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# Finite-time consensus tracking control for multi-agent systems with nonlinear dynamics under Euler digraph and switching topology

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**Abstract:** In this paper, the finite-time consensus tracking control for multi-agent systems under Euler digraph and switching topology are discussed. The new nonlinear distributed control protocol is proposed under which the systems can reach finite-time consensus tracking. Furthermore, the leader needs to connect with only one follower and which agent is connected will have no effect on finite-time consensus under Euler digraph. It is found that adding edges or increasing feedback gains will be an effective way to reduce the settling time. Two sufficient conditions are proposed to achieve finite-time consensus tracking for multi-agent systems under Euler digraph and switching topology. Finally, numerical simulations are presented to verify the effectiveness of obtained theoretical results.

*Keywords:* Multi-agent systems; Finite-time consensus; Euler digraph; Switching topology; Distributed control.

## 1. INTRODUCTION

Distributed consensus control of multi-agent systems has become a hot topic over the past decades. The general consensus control model is usually described by the following continuous-time ordinary differential equation:

$$\dot{x}_i(t) = f(x_i(t)) + g_i(t) \quad i = 1, 2, \ldots, N,$$

where $x_i(t) \in R^n$ denotes the state of agent $i$; the solution of $\dot{x}_i(t) = f(x_i(t))$ represents the intrinsic dynamics of agent $i$; $g_i(t)$ describe the effects from neighbors of agent $i$, the effects may be diffusive coupling from neighbors or the function from one/multiple leaders, i.e.,

$$g_i(t) = c \sum a_{ij} \Phi_1(x_i, x_j) + d_i \Phi_2(x_i, x_0) \quad i = 1, 2, \ldots, N,$$

and the leader $x_0$ satisfies $\dot{x}_0(t) = h(x_0(t))$ (Here we only present the ordinary differential equation of a single leader). From a mathematical point of view, the derivative of $x_i(t)$ show the instantaneous rate of change of the agent $i$'s state, it will be affected by its intrinsic dynamics and external factors, which is described by $f$ and $g_i(t)$ respectively. This shows the rationality of model to some extent. The consensus control issue can be classified into leaderless consensus control and leader-following consensus control based on whether the final consensus values are predetermined. If $f \equiv 0$ and $\Phi_2 \equiv 0$, the general consensus model becomes the leaderless consensus model, they will reach an agreement but group decision value is not known, which can be expressed by $\lim_{t\to\infty} \|x_i - x_j\| = 0$, where $\|.\|$ denotes some norm. If $f \neq 0$ and $\Phi_2 \neq 0$, it becomes the case of leader-following consensus control, and all the followers will converge to the state of leader, i.e., $lim_{t\to\infty}\|x_i - x_0\| = 0$ for $i = 1, 2, \ldots, N$, such issue is also referred to as consensus tracking control in some references such as in (Yoo. S. J (2013) - Zhang. H (2012)). If $\Phi_1(x_i, x_j) = x_j - x_i$, and $\Phi_2(x_i, x_0) = x_i - x_0$, it becomes the linear coupling among followers and between follower and leader, which have already been studied in (Y. G. Hong (2006)). If $h = f$, it denotes that the system dynamics for all the agents and the leader to be identical, if not, representing that the intrinsic dynamics of leader is different from followers', and both case have practical backgrounds such as cooperative control of unmanned air vehicles, formation control of mobile robots, etc. When $f(x_i(t)) = Ax_i(t)$, it becomes the linear dynamics, which have been studied in (Y. M. Wu (2018)).

Many sufficient criteria have been obtained for asymptotic/exponential and leaderless/leader-following consensus problems. Asymptotic consensus issue of linear/nonlinear intrinsic dynamics and linear/nonlinear coupling have also been studied in (W. Yu (2010); X. F. Zhang (2015); Olfati-Saber (2004)). In (Olfati-Saber (2004)), consensus problems for networks of dynamic agents with fixed and switching topologies have been discussed, and it has established a direct connection between the algebraic connectivity of the network and the performance (or negotiation speed) of a linear consensus protocol. Meanwhile, it indicates that increasing algebraic connectivity will improve the convergence rate. From then on, many researchers pour

