Robust Static Output Feedback Stabilization of Continuous-Time Linear Systems via Enhanced LMI Conditions

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Abstract: This paper addresses the problem of Static Output Feedback (SOF) stabilization for continuous-time linear systems subject to norm-bounded parameter uncertainties. Usually this issue leads to the feasibility of a Bilinear Matrix Inequality (BMI), which is difficult to linearize to get non conservative Linear matrix inequality (LMI) conditions. In this paper, by means of some technical lemmas, we transform the BMI into a new LMI with a line search over two scalar variables. The obtained LMI conditions are less conservative than those existing in the literature. Numerical evaluations are presented to show the superiority of the proposed method.

Keywords: Continuous-time Linear systems, Robust control, Parametric uncertainty, Static output feedback, Linear matrix inequality (LMI)

1. INTRODUCTION

Due to technical or economic difficulties, the complete knowledge of the states of the physical system can rarely be measured or available in most cases. Hence, to overcome this problem of missing of information and then to control the system, an observer-based feedback controller or a dynamic output feedback controller or a static output feedback (SOF) controller are designed. