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Coefficients and Delay Estimation of the General Form of Fractional Order Systems Using Non-Ideal Step Inputs

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Abstract: This paper proposes a novel method for the simultaneous estimation of the coefficients and the delay term of a delayed fractional order system. Because of the practicality aspect of the non-ideal step inputs, such inputs are used in this paper for the first time to identify a fractional order system. To this end, the proposed identification procedure is separately described for two types of fractional order systems, i.e., including both non-delayed and delayed systems. For the non-delayed system, a fractional order integral approach is developed, and for the delayed system, a filtering approach is investigated to make the delay term to be explicitly appeared in the parameters vector. In simulation results, some illustrative examples, covering both non-delayed and delayed systems, are given to demonstrate the validity of the proposed method.

Keywords: Fractional order systems, time delay, non-ideal step inputs.

1. INTRODUCTION

With the growth of high-tech computers in the last few decades, the usage of fractional order calculus has been increased in the different fields of science, e.g. bioengineering in Magin (2006), physics in Rudolf (2000), continuum mechanics in Carpinteri and Mainardi (2014), and biology in Magin (2010), and today we are seeing widespread usage of such calculus in the various fields of engineering, e.g. control engineering, i.e., fractional order systems and control in Wang et al. (2015), Vyawahare and Nataraj (2013), Jalloul et al. (2013), Monje et al. (2010), Azarmi et al. (2018), Azarmi et al. (2015b), Luo et al. (2010), Padula and Visioli (2011), Beschi et al. (2017), Padula and Visioli (2016), Azarmi et al. (2020), Azarmi et al. (2016), Gao (2015), Azarmi et al. (2015a), and Calderón et al. (2006).

Among the applications of fractional order calculus, in recent years, the identification of fractional order systems has attracted the attention of many researchers in the world, e.g. Tavakoli-Kakhki and Tavazoei (2014), Yakoub et al. (2015), Wang et al. (2019), Kothari et al. (2018a), Kothari et al. (2018b), and Ahmed (2020). Some pioneering works in the field of identification of fractional order systems have been done in Aoun (2005), Cois (2002), Le Lay (1998), Lin (2001), Cois et al. (2001), Chetoui et al. (2012), Sabatier et al. (2006), Ahmed (2015), Fahim et al. (2018), and Malti et al. (2006). Simultaneous estimation of the coefficients and commensurate order of a fractional order transfer function has been reported for the first time in Malti et al. (2008). The estimation of the model parameters and non-commensurate orders was performed by Tang et al. (2015), and Belkhatir and Laleg-Kirati (2018). Besides, the identification of delayed fractional order systems can be seen in the research paper of Narang et al. (2011). But so far, to the best of the authors' knowledge, the identification problem of the delayed fractional order system performed by the non-ideal step inputs has been remained unaddressed in literature. It is undeniable that in practical applications, it is not easy to apply every desired input for the identification of the system model. Because it may not operationally be possible, and also it may even cause damage to the system (Ljung, 2010; Nelles, 2002; Ljung, 1991; Narendra and Annaswamy, 1984). The use of the non-ideal step inputs for the identification of integer order systems has been reported in Ahmed (2010) and Ahmed (2016). The main contribution of this paper is to propose a novel method to identify the non-delayed and delayed fractional order systems using the non-ideal step inputs that may widely use in industry. Indeed, in the proposed method, the pivotal assumption is that the fractional orders of the system model are a piece of the user's information.

The rest of this paper is organized as follows. In Section 2, a brief mathematical background of fractional order calculus is presented. Also, the non-ideal step inputs used in the theoretical parts of the paper are introduced. In Section 3, the proposed identification methods using the non-ideal step inputs are separately presented in detail. In Section 4, two illustrative examples are given to show the effectiveness of the proposed method. Finally, the concluding remarks are given in Section 5.

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2. MATHEMATICAL BACKGROUND

In this section, the necessary preliminaries about fractional order calculus are presented, and the non-ideal step inputs, which the authors used in this paper are introduced.

2.1 Fractional Order Models

As the first step, a commonly used function in fractional order calculation is introduced, which is known as the Gamma Euler's function in literature (Podlubny, 1998).

$$\Gamma(z) \triangleq \int_0^\infty e^{-t}\, t^{z-1}\, dt\,. \tag{1}$$

The fractional order integral of order α is defined as follows:

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,,\ t>0\,,\ \alpha \in \mathbb{R}^+\,, \tag{2}$$

and one of the most popular definition used to describe the fractional order derivative of order α is the Caputo's definition, defined as

$${}_0^C D_t^\alpha f(t) = I^{\lceil\alpha\rceil-\alpha}\left\{f^{(\lceil\alpha\rceil)}(t)\right\},\ \alpha \in \mathbb{R}^+ - \mathbb{N}\,, \tag{3}$$

where $\lceil.\rceil$ is ceiling function (Podlubny, 1998). For the simplicity of the notations, which the authors used in this paper, in the rest of this paper, the Caputo's fractional order derivative of order α is shown as D^α. The other definitions of fractional order derivative are available in Podlubny (1998). The Laplace transform of the fractional order integral discussed in (2) is given by

$$L\left\{{}_0I_t^\alpha f(t)\right\} = s^{-\alpha}F(s)\,, \tag{4}$$

where $L\{f(t)\} = F(s)$ (Podlubny, 1998). Also, the Laplace transform of the Caputo's derivative of the function $f(t)$ defined in (3) is presented as

$$L\left\{{}_0^C D_t^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{\lceil\alpha\rceil-1} s^{\alpha-k-1} f^{(k)}(0)\,, \tag{5}$$

where $f^{(k)}(0)\,(\,k=1\,,\,\dots\,,\,\lceil\alpha\rceil-1\,)$ are the initial conditions of function $f(t)$ (Podlubny, 1998). The fractional order integral of Caputo's derivative of order α is given by

$${}_aI_t^\alpha\, {}_a^C D_t^\alpha f(t) = f(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{f^{(k)}(a)}{k!}(t-a)^k\,, \tag{6}$$

where the function $f(t)$ has $\lceil\alpha\rceil - 1$ continuous derivative and $\alpha \in \mathbb{R}^+ - \mathbb{N}$ (Podlubny, 1998).

2.2 Non-Ideal Step Inputs

For the identification of any system, it is known that it is required to stimulate the intended system by an appropriate input. The point is that if the input is not properly selected, then all the system modes may not be triggered, and the identification procedure may not appropriately be performed. On the other hand, it is known that it is not easy to stimulate an industrial plant by every input. Because although the input may meet some requirements, it may not be practical and it may damage the intended plant (Ljung, 2010; Nelles, 2002; Ljung, 1991; Narendra and Annaswamy, 1984). For example, an industrial heat furnace cannot be stimulated with a Pseudo-Random Binary Sequence (PRBS) input. Thus, in the identification procedure, choosing suitable inputs plays a pivotal role. Indeed, this paper attempts to use the most commonly used inputs in the industry such as staircase input, saturated ramp input and filtered step input for the identification of fractional order systems, including both non-delayed and delayed transfer functions. It is worth mentioning that the non-ideal step inputs are not limited to the three cases, which the authors considered in this manuscript (For more details see (Ahmed, 2010)). To better clarification, the mentioned non-ideal step inputs are depicted in Fig. 1.

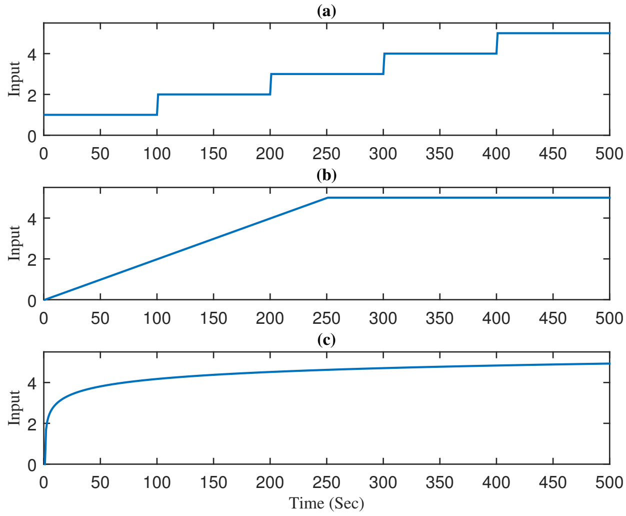

Fig. 1. Schematic of the non-ideal step inputs ((a) Staircase input, (b) Saturated ramp input, and (c) Filtered step input)

3. IDENTIFICATION USING NON-IDEAL STEP INPUTS

In this section, two methods are developed for the identification of the general forms of non-delayed and delayed fractional order systems by using the non-ideal step inputs. The method for the estimation of the coefficients of a non-delayed fractional order system is described in the first subsection of this section. This method is based on a fractional order integral approach and Least Square (LS) estimation. Additionally, the method for estimation of the coefficients and the delay term of a delayed fractional order system is explained in the second subsection of this section. This method is based on applying a fractional order low-pass filter and an estimation method, which is named Instrumental Variable (IV) in literature (Young, 1970).

3.1 Coefficients Estimation of Non-Delayed Fractional Order Systems

Let us consider the following fractional order differential equation as the model of a practical system

$$\begin{aligned} &D^{\alpha n}y(t) + a_{n-1}D^{\alpha(n-1)}y(t) + \dots + a_0 y(t) = \\ &b_{n-1}D^{\alpha(n-1)}u(t) + \dots + b_0 u(t) + e(t)\,, \end{aligned} \tag{7}$$

where $n \in \mathbb{N}$ and $\alpha\,(0 < \alpha \leq 1)$ is the fractional order. Besides, $y(t)$ and $u(t)$ are the system output and the system input, respectively (Diethelm, 2010). Note that $e(t)$ is the white noise and $[\, a_{n-1} \, ... \, a_0 \; b_{n-1} \, ... \, b_0 \,]^T$ is the vector of unknown parameters, which are estimated in the proposed identification procedure.

Due to (5) and by considering zero initial conditions, the corresponding transfer function is obtained as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{(n-1)\alpha} + ... + b_0}{s^{n\alpha} + a_{n-1}s^{(n-1)\alpha} + ... + a_0} \, . \tag{8}$$

The system model which is defined in (8) is a commensurate transfer function (Podlubny, 1998).