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attention into the study of convergence speed, which is greatly crucial for the performance of multi-agent systems. There are different ways to increase the convergence rate, such as enlarging the coupling strength, optimizing communication weights, and designing optimal network topology (E. Ghadimi (2015) - L. Xiao (2004)). Especially, when the coupling strength and communication are limited, the problem of how to distribute the weights and design a appropriate network such that a certain performance index is maximized (or minimized) is an optimization problem that will come down to the category of network design problems. Olfati-Saber and Murray have studied ultrafast consensus problem in small-world networks (Olfati-Saber (2005)). They found that consensus problem can be solved incredibly fast on certain small-world networks, which gives rise to a network design algorithm issue for ultrafast information networks, while they only illustrate this phenomenon by extensive simulation results but the rigorous theoretical explanation of what exactly affects the rate of convergence is not given. For the complexity of this problem, many researchers can only study this problem from some special cases. In (Jianxi Li (2008)), Li gives the orderings of trees, bicyclic graphs and connected graphs by algebraic connectivity. Later, many scholars focus on ordering the algebraic connectivity of some special topologies such as trees and tricycle graphs (Nair Abreu (2014) - Xueyi Huang (2015)). More generally, in (Ali Sydney (2013)), Ali Sydney provides an useful method to optimize algebraic connectivity by edge rewiring, and answer the question: “Which edge can we rewire to have the largest increase in algebraic connectivity?” After providing the general algorithm for increasing algebraic connectivity, researchers will not concentrate on some special topology such as trees, bicycle graphs and tricycle graphs. Simultaneously, we can design the topology which will have relatively high algebraic connectivity under the limit of edges and vertices by general algorithm.

Despite the fact that there exist some approaches to maximize the second smallest eigenvalue of interaction graph Laplacian, and thus we can get better convergence rate of the linear protocol proposed in (Olfati-Saber (2004)), but the state consensus can never occur in a finite time. In most cases, we hope that the consensus can occur in finite time. i.e., there exists a constant $T$ called the *settling time*, which depends on the initial values, such that

$$\lim_{t\to T} \|x_i(t) - x_j(t)\| = 0 \quad \forall i, j = 1, 2, \dots N;$$
$$x_i(t) = x_j(t), \quad \forall t \geq T,$$

what we should do is to design algorithm to make our system reach consensus in finite time. Furthermore, finite-time consensus can lead to better system performances in the disturbance rejection and robustness against uncertainty (S.P. Bhat (1998, 2000)). Thus, investigating the finite-time consensus tracking control under the new protocol will be of great importance both in theory and applications. In some practical situations, it will be appealing that the consensus can be reached in a finite time. Therefore, finite-time consensus is more attractive and there exist a number of results about finite-time convergence. In (Xiaoli Wang et al. (2010)), the finite-time $\chi$-consensus problem for a multi-agent system with first-order individual dynamics and switching interaction topologies has been discussed, it includes some special cases, such as average-consensus, max-consensus, and min-consensus problem. In (Xiaoli Wang (2008)), the finite-time consensus problem for a multi-agent systems with second-order individual dynamics has been investigated, and in (Feng Xiao (2009)), Feng et al develop a new finite-time formation control framework for multi-agent systems with a large population of members. Moreover, a class of nonlinear consensus protocols have been proposed, which ensures that the related states of all agents will reach an agreement in a finite time under suitable conditions.

From the perspective of topological structure, the undirected connected graph is the simplest and commonest case. The asymptotic consensus issue with some special digraph such as strongly connected and containing directed spanning tree have been studied in (Olfati-Saber (2004); W. Ren (2005)). In this paper, we introduce the Euler digraph. As we know, a Euler graph has an even number of edges connected to each of its vertices. This type of graph was introduced in 1736 to solve the Konigsberger bridge problem (J.A. Bondy (2010)). From then on, many scholars focus on studying the property of Euler graph, then, apply it to many practical problems such as Chinese postman problem, Euler travel problem and Eulerian graph in the application of the distribution line. Later, researchers construct Eulerian network model, which is widely used in the National Airspace System to solve the air traffic flow in congested areas. All of these will come down to optimization problems.

Due to the above superiorities and applications, we intend to consider the finite-time consensus tracking control for multi-agent systems with nonlinear dynamics under Euler digraph and switching topology. To our best knowledge, there are few papers considering this issue, which will be the subject of this paper. A novel distributed protocol is designed for the finite-time consensus tracking control, and the validity is also rigorously proved by the Lyapunov functions.

The main contributions of this paper can be summarized as follows:

1. *We proposed a novel distributed protocols for the finite-time consensus tracking control for multi-agent systems with nonlinear dynamics under Euler digraph and switching topology. Under our protocols, we can make the whole system reach to the dynamics of the leader in finite time. To reach consensus tracking in finite time, only one follower needs to be connected with the leader, and it doesn't matter which agent is connected.*
2. *Different from many papers which investigate the finite-time stability or consensus, we use an uncommon method to prove finite-time consensus. Meanwhile, adding edges or increasing control gains* $d_i \geq 0 (i = 1, 2, \dots, N)$ *will be a superior way to reduce the settling time, which is consistent with our common belief, and the reason why this phenomenon exists is rigorously proved.*

This paper is organized as follows. In Section II, some graph preliminaries and necessary lemmas are introduced. In Section III, we present a new finite-time control protocol and the main results for finite-time consensus tracking on multi-agent systems are given. Furthermore, numerical simulations are given in Section IV to show the correctness of theoretical results. Finally, the conclusion and discussions about future work are presented in Section V.

