $\dot z$ equation changed to $\dot z=z^2+z+y$ and assume that $e^T>4$.

By setting $\varepsilon = 0$, from Example 1 we have that the flow map of the reduced system is $F_r(x_1) := \begin{cases} z^2 - z \\ 1 \end{cases}$, $x_1 \in C$. Choose $V(x_1) := \frac{z^2}{2}\exp(c\tau)$ and $W(x_1,x_2) := (y+2z)^2$, then, conditions (1) of Definitions 3 and 4 hold obviously. Moreover, we obtain the conditions (2) and (3) of Definition 3 as follows: $\langle \nabla V(x_1), f_r\rangle \le (z^3 - z^2)e^{c\tau} + \frac{z^2}{2}ce^{c\tau}$, $V(g_1) - V(x_1) = 2z^2 - \frac{z^2}{2}e^{cT}$. Choose $c := 1$ and let $\rho := \frac{1}{4}$. For $x_1 \in C \cap \mathbb{B}^{\rho}_{\mathbb{W}_1}$, we have $\langle \nabla V(x_1), f_r\rangle \le (z^3 - \frac{z^2}{2})e^{c\tau} \le -\frac{1}{4}|x_1|^2_{\mathbb{W}_1}$ and $V(g_1) - V(x_1) \le -d_1|x_1|^2_{\mathbb{W}_1}$ for some $d_1 > 0$.

The conditions (2) and (3) of Definition 4 are as follows:

$$\langle \nabla_{x_2} W(x_1,x_2), \tilde{f}_2\rangle \le -2|x_2|^2_{H(x_1)},$$
$$W(g) \le (a^2 + c_1a^2)W(x_1,x_2) + \frac{1+c_1}{c_1}(b+4-2a)^2|x_1|^2_{\mathbb{W}_1}.$$

The *coupling conditions* of Definition 5 are as follows:

(1) there exists a number $k_2 > 0$ such that for all $x_1 \in C$, $f_1 \in F_1(x_1,x_2)$, $\exists f_r \in F_r(x_1)$ such that $\langle \nabla V(x_1), f_1 - f_r\rangle \le k_2|x_1|_{\mathbb{W}_1}|x_2|_{H(x_1)}$;
(2) there exist constants $k_3 > 0, k_4 > 0$ such that for all $x \in (C \cap \mathbb{B}^{\rho}_{\mathbb{W}_1}) \times X_1$ and $f_1 \in F_1(x_1,x_2)$, we have
$$\langle \nabla_{x_1} W(x_1,x_2), f_1\rangle \le k_3|x_2|^2_{H(x_1)} + k_4|x_2|_{H(x_1)}|x_1|_{\mathbb{W}_1};$$
(3) $\langle \nabla_{x_2} W(x_1,x_2), f_2 - \tilde{f}_2\rangle = 0$.

Using Theorem 1, we can conclude that $\mathbb{W}$ is UAS for sufficiently small ε and the basin of attraction can be computed via Theorem 1. ■

To show further the effectiveness of the proposed method, we next consider the switched learning inclusions with unstable modes; similar models have been studied in Poveda and Teel (2017) and Wang et al. (2020).

Example 3. Consider the switched learning inclusions with unstable modes as follows:

$$\begin{cases} \left.\begin{aligned} \dot{x}_1 &\in \begin{bmatrix} f(x) \\ \{0\} \\ [0,\delta] \end{bmatrix} \\ \varepsilon\dot{x}_2 &\in \begin{bmatrix} [0,\bar{\rho}] \\ \hat{F}_0(x,\varepsilon) \end{bmatrix} \end{aligned}\right\} x \in C_1 \times X_1 \\ \left.\begin{aligned} \dot{x}_1 &\in \begin{bmatrix} f(x) \\ \{0\} \\ [0,\delta] \end{bmatrix} \\ \varepsilon\dot{x}_2 &\in \begin{bmatrix} [0,\bar{\rho}]-1 \\ \overline{\mathrm{co}} \bigcup_{p\in\mathcal{Q}} \hat{F}_p(x,\varepsilon) \end{bmatrix} \end{aligned}\right\} x \in C_2 \times X_1 \\ \left.\begin{aligned} x_1^+ &= (\tilde{z}, \mathcal{P} \setminus \{\widetilde{\sigma}\}, \tau_1 - 1) \\ x_2^+ &= x_2 \end{aligned}\right\} x \in D \times X_2, \end{cases} \tag{16}$$