In the fractional order integral equation approach proposed in this paper, according to (6), it is assumed that $y(t)$ is the smooth signal and the system is in the rest state, i.e., the initial conditions of the system is considered to be equal to zero. Therefore, by fractional order integrating of order $n\alpha$ from the both hand sides of (7), the following equality is obtained

$$\begin{aligned} &y(t) + a_{n-1}I_t^{\alpha}y(t) + ... + a_0 I_t^{n\alpha}y(t) = \\ &b_{n-1}I_t^{\alpha}u(t) + ... + b_0 I_t^{n\alpha}u(t) + \varsigma(t) \, , \end{aligned} \tag{9}$$

where $\varsigma(t) = {}_0I_t^{n\alpha}e(t)$. The equality achieved in (9) can be compressed in the matrix form as

$$y(t) = [\, -\mathbf{I}^{n\alpha}y(t) \;\; \mathbf{I}^{n\alpha}u(t) \,] \begin{bmatrix} \mathbf{a}_{n-1} \\ \mathbf{b}_{n-1} \end{bmatrix} + \varsigma(t) \, , \tag{10}$$

where

$$\mathbf{I}^{n\alpha}y(t) = [\, I_t^{\alpha}y(t) \;\; ... \;\; I_t^{n\alpha}y(t) \,] \, , \tag{11}$$

$$\mathbf{I}^{n\alpha}u(t) = [\, I_t^{\alpha}u(t) \;\; ... \;\; I_t^{n\alpha}u(t) \,] \, , \tag{12}$$

$$\mathbf{a}_{n-1} = [\, a_{n-1} \;\; ... \;\; a_0 \,]^T \, , \tag{13}$$

and

$$\mathbf{b}_{n-1} = [\, b_{n-1} \;\; ... \;\; b_0 \,]^T \, . \tag{14}$$

From the equation presented in (10), the estimation equation can be re-formulated as follows:

$$\psi(t) = \phi(t)\theta + \varsigma(t) \, , \tag{15}$$

where $\psi(t) = y(t)$ and $\phi(t) = [\, -\mathbf{I}^{n\alpha}y(t) \;\; \mathbf{I}^{n\alpha}u(t) \,]$ is the regression vector. Additionally, $\theta = [\, \mathbf{a}_{n-1}; \mathbf{b}_{n-1} \,]$ is the unknown parameter vector, by putting two vectors $\mathbf{a}_{n-1}$ and $\mathbf{b}_{n-1}$ with the dimension of $n{\times}1$ into a vector, i.e., θ with the dimension of $2n{\times}1$.

Cumulating (15) for different time instances, yields to the following estimation equation,

$$\Psi = \Phi\theta + \mathrm{Z} \, , \tag{16}$$

where

$$\mathrm{Z} = [\, \varsigma(t_0) \;\; ... \;\; \varsigma(t_{n-1}) \,]^T \, , \tag{17}$$

and

$$\Psi = [\, \psi(t_0) \;\; ... \;\; \psi(t_{n-1}) \,]^T \, . \tag{18}$$

In (17) and (18), Z and Ψ are two vectors with the dimension of $n{\times}1$. Besides, $\Phi = [\, \phi(t_0) \; ; \; ... \; ; \; \phi(t_{n-1}) \,]$ is a matrix with the dimension of $n{\times}2n$, by putting the vectors $\phi(t_0)$, $\phi(t_1)$, ..., $\phi(t_{n-2})$, and $\phi(t_{n-1})$ with the dimension of $1{\times}2n$ into a matrix.

Now, to estimate the parameter vector θ, the LS method can be used and the unknown parameter vector is estimated as

$$\theta = \left(\phi^T(t)\phi(t)\right)^{-1}\phi^T(t)\psi(t) \, . \tag{19}$$

In the rest, the three non-ideal step inputs mentioned in Subsection 2.2 are used to attain the regression vector $\phi(t)$ in (19). To this end, the required descriptions and the mathematical formulations are mentioned with more details.

3.1.1 Staircase Input

A staircase input is the sum of a set of shifted step inputs that can be defined as

$$u(t) = \sum_{i=0}^{I} h_i \, \Omega(t - \ell_i) \, , \tag{20}$$

where h_i is the step size and $\Omega(t-\ell_j) \, (\, j \in \{0 \, , \, ... \, , \, I\} \,)$ for $\ell_j > 0$ is a shifted step input defined as

$$\Omega(t - \ell_j) = \begin{cases} 1, \; t \geq \ell_j \, , \\ 0, \; t \prec \ell_j \, . \end{cases} \tag{21}$$

For simplicity, consider a triple staircase input as

$$u(t) = h_0 \, \Omega \, (t - \ell_0) + h_1 \, \Omega \, (t - \ell_1) + h_2 \, \Omega \, (t - \ell_2) \;\; , \tag{22}$$

where $\ell_0 = 0$, whose the Laplace transform of $u(t)$ is obtained as follows:

$$U(s) = \frac{h_0}{s} + \frac{h_1}{s}e^{-\ell_1 s} + \frac{h_2}{s}e^{-\ell_2 s} \, . \tag{23}$$

According to (2) and by the fractional order integration of the staircase input $u(t)$ (20) results in

$${}_0I_t^{n\alpha}u(t) = \frac{1}{\Gamma(n\alpha+1)} \sum_{i=0}^{I} h_i \, (t - \ell_i)^{n\alpha} \, \Omega(t - \ell_i) \, . \tag{24}$$

Equation (24) is used for computing the second element of $\phi(t)$ in (15).

3.1.2 Saturated Ramp Input

A saturated ramp input is defined as

$$u(t) = \sum_{i=0}^{1} p_i \; [t - \ell_i] \; \Omega(t - \ell_i) \, , \tag{25}$$

where $p_0 = \frac{h}{\ell_1}$ and $p_1 = -p_0$. As a matter of fact, ℓ_i and h are the time of the input to reach saturation and the saturation value, respectively. Therefore, by doing some calculation, the second element of $\phi(t)$ in (15) is derived as follows:

$${}_0I_t^{n\alpha}u(t) = \frac{h}{\ell_1\Gamma(n\alpha+2)}(t^{n\alpha+1} - (t-\ell_1)^{n\alpha+1} \, \Omega(t-\ell_1)) \, . \tag{26}$$

Equation (26) helps us to build the second element of $\phi(t)$ in (15).

3.1.3 Filtered Step Input

The filtered step input considered in this paper is as follows:

$$U(s) = \frac{1}{\lambda s + 1}\frac{h}{s} \, , \tag{27}$$

where λ is the filter coefficient. Therefore, the system output, i.e., $Y(s)$, of the transfer function $G(s)$ in (8) is derived as

$$Y(s) = \frac{b_{n-1}s^{(n-1)\alpha} + ... + b_0}{s^{n\alpha} + a_{n-1}s^{(n-1)\alpha} + ... + a_0}\frac{1}{\lambda s + 1}\frac{h}{s} \, , \tag{28}$$

which yields to

$$\begin{aligned} &\lambda Y(s) + \frac{\lambda a_{n-1}}{s^{\alpha}}Y(s) + ... + \frac{\lambda a_0}{s^{n\alpha}}Y(s) + \frac{1}{s}Y(s) + \\ &\frac{a_{n-1}}{s^{\alpha+1}}Y(s) + ... + \frac{a_0}{s^{n\alpha+1}}Y(s) = \frac{h \, b_{n-1}}{s^{\alpha+2}} + ... + \frac{h \, b_0}{s^{n\alpha+2}} \, . \end{aligned} \tag{29}$$

According to (4), equality in (29) can be re-written as follows:

$$\lambda y(t) + \int_0^t y(\tau)d\tau + a_{n-1}\left(\lambda I^{\alpha} y(t) + I^{\alpha+1}y(t)\right) + \\ ... + a_0\left(\lambda I^{n\alpha}y(t) + I^{n\alpha+1}y(t)\right) = I^{\alpha+2}\, h\, b_{n-1}\,\delta(t) + \\ ... + I^{n\alpha+2}\, h\, b_0\,\delta(t)\,. \tag{30}$$

In (30), $\delta(t)$ is the Dirac delta function. From (30), the estimation equation is obtained as

$$\psi(t) = \phi(t)\theta + \varsigma(t)\,, \tag{31}$$

where

$$\psi(t) = \lambda y(t) + \int_0^t y(\tau)d\tau\,, \tag{32}$$

$$\theta = [\,\mathbf{a}_{n-1}; \mathbf{b}_{n-1}\,]\,, \tag{33}$$

and

$$\phi(t) = [\,\phi_1 \quad \phi_2\,]\,, \tag{34}$$

where

$$\phi_1 = [\,-\lambda\,{}_0\mathbf{I}_t^{n\alpha}y(t) - {}_0\mathbf{I}_t^{n\alpha+1}y(t)\,]\,, \tag{35}$$

and

$$\phi_2 = [\,\frac{h\,t^{\alpha+1}}{\Gamma(\alpha+2)} + ... + \frac{h\,t^{n\alpha+1}}{\Gamma(n\alpha+2)}\,]\,. \tag{36}$$

For more clarity, the first element of $\phi(t)$ is derived as follows:

$$\lambda\,{}_0\mathbf{I}_t^{n\alpha}y(t) + {}_0\mathbf{I}_t^{n\alpha+1}y(t) = \\ \left[\,\lambda\,{}_0I_t^{\alpha}y(t) + {}_0I_t^{\alpha+1}y(t)\ ...\ \lambda\,{}_0I_t^{n\alpha}y(t) + {}_0I_t^{n\alpha+1}y(t)\,\right]. \tag{37}$$

Note that the following calculation is helpful to build the second element of the regression vector $\phi(t)$.

$${}_0I_t^{n\alpha+2}\,h\,b_{n-1}\,\delta(t) = \\ \frac{h\,b_{n-1}}{\Gamma(n\alpha+2)}\int_0^t (t-\tau)^{n\alpha+1}\delta(\tau)d\tau = \frac{h\,b_{n-1}\,t^{n\alpha+1}}{\Gamma(n\alpha+2)}\,. \tag{38}$$

3.2 Estimation of the Coefficients and the Delay Term of Delayed Fractional Order Systems

In this subsection, the filtering method proposed in Narang et al. (2011) is developed to estimate the coefficients and the delay term of the delayed fractional order transfer functions by using the non-ideal step inputs.
Suppose a delayed fractional order differential equation as the following form,

$$a_n D^{\alpha n}y(t) + a_{n-1}D^{\alpha(n-1)}y(t) + ... + y(t) = \\ b_{n-1}D^{\alpha(n-1)}u(t-\ell) + ... + b_0 u(t-\ell) + e(t)\,, \tag{39}$$

where ℓ is an input time delay. It is worth mentioning that $[\ a_n\ ...\ 1\ b_{n-1}\ ...\ b_0\ \ \ell\,]^T$ is the vector of the unknown parameters, which are estimated in the proposed identification procedure. Due to (5) and considering zero initial condition, the corresponding transfer function to delayed fractional order differential equation in (39) is represented as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{(n-1)\alpha} + ... + b_0}{a_n s^{n\alpha} + a_{n-1}s^{(n-1)\alpha} + ... + 1}e^{-\ell s}\,. \tag{40}$$

Now, consider a fractional order filter as

$$F(s^{\alpha}) = \frac{1}{s\hat{A}(s^{\alpha})}\,, \tag{41}$$

where $\hat{A}(.)$ is the denominator of transfer function $G(s)$ in (40). Applying this filter to the numerator and denominator of (40) (Narang et al., 2011) results in

$$Y_F(s) = -\mathbf{A}\mathbf{S}^{(n-1)\alpha}\,Y_{F1}(s) + \mathbf{B}\mathbf{S}^{(n-2)\alpha}\,e^{-\ell s}\,U_{F1}(s) \\ +b_0\,e^{-\ell s}\,U_{F2}(s) + b_0\,e^{-\ell s}\,U_{F3}(s) + \varsigma(s)\,, \tag{42}$$

where

$$\mathbf{A} = [\,a_n\ a_{n-1}\ ...\ a_1\,]\,,\ \mathbf{S}^{(n-1)\alpha} = \left[\,s^{(n-1)\alpha}\ ...\ 1\,\right]^T\,, \tag{43}$$

$$\mathbf{B} = [\,b_{n-1}\ ...\ b_1\,]\,,\ \mathbf{S}^{(n-2)\alpha} = \left[\,s^{(n-2)\alpha}\ ...\ 1\,\right]^T\,, \tag{44}$$

$$Y_F(s) = F(s^{\alpha})Y(s)\,,\ Y_{F1}(s) = s^{\alpha}F(s^{\alpha})Y(s)\,, \tag{45}$$

$$U_{F1}(s) = s^{\alpha}F(s^{\alpha})U(s)\,, \tag{46}$$

and

$$U_{F2}(s) = \left(1 - \hat{A}(s^{\alpha})\right)F(s^{\alpha})\,U(s)\,,\ U_{F3}(s) = \frac{U(s)}{s}\,. \tag{47}$$

By taking the inverse Laplace transform from the both hand sides of (42) yields to

$$y_F(t) = -\mathbf{A}\mathbf{y}_{F1}^{[(n-1)\alpha]}(t) + \mathbf{B}\mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \\ +b_0 u_{F2}(t-\ell) + b_0 L^{-1}\left\{U_{F3}(s)e^{-\ell s}\right\} + \varsigma(t)\,, \tag{48}$$

where $L^{-1}\{.\}$ denote the Laplace inverse operator,

$$L^{-1}\left\{\mathbf{S}^{(n-1)\alpha}Y_{F1}(s)\right\} = \mathbf{y}_{F1}^{[(n-1)\alpha]}(t)\,, \tag{49}$$

$$L^{-1}\left\{\mathbf{S}^{(n-2)\alpha}e^{-\ell s}U_{F1}(s)\right\} = \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell)\,, \tag{50}$$

and

$$L^{-1}\left\{e^{-\ell s}U_{F2}(s)\right\} = u_{F2}(t-\ell)\,. \tag{51}$$

This identification method is used in the case of applying each of the three non-ideal step inputs, which are considered in our paper. The main problem in the following subsections is to make the delay term to be explicitly appeared in the unknown parameters vector.

3.2.1 Staircase Input

Suppose the staircase input described in (22) whose Laplace transform is in the form of (23). The Laplace inverse transform of $U_{F3}(s)e^{-\ell s}$ in (48) can be written as

$$L^{-1}\{U_{F3}(s)e^{-\ell s}\} \\ = L^{-1}\{\frac{h_0}{s^2}e^{-\ell s} + \frac{h_1}{s^2}e^{-(\ell+\ell_1)s} + \frac{h_2}{s^2}e^{-(\ell+\ell_2)s}\} \\ = u_{stair1} - \ell u_{stair2}\,, \tag{52}$$

where u_{stair1} and u_{stair2} are respectively defined as

$$u_{stair1} = h_0\,t\,\Omega(t-\ell) + h_1\,(t-\ell_1)\,\Omega(t-\ell-\ell_1) \\ + h_2\,(t-\ell_2)\,\Omega(t-\ell-\ell_2)\,, \tag{53}$$

and

$$u_{stair2} = h_0\,\Omega(t-\ell) + h_1\,\Omega(t-\ell-\ell_1) + h_2\,\Omega(t-\ell-\ell_2)\,. \tag{54}$$

According to (52), the time delay term explicitly appears in the parameters vector. Now, according to (52), Equation (48) can be re-written as follows:

$$y_F(t) = -\mathbf{A}\,\mathbf{y}_{F1}^{[(n-1)\alpha]}(t) + \mathbf{B}\,\mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \\ +b_0\,u_{F2}(t-\ell) + b_0\,u_{stair1} - b_0\,\ell u_{stair2} + \varsigma(t)\,. \tag{55}$$