Notations: Let $sig(\mathbf{x})^\alpha = (sign(x_1)|x_1|^\alpha, \ldots, sign(x_n)|x_n|^\alpha)$, and $\alpha \in R$, $R$ is the set of real numbers, and $R^+$ is positive real numbers. Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in R^n$, $R^n$ denotes the real vector space of $n$-dimensions. $\mathbf{0}$ denotes a zero vector with appropriate dimension, $\mathbf{1}_n = (1, 1, \ldots, 1)^T \in R^n$, $\underline{N} = \{1, 2, \cdots, N\}$ is an index set. $\|.\|_2$ represents the 2-norm on $R^n$.

## 2. PROBLEM FORMULATION

### 2.1 Preliminaries

A *weighted digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ consists of a *vertex set* $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ and an *edge set* $\mathcal{E} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$, where an *edge* is an ordered pair of distinct vertices of $\mathcal{V}$, and the nonsymmetric weighted adjacency matrix $\mathscr{A} = [a_{ij}]$, with $a_{ij} > 0$ if and only if $e_{ij} \in \mathscr{E}$ and $a_{ij} = 0$ if not. The in-degree and out-degree of node $v_i$ are, respectively, defined as follows:

$$deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji} \quad deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$$

The neighborhood set of node $v_i$ is denoted by $\mathscr{N}_i = \{v_j \in \mathscr{V} : (v_i, v_j) \in \mathscr{E}\}$. For a graph with 0-1 adjacency elements $deg_{out}(v_i) = |\mathscr{N}_i|$. The *degree matrix* of the digraph is a diagonal matrix $\triangle = [\triangle_{ij}]$, where $\triangle_{ij} = 0$ for $i \neq j$ and $\triangle_{ii} = deg_{out}(v_i)$. The *graph Laplacian* associated with the digraph is defined as

$$L(\mathscr{G}) = \triangle(\mathscr{G}) - \mathscr{A}(\mathscr{G}),$$

if

$$deg_{in}(v_i) = deg_{out}(v_i) \quad \forall i,$$

then, digraph $\mathscr{G}$ is *balanced*. Any undirected graph is balanced.

A *directed walk* in a digraph $\mathscr{G}$ is an alternating sequence of vertices and arcs $W := (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ such that $v_{i-1}$ and $v_i$ are the tail and head of $a_i$, respectively, $1 \leq i \leq k$. If $x$ and $y$ are the initial and terminal vertices of $W$, we refer to $W$ as a *directed (x,y)-walk*. Directed trails, closed trails in digraphs are defined analogously. A *directed Euler trail* is a directed trail which traverses each arc of the digraph exactly once, and a *directed Euler closed trail* is a directed closed trail with this same property. A digraph is *Euler digraph* if it admits a directed Euler closed trail. In a digraph $\mathscr{G}$, two vertices $x$ and $y$ are *strongly connected* if there is a directed $(x, y)$-walk and also a directed $(y, x)$-walk (that is, if each of $x$ and $y$ is reachable from the other).

### 2.2 Finite-time tracking control problem

Consider the multi-agent systems composed of one leader and $N$ followers. The dynamics of each follower is denoted by

$$\dot{x}_i(t) = f(x_i(t), t) + u_i(t), \quad i \in \underline{N}, \tag{1}$$

where $x_i(t) \in R$ is the state of agent $i$, $f : R \times R \longrightarrow R$ is the smooth nonlinear vector field, $u_i(t) \in R$ is the control input. The intrinsic dynamics of the leader is described by

$$\dot{x}_0(t) = f(x_0(t), t) \tag{2}$$

where $x_0(t) \in R$ is the state of the leader.

*Remark 1.* We take the system dynamics for all the followers and the leader to be identical(they all meet the same nonlinear dynamics $\dot{s}(t) = f(s(t), t)$), because this case has practical background such as group of birds, school of fishes etc.

In system (1), we consider the problem of designing $u_i(t)(i = 1, 2, \ldots, N)$ by using local information to render all $N$ agents to follow the leader. We make this precise by the following definition.

*Definition 2.* The finite-time tracking problem is solved if there exists a constant $T > 0$ such that

$$\lim_{t \to T} |x_i(t) - x_0(t)| = 0 \quad \forall i = 1, 2, \ldots N. \tag{3}$$

The goal is to find some appropriate controllers $u_i(t)$ for $N$ followers such that the solutions of the controlled system (1) converge to the solution of (2) in finite time.

We have defined the error as

$$e_i(t) = x_i(t) - x_0(t), \quad i = 1, 2, \ldots N. \tag{4}$$

Subtracting (2) from (1), the dynamics of error is described by:

$$\dot{e_i}(t) = f(x_i(t), t) - f(x_0(t), t) + u_i, \quad i = 1, 2, \ldots, N. \tag{5}$$

From the description above, the objective converts into finding some appropriate controllers $u_i(t)$ such that the system (5) finite-time stable. The following assumptions and lemmas play a key role in this paper.

**Assumption 1 (A1).** The topological structure $\mathscr{G}$ of followers is Euler digraph, and the leader has directed paths to at least one follower.

