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# Event-triggered Varying Speed Limit Control of Stop-and-go Traffic

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**Abstract:** This paper develops event-triggered boundary control strategies for varying speed limit (VSL) located at a freeway segment. The stop-and-go traffic oscillations are suppressed by regulating the velocity of vehicles that leave the segment. The controlled velocity signal is only updated when a event triggering condition is satisfied. Compared with the continuous input signal, the event-based controller presents as a more realistic setting to implement by VSL on a digital platform which allows the adaptation time for drivers to follow the advisory speed. The traffic dynamics of density and velocity are described with linearized Aw-Rascle-Zhang (ARZ) macroscopic traffic partial differential equation (PDE) model which results in a $2 \times 2$ coupled hyperbolic system. The event-triggered boundary controllers rely on the emulation of the full state backstepping boundary feedback and two different Lyapunov-based event-triggered strategies to determine the time instants at which the control value must be sampled/updated. One of the event-triggered strategies makes use of a dynamic triggering condition under which it is possible to state the existence of a uniform minimal dwell-time (independent of initial conditions). The exponential stability under event-triggered control is achieved and validated with numerical simulations.

*Keywords:* traffic congestion control, linear hyperbolic systems, backstepping control design, event-triggered control,

## 1. INTRODUCTION

Stop-and-go traffic is a common phenomenon appearing on congested freeways, causing increased consumption of fuel and unsafe driving conditions. The traffic instabilities, also known as "jamiton", Daganzo (1995); Belletti et al. (2015) can be described with the state-of-art Aw-Rascle-Zhang (ARZ) model Aw and Rascle (2000); Zhang (2002), which consists of second-order, nonlinear hyperbolic PDEs modeling traffic density and velocity. Many recent efforts including Belletti et al. (2015); Karafyllis and Papageorgiou (2019); Bekiaris-Liberis and Delis (2019); Yu and Krstic (2019, 2018a,b); Zhang et al. (2019) focus on macroscopic traffic PDE model and develop feedback control strategies to suppress the freeway traffic congestion and further improve various traffic evaluation indexes such as total travel time, fuel consumption and the driving comfort. Among them, boundary control strategies are implemented through freeway traffic management infrastructures including ramp metering and varying speed limit. Ramp metering regulates the flow rate on freeway by temporarily reducing or closing on-ramp flows by traffic lights so that mainline traffic congestion can be prevented and suppressed Yu and Krstic (2019); Zhang et al. (2019). Varying speed limits advise drivers to pass certain location with a desired velocity Karafyllis and Papageorgiou (2019); Yu and Krstic (2018b). However, the continuous boundary control and estimation strategies developed for the traffic problem need to be implemented into digital platforms. For instance, in ramp metering control strategies, the rate inflow is controlled throughout traffic lights modulation that cannot be carried out continuously.

Typically, periodic strategies are used to modulate the frequency of light changes. In VSL strategies, on the other hand, it is not feasible to display continuous time-varying driving speed advisories. It has to be done either periodically or on events. The drawback with periodic implementations is that one may produce unnecessary updates of the sampled controllers which translates into frequent changes of drivers' speed and may cause over utilization of computational/communication resources, more importantly, unnecessary and unsafe driving conditions. Therefore, for the arising continuous-time boundary controllers, the issue of *sampling* has to be carefully studied. Indeed, if sampling is not addressed properly, the stability and estimation properties may be lost. Therefore, sampled-data and event-triggered control can offer suitable approaches to be adopted towards digital realizations. In this work, we develop event-triggered strategies which present as a more realistic approach for implementing actuation in traffic control problems.

In particular, event-triggered control has gain a lot of the attention not only due to its efficient way of using communication and computational resources by updating the control value aperiodically (only when needed) but also due to its rigorous way of implementing continuous-time controllers into digital platforms. Nevertheless, sampled-data and event-triggered control for partial differential equations (PDEs) have not achieved a sufficient level of maturity as in finite-dimensional systems (namely networked control systems modeled by ordinary differential equations (ODEs); see e.g. Hetel et al. (2017); Tabuada (2007); Heemels et al. (2012); Girard (2015); Postoyan et al. (2015); Liu and Jiang (2015)). Few approaches on sampled-data and event-triggered control of parabolic PDEs are considered in Fridman and Blighovsky (2012); Karafyllis and Krstic (2018); Selivanov and Fridman (2016); Yao and El-Farra (2013); Espitia et al. (2019). In the context of abstract formulation of distributed parameter systems, sampled-data control is investigated in Logemann et al. (2005) and Tan et al. (2009). For hyperbolic PDEs, sampled-data control is studied in Davo et al. (2018) and Karafyllis and Krstic (2017) whereas event-triggered control has been studied e.g. in Espitia et al. (2016, 2018) under an emulation approach. In Espitia et al. (2016) event-triggered boundary controllers for linear conservation laws using output feedback are studied by following Lyapunov techniques (inspired by Bastin and Coron (2016)). In Espitia et al. (2018), the approach relies on the backstepping method for coupled system of balance laws (inspired by Vazquez et al. (2011)) which leads to a full-state feedback control that is sampled according to a dynamic triggering condition.

In this paper, we propose event-triggered boundary control strategies to reduce the stop-and-go phenomenon by stabilizing on events. This work is developed based on the results of Yu and Krstic (2018b) and Espitia et al. (2018) in such a way that we build on a $2\times2$ coupled hyperbolic PDE system for which two event-triggered boundary controllers for the VSL are proposed. They rely on the emulation of the full state backstepping boundary feedback and on the use of two different Lyapunov-based event-triggered conditions which determine the time instants at which the control value must be sampled/updated. In practice, this would translate, basically, into the way that the varying speed limit sign, located at the outlet of a freeway segment, changes the advisory speed aperiodically only when needed. The speeds of the vehicles leaving the domain of interest are thus regulated such that the upstream traffic congestion is suppressed.

The paper is organized as follows. In Section 2, we introduce the modeling of the traffic problem and some considerations related to the stop-and-go phenomenon. In Section 3, we present the trigged emulation of the backstepping control. In Section 4, we present two event-boundary control strategies. Section 5 provides the main results. Section 6 provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section 7.

## 2. PRELIMINARIES AND PROBLEM DESCRIPTION

### *2.1 Aw-Rascle-Zhang model*

We consider the macroscopic Aw-Rascle-Zhang model with a relaxation term to describe traffic dynamics of a freeway segment. The inhomogeneous ARZ model is a second-order nonlinear hyperbolic PDE system of traffic density and velocity. To study the stability of uniform density and velocity on freeway, we linearize the ARZ model around equilibrium density and velocity whose relation is given by the Greenshield model.

The Aw-Rascle-Zhang model of $(\rho(t,x), v(t,x))$-system is given by

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1}$$

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = \frac{V(\rho) - v}{\tau}. \tag{2}$$

The state variable $\rho(t,x)$ is the traffic density and $v(t,x)$ is the traffic speed, and $\tau$ is the relaxation time which shows the time scale of drivers' behavior adapting to the equilibrium density-velocity relation. The variable $p(\rho)$ is defined as the traffic pressure, an increasing function of density $p(\rho) = c_0 (\rho)^\gamma$ , and $c_0, \gamma \in \mathbb{R}^+$. The equilibrium velocity-density relationship $V(\rho)$ is given in the Greenshield model $V(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right)$. where $v_f$ is the free flow velocity, $\rho_m$ is the maximum density and $v^\star = V(\rho^\star)$. We consider a constant traffic flux $q^\star = \rho^\star v^\star$ entering the domain from $x = 0$ and there is a VSL at the outlet. Therefore, we have the following boundary conditions,

$$\rho(t,0) = \frac{q^\star}{v(t,0)}, \tag{3}$$

$$v(t,L) = U(t) + v^\star, \tag{4}$$

where $U(t)$ is defined as velocity variation from the steady state. The VSL at outlet shows static value $v^\star$ with $U(t)$ which we will design later. We linearize the model around uniform steady states $(\rho^\star,\ v^\star)$. The variations $(\tilde\rho(t,x), \tilde v(t,x))$ are defined as

$$\tilde\rho(t,x) = \rho(t,x) - \rho^\star, \quad \tilde v(t,x) = v(t,x) - v^\star. \tag{5}$$

The linearized ARZ model is given by,

$$\tilde\rho_t(t,x) + v^\star \tilde\rho_x(t,x) = -\rho^\star \tilde v_x(t,x), \tag{6}$$

$$\tilde v_t(t,x) - (\rho^\star p'(\rho^\star) - v^\star)\tilde v_x(t,x) = \frac{\tilde\rho(t,x)V'(\rho^\star) - \tilde v(t,x)}{\tau}, \tag{7}$$

with the linearized boundary conditions

$$\tilde\rho(t,0) = -\frac{\rho^\star}{v^\star}\tilde v(t,0), \quad \tilde v(t,L) = U(t).$$

### *2.2 Linearized ARZ model in Riemann coordinates*

We first write the linearized Aw-Rascle-Zhang model in the following Riemann coordinate $(\tilde w, \tilde v)$,

$$\tilde w = \frac{\gamma p^\star}{\rho^\star}\tilde\rho + \tilde v, \quad \tilde v = \tilde v, \tag{8}$$

thus we have

$$\tilde w_t(t,x) + v^\star \tilde w_x(t,x) = -c_1 \tilde w(t,x) + c_2 \tilde v(t,x), \tag{9}$$

$$\tilde v_t(t,x) - (\gamma p^\star - v^\star)\tilde v_x(t,x) = -c_1 \tilde w(t,x) + c_2 \tilde v(t,x), \tag{10}$$

$$\tilde w(t,0) = -r_0 \tilde v(t,0), \tag{11}$$

$$\tilde v(t,L) = U(t), \tag{12}$$

where the constants $c_1, c_2, r_0$ are defined as

$$c_1 = \frac{1}{\tau}\frac{v_f}{\rho_m}\frac{\rho^\star}{\gamma p^\star}, \quad c_2 = \frac{1}{\tau}\left(\frac{v_f}{\rho_m}\frac{\rho^\star}{\gamma p^\star} - 1\right), \quad r_0 = \frac{\gamma p^\star - v^\star}{v^\star}.$$

If the above constants satisfy the following inequalities, the linearized ARZ model is unstable according to the instability condition given in Yu and Krstic (2018b),

$$c_1 > \frac{1}{\tau} > 0, \quad c_2 = c_1 - \frac{1}{\tau} > 0.$$

It holds that $r_0 > 0$ for the congested regime of the traffic.

## 3. EMULATION OF THE BACKSTEPPING BOUNDARY CONTROL DESIGN

Scaling the states in (9)-(12) $\bar{w} = \exp\left(\frac{c_1}{v^\star}x\right)\tilde{w}$ and $\bar{v} = \exp\left(\frac{c_2}{\gamma p^\star - v^\star}x\right)\tilde{v}$, we reformulate the system as $2\times 2$ linear hyperbolic system with in-domain space varying coupling,

$$\bar{w}_t(t,x) + v^\star \bar{w}_x(t,x) = \bar{c}_1(x)\bar{v}(t,x), \tag{13}$$

$$\bar{v}_t(t,x) - (\gamma p^\star - v^\star)\bar{v}_x(t,x) = \bar{c}_2(x)\bar{w}(t,x), \tag{14}$$

$$\bar{w}(t,0) = -r_0\bar{v}(t,0), \tag{15}$$

$$\bar{v}(t,L) = r_1 U(t), \tag{16}$$

where the spatially-varying coefficients,

$$\bar{c}_1(x) = \exp\left(\frac{c_1}{v^\star}x - \frac{c_2}{\gamma p^\star - v^\star}x\right)c_2, \tag{17}$$

$$\bar{c}_2(x) = -\exp\left(\frac{c_2}{\gamma p^\star - v^\star}x - \frac{c_1}{v^\star}x\right)c_1, \tag{18}$$

and the constant coefficient at the outlet:

$$r_1 = \exp\left(\frac{c_2}{\gamma p^\star - v^\star}L\right). \tag{19}$$

We aim at stabilizing the closed-loop system (13)-(16) on events while sampling the continuous-time controller $U(t)$ in (16) at certain sequence of time instants $(t_k)_{k\in\mathbb{N}}$, that will be characterized later on. The control value is held constant between two successive time instants and it is sampled/updated when some triggering condition is verified. To that end, we need to suitably modify the controlled boundary condition (16). The boundary value of the state is going to be given by

$$\bar{v}(t,L) = U_d(t), \tag{20}$$

where $U_d(t) = U(t) + d(t)$, for all $t \in [t_k, t_{k+1})$, $k \geq 0$ and $d(t)$ will be seen as a deviation that will be rigorously characterized later on. We build on the following Volterra backstepping transformation:

$$\begin{aligned}\alpha(t,x) &= \bar{w}(t,x) - \int_0^x K^{11}(x,\xi)\bar{w}(t,\xi)d\xi \\ &\quad - \int_0^x K^{12}(x,\xi)\bar{v}(t,\xi)d\xi \\ \beta(t,x) &= \bar{v}(t,x) - \int_0^x K^{21}(x,\xi)\bar{w}(t,\xi)d\xi \\ &\quad - \int_0^x K^{22}(x,\xi)\bar{v}(t,\xi)d\xi,\end{aligned} \tag{21}$$

where Kernels $K^{ij}(x,\xi)$, $i,j = 1,2$ evolve in the triangular domain $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq L\}$ and are solution of the linear hyperbolic PDE kernel equations given in (Yu and Krstic, 2018b, Section 4). Therefore, with the transformation (21), one maps the system (13)-(15) with boundary input (20), into the following target system:

$$\alpha_t(t,x) + v^\star\alpha_x(t,x) = 0, \tag{22}$$

$$\beta_t(t,x) - (\gamma p^\star - v^\star)\beta_x(t,x) = 0, \tag{23}$$

$$\alpha(t,0) = -r_0\beta(t,0), \tag{24}$$

$$\beta(t,L) = r_1 d(t), \tag{25}$$

where $\alpha,\beta : \mathbb{R}^+ \times [0,L] \to \mathbb{R}$. Moreover, it is worth recalling that the transformation (21) is invertible whose inverse is given as follows:

$$\begin{aligned}\bar{w}(t,x) &= \alpha(t,x) + \int_0^x L^{11}(x,\xi)\alpha(t,\xi)d\xi \\ &\quad + \int_0^x L^{12}(x,\xi)\beta(t,\xi)d\xi \\ \bar{v}(t,x) &= \beta(t,x) + \int_0^x L^{21}(x,\xi)\alpha(t,\xi)d\xi \\ &\quad + \int_0^x L^{22}(x,\xi)\beta(t,\xi)d\xi,\end{aligned} \tag{26}$$

where $L^{ij}(x,\xi)$ , $i,j = 1,2$ are solution of linear hyperbolic PDE kernel equations given in (Vazquez et al., 2011, Section 3).

In this regard, the continuous-time control $U(t)$ (which is going to be sampled on events) is given as follows:

$$U(t) = \frac{1}{r_1}\left(\int_0^L L^{21}(L,\xi)\alpha(t,\xi)d\xi + \int_0^L L^{22}(L,\xi)\beta(t,\xi)d\xi\right) \tag{27}$$

and its emulated or sampled version is given as follows:

$$U_d(t) = \frac{1}{r_1}\left(\int_0^L L^{21}(L,\xi)\alpha(t_k,\xi)d\xi + \int_0^L L^{22}(L,\xi)\beta(t_k,\xi)d\xi\right) \tag{28}$$

for all $t \in [t_k, t_{k+1})$. Note that $U_d(t) = U(t) + d(t)$ where $d$ is given by:

$$\begin{aligned}d(t) =& \frac{1}{r_1}\int_0^L L^{21}(L,\xi)\left(\alpha(t_k,\xi) - \alpha(t,\xi)\right)d\xi \\ &- \frac{1}{r_1}\int_0^L L^{22}(L,\xi)\left(\beta(t_k,\xi) - \beta(t,\xi)\right)d\xi.\end{aligned} \tag{29}$$

Here, $d(t)$ represents an actuation deviation between the continuous controller and the event-triggered one. Hence, with $U_d$ given by (28), one can realize the backstepping transformation. Since $U_d(t)$ is given in the form of the transformed states, we can represent the $\alpha,\beta$ with inverse Volterra operator on $(\tilde{\rho},\tilde{v})$.