According to (55), the estimation equation is obtained as

$$\psi(t) = \phi(t)\theta + \varsigma(t)\,, \tag{56}$$

where

$$\psi(t) = y_F(t)\,, \tag{57}$$

$$\theta = [\,\mathbf{A};\ \mathbf{B};\ b_0;\ b_0\ell\,]\,, \quad (58)$$

and

$$\phi(t) = [\,\phi_1 \quad \phi_2\,]\,, \quad (59)$$

where

$$\phi_1 = \left[\,-\mathbf{y}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell)\,\right], \quad (60)$$

and

$$\phi_2 = [\,u_{F2}(t-\ell) + u_{stair1} \quad -u_{stair2}\,]\,. \quad (61)$$

From (56) and (58), the delay term can be obtained as $\ell = \frac{\theta(2n+1,1)}{\theta(2n,1)}$, i.e., by dividing the $(2n+1)^{th}$ term of the vector θ to the $(2n)^{th}$ term of the vector θ. It is worth mentioning that because of the filtering procedure for estimating the coefficients and the delay term of the delayed fractional order system (39), the white noise $e(t)$ converted to colored noise. As it is known, the LS method gives a biased-estimation in the presence of colored noise. According to this point, the IV method is used in this paper to obtain a non-biased estimation. The instrument for the staircase input is expressed as follows:

$$\phi_{IV}(t) = [\,\phi_{IV1} \quad \phi_{IV2}\,]\,, \quad (62)$$

where

$$\phi_{IV1} = \left[\,-\hat{\mathbf{y}}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell)\,\right], \quad (63)$$

and

$$\phi_{IV2} = [\,u_{F2}(t-\ell) + u_{stair1} \quad -u_{stair2}\,]\,. \quad (64)$$

In (63), $\hat{\mathbf{y}}$ is the value of the estimated output $\mathbf{y}$ in each loop. Then, the parameter vector θ in (56) and (58) is estimated as follows:

$$\theta = \left((\phi_{IV}(t))^T\phi(t)\right)^{-1}(\phi_{IV}(t))^T\psi(t)\,. \quad (65)$$

3.2.2 Saturated Ramp Input

Consider the saturated ramp defined in (25), the Laplace transform of (25) is given by

$$U(s) = \frac{h}{\ell_1}\frac{1}{s^2} - \frac{h}{\ell_1}\frac{1}{s^2}e^{-\ell_1 s}\,. \quad (66)$$

Considering (66) and by taking the Laplace inverse of $U_{F3}(s)e^{-\ell s}$ results in

$$L^{-1}\left\{U_{F3}(s)e^{-\ell s}\right\} = L^{-1}\left\{\frac{h}{\ell_1}\frac{1}{s^3}e^{-\ell s} - \frac{h}{\ell_1}\frac{1}{s^3}e^{-(\ell+\ell_1)s}\right\} = u_{ramp1} - \ell u_{ramp2} + \ell^2 u_{ramp3}\,, \quad (67)$$

where

$$u_{ramp1} = \frac{h}{2\ell_1}\,t^2\,\Omega\,(t-\ell) - \frac{h}{2\ell_1}\,(t-\ell_1)^2\,\Omega\,(t-\ell-\ell_1)\,, \quad (68)$$

$$u_{ramp2} = \frac{h}{\ell_1}\,t\,\Omega\,(t-\ell) - \frac{h}{\ell_1}\,(t-\ell_1)\,\Omega\,(t-\ell-\ell_1)\,, \quad (69)$$

and

$$u_{ramp3} = \frac{h}{2\ell_1}\Omega\,(t-\ell) - \frac{h}{2\ell_1}\Omega\,(t-\ell-\ell_1)\,. \quad (70)$$

Now, by defining G_{u1} as the following vector,

$$G_{u1} = [\,u_{F2}(t-\ell) + u_{ramp1} \quad -u_{ramp2} \quad u_{ramp3}\,]\,, \quad (71)$$

the regression vector and the parameter vector of the estimation equation (56) are respectively equal to

$$\phi(t) = \left[\,-\mathbf{y}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \quad G_{u1}\,\right], \quad (72)$$

and

$$\theta = \left[\,\mathbf{A}; \mathbf{B}; b_0; b_0\ell; b_0\ell^2\,\right]. \quad (73)$$

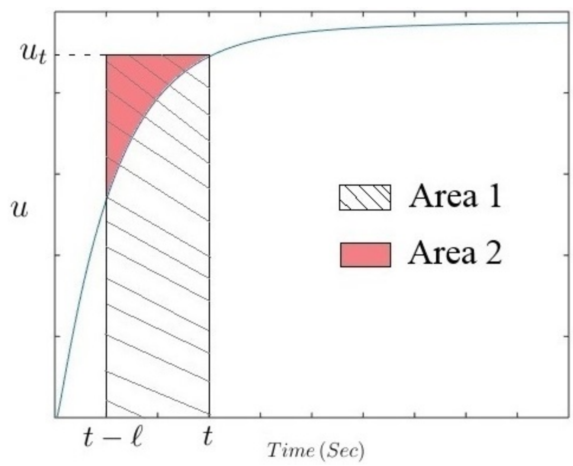

Fig. 2. Schematic for calculation of $u_{F3}(t-\ell)$ in (76)

Consequently, the delay term can be obtained as $\ell = \frac{\theta(2n+1,1)}{\theta(2n,1)}$, i.e., by dividing the $(2n+1)^{th}$ term of the vector θ to the $(2n)^{th}$ term of the vector θ, or $\ell = \frac{\theta(2n+2,1)}{\theta(2n+1,1)}$, i.e., by dividing the $(2n+2)^{th}$ term of the vector θ to the $(2n+1)^{th}$ term of the vector θ.
According to the discussion done in the previous subsection, the instrument for the saturated ramp is expressed as

$$\phi_{IV}(t) = \left[\,-\hat{\mathbf{y}}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \quad G_{u1}\,\right], \quad (74)$$

and the vector θ can similarly be obtained from (65).