**Assumption 2 (A2).** Suppose that there exist $\mu_i$, such that $|f(x_i(t), t) - f(x_0, t)| \leq \mu_i |e_i|$, and let $\mu = max\{\mu_1, \ldots, \mu_N\}$, $(i = 1, 2, \ldots, N)$.

*Lemma 3.* (J.A. Bondy (2010)) If $\mathscr{G}$ is a Euler digraph, then $\mathscr{G}$ is strongly connected and for any node $v_i \in \mathcal{V}$, $deg_{in}(v_i) = deg_{out}(v_i)$, $i = 1, 2, \ldots, N$.

*Lemma 4.* **(Weyl's inequality)** Let $E, F \in M_N$ be Hermitian where $M_N$ is the linear space consists of all the $N \times N$ matrices, and let the respective eigenvalues of $E, F$ and $E + F$ be $\{\lambda_i(E)\}_{i=1}^N$, $\{\lambda_i(F)\}_{i=1}^N$, and $\{\lambda_i(E + F)\}_{i=1}^N$. Suppose that $\lambda_1(E) \leq \lambda_2(E) \leq \ldots \leq \lambda_N(E)$, $\lambda_1(F) \leq \lambda_2(F) \leq \ldots \leq \lambda_N(F)$, and $\lambda_1(E + F) \leq \lambda_2(E + F) \leq \ldots \leq \lambda_N(E + F)$. Then

$$\lambda_i(E + F) \leq \lambda_{i+j}(E) + \lambda_{N-j}(F), \quad j = 0, 1, \ldots, N - i,$$

for each $i = 1, 2, \ldots, N$. Also,

$$\lambda_{i-j+1}(E) + \lambda_j(F) \leq \lambda_i(E + F), \quad j = 1, \ldots, i,$$

for each $i = 1, 2, \ldots, N$.

*Lemma 5.* (S.P. Bhat (2000)) Suppose the Lyapunov function $V(x)$ satisfies

$$\dot{V}(x) \leq -cV^\alpha(x), \quad 0 < \alpha < 1, c > 0.$$

Then $V(x) \equiv 0$, if $t \geq V(0)^{1-\alpha}/c(1 - \alpha)$.

Many papers investigating the finite-time stability or consensus are based on this result (see Xiaoli Wang et al. (2010)and Shuanghe Yu (2015)). In this paper, we will use the following result to realize finite-time stability and this result has been proposed in (Y. J. Shen (2008)).

*Lemma 6.* (Y. J. Shen (2008)) Suppose the Lyapunov function $V(x)$ defined on a neighborhood $U$ of the origin, and

$$\dot{V}(x) \leq -cV^\alpha(x) + kV(x), \quad 0 < \alpha < 1, c > 0, k > 0,$$

then, the origin is finite-time stable. The set

$$\Omega = \left\{x|V^{1-\alpha}(x) < c/k\right\} \cap U$$

is contained in the domain of attraction of the origin. The settling time satisfies

$$T(x) \leq \frac{ln(1 - \frac{k}{c}V(0)^{1-\alpha})}{k(\alpha - 1)}, x \in \Omega.$$

*Lemma 7.* (Hardy. G. H (1952)) Let $y_1, \ldots, y_n \geq 0$ and $0 < p \leq 1$. Then $\sum_{i=1}^{n} y_i^p \geq (\sum_{i=1}^{n} y_i)^p$.

*Lemma 8.* (Y. G. Hong (2006)) If $L$ is the symmetric Laplacian matrix of a connected undirected graph $\mathbf{G}$, and the matrix $E = diag(e_1, e_2, \ldots, e_N)$ with $e_i \geq 0$ for $i = 1, 2, \ldots, N$, and at least one element in E is positive, then $L + E > 0$.

## 3. MAIN RESULTS

In this section, we give the results of finite-time tracking control for multi-agent systems with nonlinear dynamics under Euler digraph and switching topology. The finite-time Lyapunov stability theory is used to prove our results.

### 3.1 Multi-agent systems under Euler digraph

In the subsection, the multi-agent systems with Euler digraph is considered. We use the following control law for agent $i$:

$$u_i(t) = -\beta \sum_{j \in \mathcal{N}_i} l_{ij} sig\Big(x_j(t) - x_i(t)\Big)^{\alpha} - d_i sig\Big(x_i(t) - x_0(t)\Big)^{\alpha}, i = 1, 2, \ldots, N, \tag{6}$$

where $d_i$ is the feedback gains, $d_i > 0$ when the agent $i$ is a neighbor of the leader and $d_i = 0$ otherwise. $\mathcal{N}_i$ is the neighbor set of node $i$ (from $i$ to $j$). $L = (l_{ij})_{N \times N}$ is the coupling configuration matrix representing the topological structure of the system, where $l_{ij}$ are defined as follows:

$$l_{ij} = \begin{cases} -a_{ij} > 0, \ i \neq j \ and \ j \in \mathcal{N}_i \\ \sum\limits_{j \in \mathcal{N}_i} a_{ij}, \quad i = j \\ 0, \qquad \text{otherwise} \end{cases}, \tag{7}$$

which ensures the property that $\sum\limits_{j=1}^{N} l_{ij} = 0, i = 1, 2, \ldots N$.