## 4. EVENT-TRIGGERED BOUNDARY CONTROL

Let us define two event-triggered boundary control strategies in this paper. They enclose an event-trigger mechanism containing a suitable triggering condition (which determines the time instant at which the controller needs to be sampled/updated) and the backstepping feedback controller (28).

The boundary feedback law is defined

$$U_d(t) = \frac{1}{r_1}\left(\int_0^L L^{21}(L,\xi)\alpha(t_k,\xi)d\xi + \int_0^L L^{22}(L,\xi)\beta(t_k,\xi)d\xi\right) \tag{30}$$

for all $t \in [t_k, t_{k+1})$, where the events $(t_k)_{k\in\mathbb{N}}$ is determined with two different triggering conditions, introduce in the following. Therefore both the static and dynamic event-triggering mechanisms are developed for the boundary feedback law.

### 4.1 Static triggering condition and time regularization

For the first event-triggered boundary control, we consider a triggering condition which relies on the evolution of the square of the actuation deviation (29) and the evolution of the following Lyapunov function candidate of the target system (22)-(25) (Bastin and Coron, 2016, Section 2):

$$V(\alpha,\beta) := \int_0^L \Big( \tfrac{A}{v^\star}\alpha^2(x)\exp(-\tfrac{\mu x}{v^\star}) + \tfrac{B}{(\gamma p^\star - v^\star)}\beta^2(x)\exp\left(\tfrac{\mu x}{(\gamma p^\star - v^\star)}\right)\Big)dx \tag{31}$$

with positive constant coefficients $A$, $B$ and $\mu$.

**Definition 1.** *[Static triggering condition] Let $\sigma \in (0,1)$, $T>0$, $\mu>0$, $B>0$. Let $t \mapsto V(\alpha(t,\cdot),\beta(t,\cdot))$ be given by* (31). *The event-triggered boundary control is defined in a static event-trigger mechanism. The times of the events $t_k \geq 0$ with $t_0 = 0$ form a finite or countable set of times which is determined by the following rules for some $k \geq 0$:*

*I) if $\{t > t_k + T | B\exp\left(\frac{\mu L}{\gamma p^\star - v^\star}\right) r_1^2 d^2(t) \geq \sigma\mu V(t)\} = \emptyset$ then the set of the times of the events is $\{t_0,...,t_k\}$.*

*II) if $\{t > t_k + T | B\exp\left(\frac{\mu L}{\gamma p^\star - v^\star}\right) r_1^2 d^2(t) \geq \sigma\mu V(t)\} \neq \emptyset$, then the next event time is given by:*

$$t_{k+1} = \inf\{t > t_k + T | B\exp\left(\tfrac{\mu L}{\gamma p^\star - v^\star}\right) r_1^2 d^2(t) \geq \sigma\mu V(t)\}. \tag{32}$$

Note that we have included $T>0$ within the definition of the next triggering (or event) time (32). It can be viewed as a timer threshold (or waiting time) that enforces a positive minimal inter-event time of at least $T$ units of time [1]. Only after the waiting time, the triggering condition is checked. One reason of having included $T$ is due to the difficulty to prove the existence of a minimal dwell-time as long as the next event time is just defined as $t_{k+1} = \inf\{t > t_k | B\exp\left(\frac{\mu}{(\gamma p^\star - v^\star)}\right) r_1^2 d^2(t) \geq \sigma\mu V(t)\}$ for which, without any time regularization, there may not be guarantees for avoiding the so-called Zeno phenomenon. In this framework, Zeno phenomenon would mean infinite triggering times in a finite-time interval. It represents infeasible practical implementations into digital platforms because it would be required to sample/update the controller infinitely fast. It is important, however, to remark that $T$ has to be suitably chosen. To that end, one option is to look at periodic implementations for the system (13)-(16) where the control is updated periodically in a sample-and-hold fashion while meeting stability guarantees. Therefore, inspired by Davo et al. (2018) (which deals with sampled-data control and LMI-based conditions to find periods), we will follow the idea of using *Looped functionals* in order to take into account the time regularization and its impact into the stability analysis.

### 4.2 Dynamic triggering condition

The second event-triggering condition is based on the evolution of the square of the actuation deviation (29) and the evolution of a dynamic variable which can be viewed as a filtered value of the static triggering condition.