3.2.3 Filtered Step Input

Consider the filtered step input in (27), for the filtered step input, the graphical information is used to make the delay term to be explicitly appeared in the parameters vector (Narang et al., 2011). In this case, the inverse Laplace transform of term $U_{F3}(s)e^{-\ell s}$ (48) is obtained as

$$L^{-1}\left\{U_{F3}(s)e^{-\ell s}\right\} = L^{-1}\left\{\frac{1}{\lambda s+1}\frac{1}{s^2}e^{-\ell s}\right\} = u_{F3}(t-\ell)\,, \quad (75)$$

where

$$u_{F3}(t-\ell) = u_{F3}(t) - u_t\ell + \int_{t-\ell}^{t}\left[u_t - u(t_k)\right]dt_k\,, \quad (76)$$

and $u(t_k)$ equals

$$u(t_k) = \left(1 - e^{-\frac{1}{\lambda}t_k}\right)\Omega(t_k)\,, \quad (77)$$

in which $u_{F3}(t) = \int_0^t u(t)dt$ in (76). Also, u_t in (76) is the value of the input signal $u(t)$ at each time instant t. Note that the value of the constant u_t will be changed for each time instant. For better clarification, according to (76), consider Area 1 as $u_t\ell$ and Area 2 as $\int_{t-\ell}^{t}\left[u_t - u(t_k)\right]dt_k$. These two areas are simultaneously depicted in Fig. 2.
Now, by defining G_{u2} as the following equation,

$$G_{u2} = u_{F2}(t-\ell) + u_{F3}(t) + \int_{t-\ell}^{t}\left[u_t - u(t_k)\right]dt_k\,, \quad (78)$$

and according to (78), the regression vector and the unknown parameter vector of the estimation equation (56) are obtained as

$$\phi(t) = \left[-\mathbf{y}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \quad G_{u2} \quad -u_t \right], \tag{79}$$

and

$$\theta = [\,\mathbf{A};\, \mathbf{B};\, b_0;\, b_0\ell\,], \tag{80}$$

respectively. Therefore, the delay term can be obtained as $\ell = \frac{\theta(2n+1,1)}{\theta(2n,1)}$, i.e., by dividing the $(2n+1)^{th}$ term of the vector θ to the $(2n)^{th}$ term of the vector θ.
The instrument for the filtered step is expressed as

$$\phi_{IV}(t) = \left[-\hat{\mathbf{y}}_{F1}^{[(n-1)\alpha]}(t) \quad \mathbf{u}_{F1}^{[(n-2)\alpha]}(t-\ell) \quad G_{u2} \quad -u_t \right], \tag{81}$$

and the vector θ can similarly be derived from (65). According to the point discussed at the end of Subsection 3.2.1, in the proposed identification procedure, the unknown parameters vector and the delay term are iteratively estimated in a loop. The procedure is iterated until the difference between the last two estimated values reaches the stopping condition chosen by the designer.
Since there is a delay term in the regression vectors in (62), (74), (81), and also the filter $F(s^\alpha)$ in (41) needs the denominator coefficients of the transfer function $G(s)$ in (39), an initial guess is needed for estimation the denominator coefficients and the delay term. If the integer order model exists, appropriate initial values would be the denominator coefficients and the delay value of the integer order model. Generally, by paying attention to the stability of the filter described in (41), any initial guess can be practically applied (Narang et al., 2011).

4. SIMULATION RESULTS

To illustrate the effectiveness of the identification methods proposed in this paper, two fractional order systems are considered as the following form, i.e., a non-delayed fractional order transfer function $G_1(s)$ in (82) and a delayed fractional order transfer function $G_2(s)$ in (83).

$$G_1(s) = \frac{1}{s^{0.5}+1}. \tag{82}$$

$$G_2(s) = \frac{1}{10\, s^{0.75}+1} e^{-6s}. \tag{83}$$

At first, these systems are simulated, and in the next step, they are identified in a noisy context. For all the discussed cases, the sample time and the signal to noise ratio are considered as $1(Sec)$ and 30, respectively. The integral in (78) is numerically evaluated, and all the fractional order integrals and derivatives in all the regression vectors are evaluated by the numerical approximations proposed in Podlubny (1998) and Diethelm et al. (2005). During the identification procedure, the particular assumption is that the fractional order of the system model is known.

4.1 Staircase Input

For the simulation, the triple staircase input, which is previously defined in (23), is supposed as

$$U(s) = \frac{1}{s} + \frac{1}{s} e^{-\ell_1' s} + \frac{1}{s} e^{-\ell_2' s}, \tag{84}$$

where $\ell_2' = 2 \times \ell_1' = 200$ for the non-delayed plant and $\ell_2' = 2 \times \ell_1' = 400$ for the delayed system. The results of applying this kind of non-ideal step input are mentioned in Table 1. Also, to show the effectiveness of the proposed method in identifying the non-delayed and delayed fractional order systems assumed in (82) and (83), the original and the estimated system's output are simultaneously shown in Fig. 3.