*Theorem 9.* Suppose that **(A1)**, **(A2)** holds. For system (1), the local finite-time tracking problem is solved under the protocol (6).

**Proof.** Consider the Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^2 \tag{8}$$

The derivative of $V(t)$ along the trajectories of (5) is

$$\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{N} e_i \dot{e}_i \\
=& \sum_{i=1}^{N} e_i \Big( f(x_i(t), t) - f(x_0(t), t) - \beta \sum_{j=1}^{N} l_{ij} sig(e_j - e_i)^{\alpha} \\
& - d_i sig(e_i)^{\alpha} \Big) \\
\leq & \sum_{i=1}^{N} \mu |e_i|^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} \beta l_{ij} sig(e_j - e_i)^{\alpha} e_i - \sum_{i=1}^{N} d_i |e_i|^{1+\alpha} \\
=& \sum_{i=1}^{N} \mu |e_i|^2 + \beta \sum_{i=1}^{N}\sum_{j=1}^{N} (l_{ij}^{\frac{2}{1+\alpha}} |e_j - e_i|^2)^{\frac{1+\alpha}{2}} - \sum_{i=1}^{N} d_i |e_i|^{1+\alpha} \\
\leq & \sum_{i=1}^{N} \mu |e_i|^2 + \beta \Big( \sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij}^{\frac{2}{1+\alpha}} (e_j - e_i)^2 \Big)^{\frac{1+\alpha}{2}} \\
& - \beta \sum_{i=1}^{N} (1/\beta)^{\frac{2}{1+\alpha}} d_i^{\frac{2}{1+\alpha}} |e_i|^2)^{\frac{1+\alpha}{2}} \\
=& \mu V(t) - \beta \Big( (e^T \bar{L} e)^{\frac{1+\alpha}{2}} + (e^T D e)^{\frac{1+\alpha}{2}} \Big) \\
\leq & \mu V(t) - \beta \Big( e^T (\bar{L} + D) e \Big)^{\frac{1+\alpha}{2}} \\
=& \mu V(t) - \beta (e^T (\frac{\bar{L} + \bar{L}^T}{2} + D) e)^{\frac{1+\alpha}{2}} \\
=& \mu V(t) - \beta (e^T M e)^{\frac{1+\alpha}{2}},
\end{aligned}$$

where $M = \frac{\bar{L} + \bar{L}^T}{2} + D, \bar{L} = [l_{ij}^{\frac{2}{1+\alpha}}]_{N \times N}$ is the Laplacian of Euler digraph $\bar{\mathscr{G}}$, which has the same topological structure as $\mathscr{G}$, $D = diag\left\{ (\frac{d_1}{\beta})^{\frac{2}{1+\alpha}}, (\frac{d_2}{\beta})^{\frac{2}{1+\alpha}}, \ldots, (\frac{d_N}{\beta})^{\frac{2}{1+\alpha}} \right\}$, $d_i \geq 0, i = 1, 2, \ldots, N$. $e = (e_1, e_2, \ldots, e_N)^T$.

According to Lemma 3, $\bar{\mathscr{G}}$ is strongly connected and balanced, from Olfati-Saber (2004), we know that matrix $\frac{\bar{L} + \bar{L}^T}{2}$ is a valid Laplacian matrix for an undirected graph $\mathbf{G}$, and by Lemma 8, we can easily get $e^T M e \geq k_1 e^T e, k_1 > 0$. Thus

$$\begin{aligned}
\dot{V}(t) &\leq \gamma V(t) - \beta (k_1 e^T e)^{\frac{1+\alpha}{2}} \\
&= \gamma V(t) - \beta k_1^{\frac{1+\alpha}{2}} (e^T e)^{\frac{1+\alpha}{2}} \\
&= \gamma V(t) - 4\beta k_1^{\frac{1+\alpha}{2}} V(t)^{\frac{1+\alpha}{2}}
\end{aligned}$$

Let $k = \gamma, c = 4\beta k_1^{\frac{1+\alpha}{2}}$, then, we get

$$\dot{V}(t) \leq -cV^{\frac{1+\alpha}{2}} + kV, \quad 0 < \alpha < 1.$$

For $0 < \alpha < 1$, then, $0 < \frac{1+\alpha}{2} < 1$, according to Lemma 6, $V(t)$ will be zero in finite time $T = \frac{2ln(1 - \frac{k}{c}V(0)^{\frac{1-\alpha}{2}})}{k(\alpha - 1)}$, which means $e_i(t)(i = 1, 2, \ldots, N)$ will be zero in finite time. Thus, this system solves a finite-time consensus tracking problem.

*Remark 10.* To solves a finite-time tracking problem, it is not necessary that the leader has directed paths to all followers, and one follower is enough, and it doesn't matter which follower is connected by leader. This condition is superior to many papers about leader-following consensus of multi-agent systems such as in (Weijun Cao (2015); Chang Chun Hua (2016)).

In the following, we consider how to reduce the settling time. Increasing feedback gains and communication connections will be a superior way to reduce the settling time, which is consistent with our common belief. As we know, increasing communication connections means adding edges of the topological structure. So, the settling time of the system with dense graphs will be relatively short. For system with sparse graphs, its settling time will be longer. Nextly, rigorous proof will be given in the following section.

*Definition 11.* **(Average degree)** Suppose an undirected graph $\mathbf{G}=(\mathcal{V},\mathcal{E})$, $v$ and $\varepsilon$ are its number of vertices and edges, respectively. The average degree of the graph is $2v/\varepsilon$.

*Corollary 12.* For system (1), adding edges or increasing feedback gains will decrease the settling time to reach finite-time consensus tracking.

**Proof.** We get the settling time $T = \frac{(\alpha+1)ln(1-\frac{k}{c}V(0)^{\frac{1-\alpha}{2}})}{k(\alpha-1)}$ from theorem 9. If $c = 4\beta k_1^{\frac{1+\alpha}{2}}$ increases, the settling time $T$ will reduce. Increasing $k_1$ will be the only way to increase $c$. Since $k_1$ is the smallest eigenvalue of matrix $M = \frac{\bar{L}+\bar{L}^T}{2} + D$, $D = diag\left\{(\frac{d_1}{\beta})^{\frac{2}{1+\alpha}}, (\frac{d_2}{\beta})^{\frac{2}{1+\alpha}}, \dots, (\frac{d_N}{\beta})^{\frac{2}{1+\alpha}}\right\}$, $d_i \geq 0, i = 1, 2, \dots, N$. Our objective is to improve the smallest eigenvalue of matrix $M$. Firstly, we prove that adding edges will improve the smallest eigenvalue of matrix $M$, after adding edges, the digraph is also an Euler digraph. Without loss of generality, assume that there is no edge between vertices 1 and 2 in graph $\mathscr{G}_1$ , and graph $\mathscr{G}_2$ is generated by adding a new edge between vertices 1 and 2(from 1 to 2 and from 2 to 1) in graph $\mathscr{G}_1$. Let $M_1 = \frac{\bar{L}_1+\bar{L}_1^T}{2} + D$ and $M_2 = \frac{\bar{L}_2+\bar{L}_2^T}{2} + D$, where $\frac{\bar{L}_2+\bar{L}_2^T}{2} = \frac{\bar{L}_1+\bar{L}_1^T}{2} + \delta L$, and

$$\delta L = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then, $M_2 = M_1+\beta^2\delta L$. According to Lemma 8, we can get $\lambda_1(M_1) > 0$, and by Weyl's inequality, $\lambda_1(M_2) = \lambda_1(M_1 + \beta^2\delta L) \geq \lambda_1(M_1) + \beta^2\lambda_1(\delta L) = \lambda_1(M_1)$.

Nextly, we will prove that increasing feedback gains will improve the smallest eigenvalue of matrix $M$. Without loss of generality, we only increase $d_1$, $\delta D = diag(\delta d_1, 0, \dots, 0)$, $\delta M = \beta\delta D$, according to Weyl's inequality, $\lambda_1(M + \delta M) \geq \lambda_1(M) + \lambda_1(\delta M) = \lambda_1(M) + \lambda_1(\beta\delta D) = \lambda_1(M) + \beta\lambda_1(\delta D) = \lambda_1(M)$. This completes the proof.

*Remark 13.* From the corollary above, we can get that the system whose topological graph have relatively high average degree will reach consensus within shorter time. Average degree can be an important index to compare the settling time among different systems.

### *3.2 Multi-agent systems under switching topology*

In practice, it is hard to ensure that all of the existing communication links will not fail due to the existence of an obstacle between two agents, such as failure of physical devices or limited sensing range. At the same time, some new communication links may appear between any two agents. These uncertain factors make some edges be added or removed from the topology of the system. Thus, it is reasonable to assume that the interaction topology is time-varying.

Let $\mathscr{G}_c = (\mathscr{V}, \mathscr{E}, \mathscr{A}_{\delta(t)})$ be a *balanced* graph set of order $N$. The set $\mathscr{G}_c$ is a finite set because at most a graph of order $N$ is complete and has $N(N-1)$ directed edges. Define the finite set $\Gamma = \{0, 1, \dots, m\}$ representing the index set of graph $\mathscr{G}_c$. We introduce a switching signal $\delta(t) : [0, \infty) \to \Gamma$ and a switching time sequence $t_0 = 0, t_1, \dots, t_s \dots$ at which the interaction topology changes. For any $t \in [t_s, t_{s+1})$, the topology $\mathscr{G}_{\delta(t)} = \mathscr{G}_s \in \mathscr{G}_c$ is fixed and the adjacency weight between agent $i$ and $j$ is $a_{ij}^s$. Suppose that the agent $i$ of systems (1) employs the following control protocol:

$$\begin{aligned} u_i^s(t) = &-\beta\sum_{j\in\mathcal{N}_i} l_{ij}^s sig\big(x_j(t) - x_i(t)\big)^\alpha \\ &- d_i^s sig\big(x_i(t) - x_0(t)\big)^\alpha, \quad i = 1, 2, \dots, N, \end{aligned} \tag{9}$$

then, the error system (5) will be described by:

$$\begin{aligned} \dot{e}_i^s(t) =&f(x_i(t), t) - f(x_0(t), t) \\ &- \beta\sum_{j\in\mathcal{N}_i} l_{ij}^s sig\big(x_j(t) - x_i(t)\big)^\alpha \\ &- d_i^s sig\big(x_i(t) - x_0(t)\big)^\alpha, \quad i = 1, 2, \dots, N, \end{aligned} \tag{10}$$

we will analysis the finite-time stability of switching system (10). Some necessary assumptions and lemmas are given as follows.

**Assumption 3 (A3).** (Hölder continuity) Suppose that there exist $C$, such that $|f(x_i(t), t) - f(x_0, t)| \leq$C$|e_i^s|^\gamma$, $0 < \gamma \leq 1$. When $\gamma = 1$, it is the condition of (**A2**).

**Assumption 4 (A4).** For any $t \in [t_s, t_{s+1})$, the topology $\mathscr{G}_{\delta(t)}$ of followers is *balanced*, and the leader has directed paths to all the followers.

*Remark 14.* Assumption 4 do not require that the topological structure $\mathscr{G}_{\delta(t)}$ of followers is strongly connected for any $t \in [t_s, t_{s+1})$. The topological structure of followers can even be unconnected, which means $deg_{in}(v_i) = deg_{out}(v_i) = 0$ for some $i \in \underline{N}$.

*Lemma 15.* (Xiaoli Wang et al. (2010)) Considers a family of systems

$$\dot{x} = f_\delta(x), f_\delta(0) = 0, x \in R^n. \tag{11}$$

Let $\Gamma$ denotes the finite switching index set, $\delta(t) : [0, \infty) \to \Gamma$ be a piecewise constant function of time, $f_k$ be a continuous with respect to $x$ for fixed $k \in \Gamma$, and $\tau$ be the dwell time. If the switched system (10) is asymptotically stable, and for any fixed $k \in \Gamma, \dot{x} = f_k(x)$ is finite-time stable, then system (10) is finite-time stable.

*Theorem 16.* Suppose that **(A3)**, **(A4)** holds. For system (1), if the feedback gains $d_i^s > C, i = 1, 2, \dots, N$, and the sum of time-interval is sufficient large. Then the local finite-time tracking problem is solved under the protocol (9).

**Proof.** Consider the Lyapunov functional candidate:

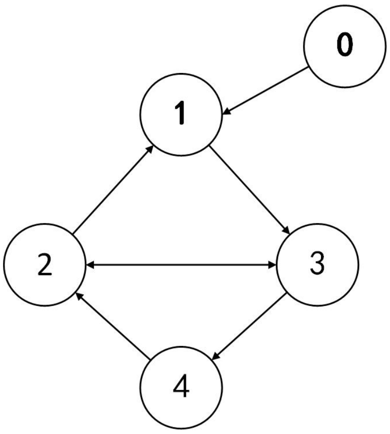


Fig. 1. Topological structure.

$$V^s(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^s(t)^2 \tag{12}$$

Similar to the proof of Theorem 9, we can obtain

$$\begin{aligned}\dot{V}^s(t) =& \sum_{i=1}^{N} e_i^s \dot{e}_i^s \\ =& \sum_{i=1}^{N} e_i^s \Big(\dot{f}(s)e_i^s - \beta \sum_{j=1}^{N} l_{ij}^s sig(e_j^s - e_i^s)^\alpha - d_i^s sig(e_i^s)^\alpha\Big) \\ \le& \gamma V^s(t) - \beta(e^T M^s e)^{\frac{1+\alpha}{2}},\end{aligned}$$

where $M = \frac{\bar{L^s} + \bar{L^s}^T}{2} + D$, we can easily get

$$\dot{V}(t) \le -c^s V^{\frac{1+\alpha}{2}} + k^s V, \quad 0 < \alpha < 1.$$

Also, we can obtain

$$\begin{aligned}\dot{V^s}(t) =& \sum_{i=1}^{N} e_i^s \dot{e_i^s} \\ =& \sum_{i=1}^{N} e_i^s \Big(\dot{f}(s)e_i^s - \beta \sum_{j=1}^{N} l_{ij}^s sig(e_j^s - e_i^s)^\alpha - d_i^s sig(e_i^s)^\alpha\Big) \\ \le& \sum_{i=1}^{N} (C - d_i^s)|e_i^s|^{\alpha+1} - \beta\Big(\sum_{i=1}^{N}\sum_{j=1}^{N} (l_{ij}^s)^{\frac{2}{1+\alpha}} (e_j^s - e_i^s)^2\Big)^{\frac{1+\alpha}{2}} \\ <& \, 0.\end{aligned}$$

Thus, when $s$ is fixed, the switched system (10) is asymptotically stable. According to Lemma 15, system (10) are finite-time stable. This completes the proof.