where $x_1 := (\tilde{z}, \widetilde{\sigma}, \tau_1)$, $x_2 := (\tau_2, \tilde{y})$, $\tilde{z} := (\chi, \zeta, \widetilde{w}, \upsilon)$, $\tilde{y} := (\vartheta_1, \vartheta_2, \xi_2)$, $x := (x_1,x_2)$, $\chi := \bar{x} - \bar{x}^*$,

$$f(x) := \begin{bmatrix} P_\Omega(\chi+\bar{x}^* - \widetilde{G}(\chi+\bar{x}^*, r\vartheta_1 + R\vartheta_2)) - (\chi+\bar{x}^*) + \widetilde{w} \\ -\bar{\kappa}_c\zeta + \widetilde{w} \\ -\tilde{\kappa}_c\zeta - \bar{\beta}SGN(\zeta) + \upsilon + \tilde{g} \\ \mathbb{B}^N_{\bar{d}_2} \end{bmatrix},$$

$$\hat{F}_0(x,\varepsilon) := \begin{bmatrix} -\vartheta_1 + r^T\varphi(\chi+\bar{x}^*) \\ -\vartheta_2 - \xi_2 - R^T\mathbf{L}_0R\vartheta_2 + R^T\varphi(\chi+\bar{x}^*) \\ R^T\mathbf{L}_0R\vartheta_2 \end{bmatrix},$$

$$\hat{F}_p(x,\varepsilon) := \begin{bmatrix} -\vartheta_1 + r^T\varphi(\chi+\bar{x}^*) \\ -\vartheta_2 - \xi_2 - R^T\mathbf{L}_pR\vartheta_2 + R^T\varphi(\chi+\bar{x}^*) \\ R^T\mathbf{L}_pR\vartheta_2 \end{bmatrix},$$

and $C_1 := (\mathbb{R}^{3Nn} \times \mathbb{B}^N_{\bar{d}_1}) \times \{0\} \times [0,N_0]$, $C_2 := (\mathbb{R}^{3Nn} \times \mathbb{B}^N_{\bar{d}_1}) \times \mathcal{Q} \times [0,N_0]$, $C := C_1 \cup C_2$, $D := (\mathbb{R}^{3Nn} \times \mathbb{B}^N_{\bar{d}_1}) \times \mathcal{P} \times [1,N_0]$, $X_1 := [0,T_0] \times \mathbb{R}^{(2N-1)m}$, $X_2 := [0,T_0] \times \mathbb{R}^{(2N-1)m}$, $\mathcal{P} := \{0\} \cup \mathcal{Q}$.

Note that $\mathbf{L}_0$ and $\mathbf{L}_p$ are symmetric matrices. $\mathbf{0}$ is the simple eigenvalue of $\mathbf{L}_0$. $\mathbf{L}_q\mathbf{1}_N = \mathbf{1}_N^T\mathbf{L}_q = \mathbf{0}, q \in \mathcal{P}$. r, R are the corresponding eigenvectors of the eigenvalues of $\mathbf{L}_0$ and satisfy $r := \frac{\mathbf{1}_N}{\sqrt{N}}$, $r^TR = \mathbf{0}$, $R^TR = I_{N-1}$ and $RR^T = I_N - \frac{\mathbf{1}_N\mathbf{1}_N^T}{N}$. $\nabla_{\bar{x}_i}J_i(\bar{x}_i,\bar{x}_{-i}) = \widetilde{G}_i(\bar{x}_i, \sigma(\bar{x}))$, where $\bar{x}_i \in \Omega_i$, $\Omega := \prod_{i=1}^N \Omega_i$, $\bar{x} := (\bar{x}_1,\ldots,\bar{x}_N)$, $\sigma(\bar{x}) := \frac{1}{N}\sum_{i=1}^N \varphi_i(\bar{x}_i)$, φ_i, J_i are continuously differentiable functions and $J_i(\bar{x}_i,\bar{x}^*_{-i}) - J_i(\bar{x}^*_i,\bar{x}^*_{-i}) \ge \nabla_{\bar{x}_i}J_i(\bar{x}^*_i,\bar{x}^*_{-i})(\bar{x}_i - \bar{x}^*_i)$. $P_\Omega(\cdot)$ is a projection operator given by $P_\Omega(x) := arg\min_{y\in\Omega}|x-y|$ and it has the following property:

$$|P_\Omega(x) - P_\Omega(y)| \le |x-y|,\ \forall x,y \in \mathbb{R}^n. \tag{17}$$