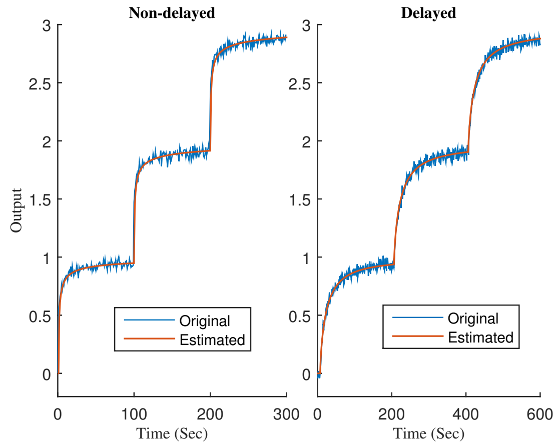

Fig. 3. The original and the estimated output of the non-delayed and delayed fractional order systems by applying the triple staircase input (84)

Table 1. The error of estimation, transient time response (from the beginning to 95 percentage of the steady-state value), and the values of the estimated parameters by applying the triple staircase input (84)

	a	b	ℓ	RMSE	IAE
$G_1(s)$	0.9720	0.9746	–	0.0051	1.1263
$G_2(s)$	9.9939	1.0019	5.8076	0.0084	2.0940

4.2 Saturated Ramp Input

In this part of the simulation, the saturated ramp input, which is previously defined in (66), is considered as

$$U(s) = \frac{5}{150}\frac{1}{s^2} - \frac{5}{150}\frac{1}{s^2} e^{-150s}, \tag{85}$$

in which the saturation level is considered to be equal to 5 and the input is saturated in the time instant $t = 150(Sec)$. The results of applying this kind of non-ideal step input are reported in Table 2. Also, the original and the estimated system's output are simultaneously illustrated in Fig. 4.

Table 2. The error of estimation, transient time response (from the beginning to 95 percentage of the steady-state value), and the values of the estimated parameters by applying the saturated ramp input (85)

	a	b	ℓ	RMSE	IAE
$G_1(s)$	1.0043	1.0034	–	0.0019	0.3790
$G_2(s)$	9.9796	0.9995	5.9189	0.0131	3.4103

4.3 Filtered Step Input

In this subsection, for the simulation, the filtered step input, which is previously defined in (27), is assumed as

$$U(s) = \frac{1}{7s+1}\frac{1}{s}, \tag{86}$$

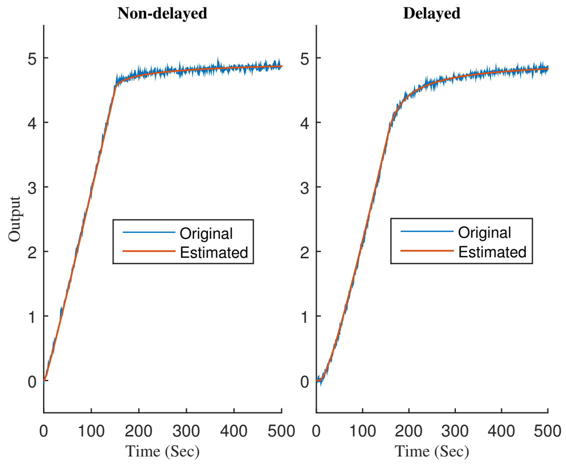

Fig. 4. The original and the estimated output of the non-delayed and delayed fractional order systems by applying the saturated ramp input (85)

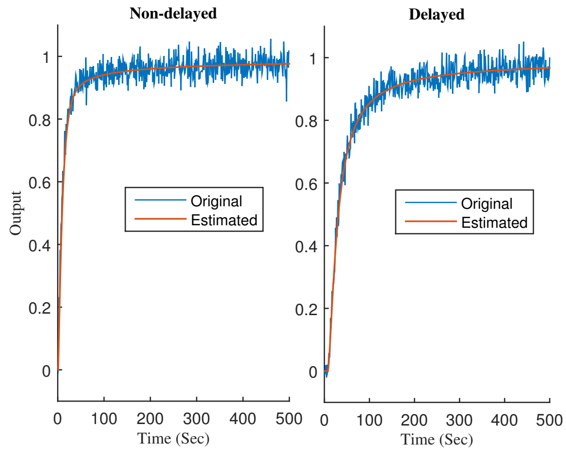

Fig. 5. The original and the estimated output of the non-delayed and delayed fractional order systems by applying the filtered step input (86)

in which the filter coefficient λ and the step size h are assumed to be equal to 7 and 1, respectively. The results of applying this kind of non-ideal step input are summarized in Table 3. The original and the estimated system's output are simultaneously depicted in Fig. 5.

Table 3. The error of estimation, transient time response (from the beginning to 95 percentage of the steady-state value), and the values of the estimated parameters by applying the filtered step input (86)

	a	b	ℓ	RMSE	IAE
$G_1(s)$	1.0059	1.0020	–	0.0045	0.4980
$G_2(s)$	10.2878	1.0003	6.2734	0.0047	1.0195

5. CONCLUSION

In this paper, due to the usage of the non-ideal step inputs in practical applications, three non-ideal step inputs were used for the identification of non-delayed and delayed fractional order systems in a general form. For the identification of non-delayed systems, the fractional order integral approach was developed, and for the identification of delayed systems, a fractional order low-pass filter was used to make the delay term to be explicitly appeared in the parameters vector. Finally, some numerical simulations were provided to demonstrate the effectiveness of the proposed method in estimating the parameters of fractional order systems, including both non-delayed and delayed transfer functions.
Simultaneous estimation of the coefficients and the delay term of unstable delayed fractional order systems using the non-ideal step inputs can be considered as an interesting research topic for future work.

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