**Definition 2.** *[Dynamic triggering condition] Let $\sigma \in (0,1)$, $\theta > 0$, $\eta > 0$, $\mu > 0$, $\kappa_1 > 0$, $m^0 \in \mathbb{R}_0^-$, $B > 0$. Let $t \mapsto V(\alpha(t,\cdot),\beta(t,\cdot))$ be given by* (31). *The event-triggered boundary control with dynamic triggering condition is defined in the following. The times of the events $t_k \geq 0$ with $t_0 = 0$ form a finite or countable set of times which is determined by the following rules for some $k \geq 0$:*

*I) if $\{t > t_k | \theta B\exp\left(\frac{\mu L}{(\gamma p^\star - v^\star)}\right) r_1^2 d^2(t) \geq -m(t)\} = \emptyset$ then the set of the times of the events is $\{t_0,...,t_k\}$.*

*II) if $\{t > t_k | \theta B\exp\left(\frac{\mu L}{(\gamma p^\star - v^\star)}\right) r_1^2 d^2(t) \geq -m(t)\} \neq \emptyset$, then the next event time is given by:*

$$t_{k+1} = \inf\{t > t_k | \theta B\exp\left(\tfrac{\mu L}{(\gamma p^\star - v^\star)}\right) r_1^2 d^2(t) \geq -m(t)\} \tag{33}$$

*where $m$ satisfies the ordinary differential equation,*

$$\dot m(t) = -\eta m(t) + \Big(B\exp\left(\tfrac{\mu L}{(\gamma p^\star - v^\star)}\right) r_1^2 d^2(t) - \sigma\mu V(t) - \kappa_1\alpha^2(t,L)\Big) \tag{34}$$

*for a given $\eta \geq \mu(1-\sigma)$ and $m(0) = m^0$.*

Note that in Definition 2, we have not imposed any time regularization. The reason is that, under (33), it is possible to show the existence of a minimal dwell-time between two event time instants. With this strategy, we aim at reducing execution times, i.e. triggering less often than with a static triggering mechanism.

## 5. MAIN RESULTS

In this section we present our main results: the avoidance of the Zeno phenomenon and the exponential stability of the closed-loop system.
The following estimate is useful to study the growth-in-time of the actuation deviation. This is instrumental to derive the existence of a waiting time $T$ (for the time regularization in the static triggering condition) and to establish the existence of a minimal dwell-time (for the dynamic triggering condition).

**Lemma 1.** *For $d(t)$ given by* (29), *it holds for all $t \in (t_k, t_{k+1})$,*

$$(\dot d(t))^2 \leq \varepsilon_0 V(t) + \varepsilon_1\alpha^2(t,L) + \varepsilon_2 d^2(t), \tag{35}$$

*with $\varepsilon_0 := \frac{4}{r_1^2}\min\{\frac{1}{v^\star}\exp\left(-\frac{\mu L}{v^\star}\right), \frac{r_0^2+1}{(\gamma p^\star - v^\star)}\}^{-1} \times \max\{(v^\star)^2\tilde{L_\xi^{21}}, (\gamma p^\star - v^\star)^2\tilde{L_\xi^{22}}\}$, $\varepsilon_1 := \frac{1}{r_1^2}4(v^\star)^2\left(L^{21}(L,L)\right)^2$ and $\varepsilon_2 := 2(\gamma p^\star - v^\star)^2\left(L^{22}(L,L)\right)^2$ where $\tilde{L_\xi^{21}} := \int_0^L (L_\xi^{21}(L,\xi))^2 d\xi$, $\tilde{L_\xi^{22}} := \int_0^L (L_\xi^{22}(L,\xi))^2 d\xi$.*

### 5.1 Avoidance of Zeno phenomena

*Static triggering condition*

For the event-triggered boundary control (30),(32) we have imposed a time regularization so that $t_{k+1} - t_k \geq T$, for all $k \geq 0$. As we discussed before, due to this fact, the Zeno phenomenon is immediately excluded.

[1] In event-triggered control literature, adding such a waiting time is often refereed as a time regularization.

*Dynamic triggering condition*

Let us state that under the event triggered control (30),(33), there exists a minimal dwell-time and therefore one avoids the Zeno phenomenon.