Assumption 4. For any $\bar{x}_i, \bar{x}^*_i \in \mathbb{R}^{n_1}, s_i, \bar{s}_i \in \mathbb{R}^{m_1}$, there exist constants $\ell_1, \ell_2 > 0$ such that $|\widetilde{G}_i(\bar{x}_i,s_i) - \widetilde{G}_i(\bar{x}_i,\bar{s}_i)| \le \ell_1|s_i - \bar{s}_i|$ and $|\nabla_{\bar{x}_i}J_i(\bar{x}_i) - \nabla_{\bar{x}_i}J_i(\bar{x}^*_i)| \le \ell_2|\bar{x}_i - \bar{x}^*_i|$. □

Set $\varepsilon = 0$, then from (16) we can obtain $H(x_1) \in \begin{bmatrix} [0,T_0] \\ h_1(x_1) \end{bmatrix}$, where $h_1(x_1) := \begin{bmatrix} r^T\varphi(\chi+\bar{x}^*) \\ \{0\} \\ R^T\varphi(\chi+\bar{x}^*) \end{bmatrix}$ and we assume that $h_1(x_1)$ is globally Lipschitz in x_1. Given $\mathbb{W}_1 := (\{\mathbf{0}\} \times \mathbb{B}^N_{\bar{d}_1}) \times \mathcal{P} \times [0,N_0]$ and $\mathbb{W}_2 := [0,T_0] \times \{h_1(x_1) : \chi = 0\}$. Choose

$$V(x_1) := \frac{1}{2}\chi^T\chi + \sum_{i=1}^N [J_i(\chi_i + \bar{x}^*_i, \bar{x}^*_{-i}) - J_i(\bar{x}^*_i,\bar{x}^*_{-i}) - \nabla_{\bar{x}_i}J_i(\bar{x}^*_i,\bar{x}^*_{-i})\chi_i]$$
$$+ c_w\Big(\frac{1}{2}\zeta^T\zeta + \frac{1}{2}\widetilde{w}^TQ\widetilde{w} - \bar{\delta}\zeta^T\widetilde{w} + \sum_{i=1}^N m_i|\zeta_i| - \upsilon^TQ\zeta\Big),$$
$$W(x_1,x_2) := \widetilde{W}_{\widetilde{\sigma}}(\tilde{y} - h_1(x_1))\exp(\ln(\tilde{\mu})\tau_1 + (c_6 + c_7)\tau_2),$$

where $c_w, \bar{\delta} > 0$, $k^c := diag\{k^c_i\}$, $\kappa := diag\{\kappa_i\}$, $\beta := diag\{\bar{\beta}_i\}$, $m_i := \frac{(\bar{\delta}(k^c_i+\kappa_i)+1)\bar{\beta}_i}{k^c_i\kappa_i}$, $\bar{\kappa}_c := (k^c + \kappa) \otimes I_n$, $\tilde{\kappa}_c := (k^c\kappa) \otimes I_n$, $\bar{\beta} := \beta \otimes I_n$, $Q := \Pi_1\Pi_2$, $\Pi_1 := (\bar{\delta}(k^c + \kappa) + I_N) \otimes I_n$, $\Pi_2 := \tilde{\kappa}_c^{-1}$. $\tilde{\mu} > 1$ and satisfies $\widetilde{W}_p(y) \le \tilde{\mu}\widetilde{W}_q(y)$, $p,q \in \mathcal{P}$, $y := \tilde{y} - h_1(x_1)$, $\widetilde{W}_{\widetilde{\sigma}}(y) := y^TP_0y, \widetilde{\sigma} = 0$ and $\widetilde{W}_{\widetilde{\sigma}}(y) := y^TP_1y, \widetilde{\sigma} \in \mathcal{Q}$, $P_0, P_1 > 0$.