*Remark 17.* In the proof of theorem 16, let $\gamma = \alpha \in (0, 1)$ when proving the switched system (10) is asymptotically stable, and let $\gamma = 1$ when proving the system(10) is finite-time stable for any fixed $s \in \Gamma$.

## 4. SIMULATION

In this section, simulation examples are given to demonstrate the theoretical results. There are five agents in the system. We choose agent 0 as leader, and the remainder as followers. Their topology is shown in Fig. 1, in which the topological structure among followers is Euler digraph and we select the follower 1 to be connected by the leader. The coupling information among followers is fixed, and suppose that the weight among nodes is 1. Take the topology 1 which is an Euler digraph in Fig. 1 for example, we have the adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

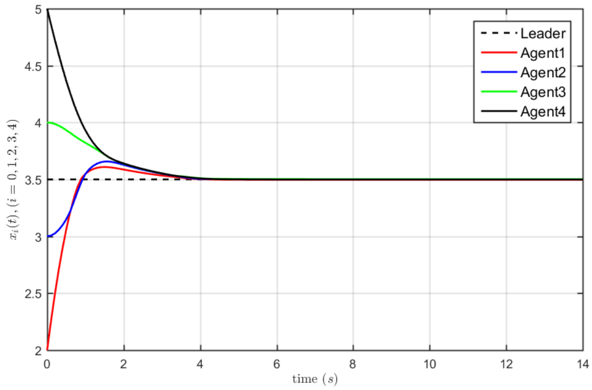


Fig. 2. Trajectories of the five agents.

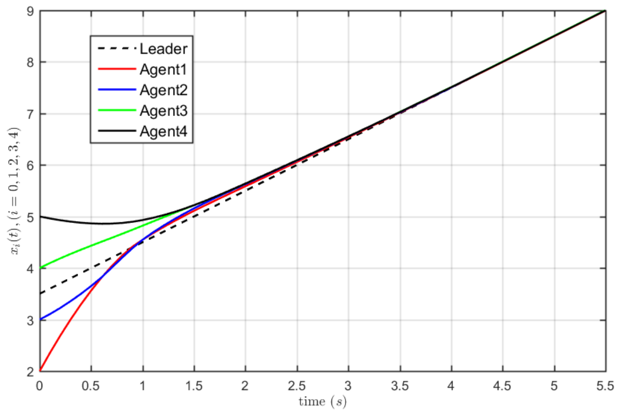


Fig. 3. Trajectories of the five agents.

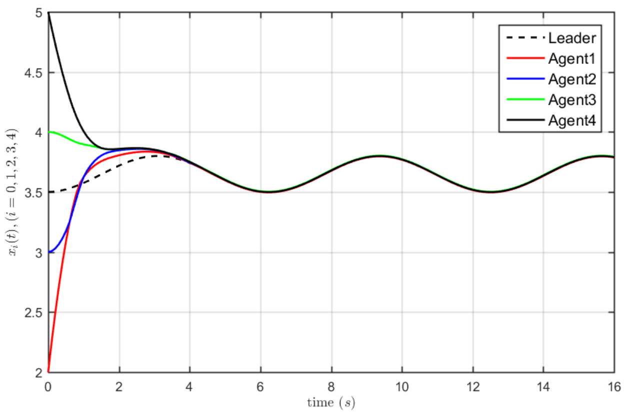


Fig. 4. Trajectories of the five agents.

and choose $f(x) = 0$ and 1, then, we can obtain their trajectories in Figs 2 and 3, respectively, which shows that the trajectories of followers can track the leader's trajectory in finite time. In the figures, the black imaginary line represents the trajectory of the leader, and the solid lines of different colors represent the trajectories of the followers. Then, let the intrinsic dynamics of the leader $x_0(t) = 3.5 + 0.15sin(t)$, we can get their trajectories in Fig. 4.

## 5. CONCLUSION

In this paper, we have discussed the finite-time consensus tracking control for multi-agent systems with nonlinear dynamics under Euler digraph and switching topology. By using finite-time stability theory, this paper propose a new nonlinear distributed control protocol under which the systems can reach finite-time consensus tracking control. In order to reduce the settling time, we have found two methods(adding edges or increasing feedback gains) to address this issue. Meanwhile, only one follower needs to be connected with the leader, and which agent is connected by leader will not affect the finite time consensus tracking problem. Finally, the simulation results are presented to demonstrate the effectiveness of the theoretical results.

There are still many challenging problems to be investigated, like, finite-time and fixed-time consensus tracking control with time delays and finite-time consensus tracking control of multi-agent with high-order agent dynamics. These issues will be carried out in the future work.

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