**Theorem 1.** *Under the the event-triggered boundary control* (30),(33) *in Definition 2, with positive scalars, $\sigma \in (0,1)$, $\theta$, $\eta$, $\mu$, $B$, $\kappa_1$, and $\varepsilon_0$, $\varepsilon_1$, and $\varepsilon_2$ (from Lemma 1) satisfying the following conditions,*

$$\begin{aligned} \theta B \exp\left(\frac{\mu L}{\gamma p^\star - v^\star}\right)\varepsilon_0 \leq (1-\sigma)\sigma\mu, \\ \theta B \exp\left(\frac{\mu L}{\gamma p^\star - v^\star}\right)\varepsilon_1 \leq (1-\sigma)\kappa_1, \end{aligned} \tag{36}$$

*there exists a minimal dwell-time $\tau^* > 0$ between two triggering times, i.e. there exists a constant $\tau^* > 0$ (independent of the initial conditions) such that $t_{k+1} - t_k \geq \tau^*$, for all $k \geq 0$. Moreover, the minimal dwell-time is given by:*

$$\tau^* := \int_0^1 \frac{1}{a_0 + a_1 s + a_2 s^2} ds, \tag{37}$$

*with* $a_0 := \left(1 + \varepsilon_2 + \frac{(1-\sigma)}{\theta} + \eta\right)\frac{(1-\sigma)}{\sigma}$, $a_1 := 1 + \varepsilon_2 + \frac{2(1-\sigma)}{\theta} + \eta$ *and* $a_2 := \frac{\sigma}{\theta}$.

Since there is a minimal dwell-time (which is uniform and does not depend on the initial conditions), no Zeno solution can appear.

*5.2 Stability results*

We perform a Lyapunov-based analysis by taking into account the two event-triggered strategies proposed in this paper.

*Static triggering condition*

We study first the stability of the closed-loop system under the event-triggered boundary control (30),(32).

**Theorem 2.** *Let $\sigma \in (0,1)$, $\mu > 0$, $T > 0$, $\bar{\gamma}_0, \bar{\gamma}_1 > 0$, $A := \bar{\gamma}_0 \exp\left(\frac{\mu L}{v^\star}\right)$, $B := \bar{\gamma}_0(r_0^2 \exp\left(\frac{\mu L}{v^\star}\right) + 1)$, $\varepsilon_0, \varepsilon_1$ and $\varepsilon_2$ given in Lemma 1. If the following conditions hold,*

$$\begin{aligned} B\exp\left(\frac{\mu L}{(\gamma p^\star - v^\star)}\right) - \bar{\gamma}_1 \leq 0, \quad -\sigma\mu + T\varepsilon_0 r_1^2 \bar{\gamma}_1 \leq 0, \\ -\bar{\gamma}_0 + T\varepsilon_1 r_1^2 \bar{\gamma}_1 \leq 0, \qquad \Pi(T) \leq 0, \end{aligned} \tag{38}$$

*where:*

$$\begin{aligned} \Pi(T) :=& B\exp\left(\frac{\mu L}{(\gamma p^\star - v^\star)}\right) r_1^2 - \bar{\gamma}_1 r_1^2 \\ &+ T r_1^2 \bar{\gamma}_1 (1 + \mu(1-\sigma) + \varepsilon_2) \end{aligned}$$

*then, the closed-loop system* (13)-(15),(20) *with event-triggered control* (30),(32) *is exponentially stable.*

*Dynamic triggering condition*

We study then the stability of the closed-loop system with event-triggered boundary control (30),(33).

**Theorem 3.** *Let $\sigma \in (0,1)$, $\mu > 0$, $\eta \geq \mu(1-\sigma)$, $A = \bar{\gamma}_0 \exp\left(\frac{\mu L}{v^\star}\right)$, $B = \bar{\gamma}_0(r_0^2 \exp\left(\frac{\mu L}{v^\star}\right) + 1)$ and $\kappa_1 > 0$ (from Definition 1). If $\kappa_1 \leq \bar{\gamma}_0$ and $\theta > 0$ is such that conditions* (36) *hold, then the closed-loop system* (13)-(15), (20) *with event-triggered control* (30),(33) *is exponentially stable.*

## 6. NUMERICAL SIMULATIONS

We consider the ARZ model with initial conditions $\rho(0,x) = \frac{p^\star}{3}\sin(\frac{3\pi}{L}x) + \rho^\star$ and $v(0,x) = -\frac{v^\star}{5}\sin(\frac{3\pi}{L}x) + v^\star$ where the steady-states in congested regime are $(\rho^\star, v^\star) =$ (120 veh/km, 36 km/h). We take $\gamma = 1$ and choose $p(v^\star) = c_0(\rho^\star)^\gamma = 0.9$. The length of the freeway section is chosen to be $L = 1$ km. The free speed is $v_f = 144$ km/h and the maximum density is $\rho_m = 160$ vehicles/km. The relaxation time is $\tau = 1.5$ min. We perform the simulation on a time horizon of 18 min.

We stabilize the system on events under the event-triggered boundary control (30),(32) with static triggering condition and time regularization. To set the parameters of the triggering condition, we first perform a line search on $\mu$ and on $T$ while finding $\bar{\gamma}_0$ and $\bar{\gamma}_1$. This gives $T = 0.13$ min , $\mu = 0.1$, $\bar{\gamma}_0 = 0.0204$ and $\bar{\gamma}_1 = 0.0487$ along with $\sigma = 0.9$ such that conditions of Theorem 2 hold. Then, we stabilize on events with (30),(33) (with dynamic triggering condition). The parameters are chosen as $\kappa_1 = 0.0102$, $\theta = 0.5$, $\eta = 0.33$ such that Theorems 1 and 3 apply. Moreover, we compute the minimal dwell-time between two triggering times according to (37), i.e. $\tau^* = 0.87$ min. Note that it is larger than $T$. In fact, $\tau^*$ can also be used as a suitable waiting time (much less conservative) for time regularization or in even in periodic schemes where the control is sampled/updated in s sample-and-hold fashion. Finally, Figure 1 shows time-evolution of the control functions. The red curve corresponds to the continuous-time control (27). The black curve corresponds to the event-triggered control (30) with static triggering condition in (32). Finally, The blue curve corresponds to the event-triggered control (30) with dynamic triggering condition in (33). Note that with a dynamic triggering condition, we are able to stabilize on events while reducing more execution times (than with a static triggering strategy) and also while meeting the theoretical guarantees. As motivated throughout the paper, with this approach we can update the control value only when needed which translates into the way that the varying speed limit sign, located at the outlet of a freeway segment, changes the advisory speed and only when needed.

## 7. CONCLUSION

In this work, we have proposed two event-triggered boundary control strategies for varying speed limit sign to reduce oscillations of the stop-and-go traffic congestion problem. Exponential stability and Zeno free behavior are achieved. The results can be adopted to implement in practice the ramp metering control strategy for the traffic congestion problem. We expect to address observer-based event-triggered boundary with validations on freeway data. In addition, self-triggered and periodic-event-triggered boundary control are of great interest. The latter suggests to monitor periodically the triggering condition while the actuation remains to be on events.

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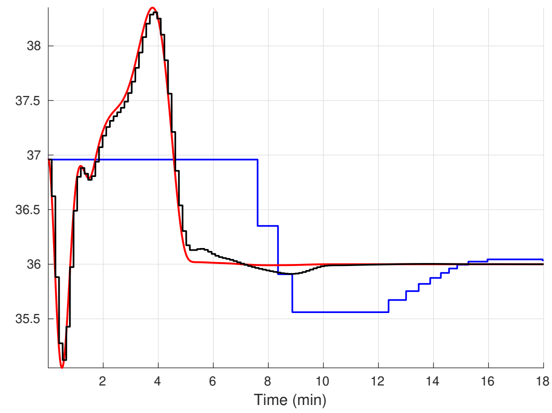


Fig. 1. Time-evolution of the velocity control input: the continuous-time control $U$ (27) (depicted in red line), the event-triggered boundary control $U_d$ (30),(32) (static triggering condition with time regularization $T = 0.13$) (depicted in black line) and the event-triggered boundary control $U_d$ (30),(33) (dynamic triggering condition) (depicted in blue line).


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