Remark 1. From (Wang et al., 2020, Lemmas 1 and 2) we know the function $\widetilde{W}_{\widetilde{\sigma}}(\tilde{y} - h_1(x_1))$ is exponentially increasing (rate of growth $c_7 > 0$) when $\widetilde{\sigma} \in \mathcal{Q}$, while it is exponentially decaying (rate of decay $c_6 > 0$) when $\widetilde{\sigma} = 0$. □

Next we will verify the conditions of Theorem 4.

$$\partial V(x_1) = \begin{bmatrix} \begin{bmatrix} \chi + F_J(\chi,\bar{x}^*) - \widetilde{F}(\bar{x}^*) \\ c_w(\zeta - \bar{\delta}\widetilde{w} + M^TSGN(\zeta) - Q^T\upsilon) \\ c_w(-\bar{\delta}\zeta + Q^T\widetilde{w}) \\ -c_w\zeta^TQ^T \end{bmatrix} \\ 0 \\ 0 \end{bmatrix},$$

where

$$F_J(\chi,\bar{x}^*) := (\nabla_{\bar{x}_1} J_1(\chi_1+\bar{x}_1^*,\bar{x}_{-1}^*),\ldots,\nabla_{\bar{x}_N} J_N(\chi_N+\bar{x}_N^*,\bar{x}_{-N}^*)),$$

$$\widetilde{F}(\bar{x}^*) := (\nabla_{\bar{x}_1} J_1(\bar{x}_1^*,\bar{x}_{-1}^*),\ldots,\nabla_{\bar{x}_N} J_N(\bar{x}_N^*,\bar{x}_{-N}^*)).$$

i) If $\widetilde{\sigma} = 0$,

conditions (1) of Definitions 3 and 4 hold obviously. The conditions for V are given as follows:

$$\begin{aligned}\min_{\tilde{\upsilon}\in\partial V(x_1)} \langle \tilde{\upsilon}, f_r\rangle &\le -\frac{\bar{c}}{4}\chi^T\chi - \frac{c_w}{2}\zeta^T\bar{\eta}\zeta - \frac{c_w\bar{\delta}}{8}\widetilde{w}^T\widetilde{w}\\ &\le -\lambda_1|x_1|^2_{\mathbb{W}_1},\ \exists\lambda_1>0,\\ V(g_1)-V(x_1) &\le 0.\end{aligned}$$

The conditions for W are given as follows: From Remark 1, it follows that

$$\begin{aligned}\langle\nabla_{x_2}W(x_1,x_2),\tilde{f}_2\rangle =& \Big(\frac{\partial\widetilde{W}_{\widetilde{\sigma}}(\tilde{y}-h_1(x_1))}{\partial\tilde{y}}\Big)^T\dot{\tilde{y}}e^{(\ln(\tilde{\mu})\tau_1+(c_6+c_7)\tau_2)}\\ &+(c_6+c_7)\dot{\tau}_2W(x_1,x_2)\\ \le& -(c_6-\bar{\rho}(c_6+c_7))W(x_1,x_2)\\ \le& -\lambda_2|x_2|^2_{H(x_1)},\ \exists\lambda_2>0,\end{aligned}$$

here we have used $c_6-\bar{\rho}(c_6+c_7)>0$. Moreover, we have

$$\begin{aligned}W(g) \le& \tilde{\mu}\widetilde{W}_{\widetilde{\sigma}}(\tilde{y}-h_1(x_1))\exp(\ln(\tilde{\mu})(\tau_1-1)+(c_6+c_7)\tau_2)\\ =& W(x_1,x_2).\end{aligned}$$

The *coupling conditions* are given as follows:

(1) $\max_{p\in\partial V(x_1)}\langle p, f_1-f_r\rangle \le \ell_1(\ell_2+1)|x_1|_{\mathbb{W}_1}|x_2|_{H(x_1)}$ can be easily obtained by using (17) and Assumption 4;

(2)

$$\begin{aligned}\langle\nabla_{x_1}W(x_1,x_2),f_1\rangle \le& c_0|\tilde{y}-h_1(x_1)|\Big|\frac{d(h_1(x_1))}{dx_1}\Big|(|P_\Omega(\chi)\\ &-(\chi+\bar{x}^*)|+|\widetilde{w}|)+\delta\ln(\tilde{\mu})W(x_1,x_2)\\ =& c_0|\tilde{y}-h_1(x_1)|\Big|\frac{d(h_1(x_1))}{dx_1}\Big|(|P_\Omega(\chi)-(\chi+\bar{x}^*)\\ &-P_\Omega(\chi^*)+(\chi^*+\bar{x}^*)|+|\widetilde{w}|)+\delta\ln(\tilde{\mu})W(x_1,x_2)\\ \le& c_2|x_1|_{\mathbb{W}_1}|x_2|_{H(x_1)}+c_1|x_2|^2_{H(x_1)},\end{aligned}$$

where $P_\Omega(\chi) := P_\Omega(\chi+\bar{x}^*-\widetilde{G}(\chi+\bar{x}^*, r\vartheta_1+R\vartheta_2))$,
$P_\Omega(\chi^*) := P_\Omega(\chi^*+\bar{x}^*-\widetilde{G}(\chi^*+\bar{x}^*, \frac{1_N}{N}\sum_{i=1}^N \varphi_i(\bar{x}_i^*)))$, and we also used that $P_\Omega(\chi^*)-(\chi^*+\bar{x}^*)=0$, $\chi^*=0$ in the equilibrium set. Moreover, inequality (17) has been used here;

(3) $\langle\nabla_{x_2}W(x_1,x_2), f_2-\tilde{f}_2\rangle = 0$ can be easily verified;

ii) if $\widetilde{\sigma}\in\mathcal{Q}$, similarly, we can verify that the conditions in Theorem 4 hold.

Then using Theorem 4, the set $\mathbb{W}$ is UGAS for sufficiently small ε. ■

5. CONCLUSIONS

We have studied stability analysis of singularly perturbed systems in the hybrid systems framework based on continuously differentiable and non-smooth Lyapunov functions. Using these Lyapunov functions, UAS and UGAS results have been established for such systems. In addition, an estimate for the basin of attraction was given for the local case. Compared with the existing stability results for such systems, our conclusions are stronger under stronger assumptions. The obtained results were illustrated by three examples.

REFERENCES

Bacciotti, A. and Ceragioli, F. (1999). Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 4, 361–376.

Chow, J.H. and Kokotovic, P.V. (1981). A two-stage Lyapunov-Bellman feedback design of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 26(3), 656–663.

Cochran, J. and Krstić, M. (2009). Nonholonomic source seeking with tuning of angular velocity. *IEEE Transactions on Automatic Control*, 54(4), 717–731.

Frihauf, P., Krstić, M., and Başar, T. (2012). Nash equilibrium seeking in noncooperative games. *IEEE Transactions on Automatic Control*, 57(5), 1192–1207.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems*. Princeton University Press, NJ.

Grujić, L.T. (1981). Uniform asymptotic stability of nonlinear singularly perturbed and large scale systems. *International Journal of Control*, 33(3), 481–504.

Kokotovic, P.V., O'Malley, R., and Sannuti, P. (1976). Singular perturbations and order reduction in control theory-an overview. *Automatica*, 12(2), 123–132.

Poveda, J.I. and Teel, A.R. (2017). A framework for a class of hybrid extremum seeking controllers with dynamic inclusions. *Automatica*, 76, 113–126.

Saberi, A. and Khalil, H. (1984). Quadratic-type Lyapunov functions for singularly perturbed systems. *IEEE Transactions on Automatic Control*, 29(6), 542–550.

Saksena, V., O'Reilly, J., and Kokotovic, P.V. (1984). Singular perturbations and time-scale methods in control theory: Survey 1976-1983. *Automatica*, 22(3), 273–293.

Sanfelice, R.G., Goebel, R., and Teel, A.R. (2007). Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12), 2282–2297.

Sanfelice, R.G. and Teel, A.R. (2011). On singular perturbations due to fast actuators in hybrid control systems. *Automatica*, 47, 692–701.

Sanfelice, R.G., Teel, A.R., Goebel, R., and Prieur, C. (2006). On the robustness to measurement noise and unmodeled dynamics if stability in hybrid systems. *In Proceedings of the American Control Conference*, 17, 4061–4066.

Teel, A.R. (2000). Lyapunov methods in nonsmooth optimization, part I: Quasi-Newton algorithms for Lipschitz, regular functions. *In Proceedings of 39th IEEE Conference on Decision and Control*, 1, 112–117.

Wang, W., Teel, A.R., and Nešić, D. (2012). Analysis for a class of singularly perturbed hybrid systems via averaging. *Automatica*, 48, 1057–1068.

Wang, X.F., Teel, A.R., Liu, K.Z., and Sun, X.M. (2020). Stability analysis of distributed convex optimization under persistent attacks: A hybrid systems approach. *Automatica*, 111, 1–7.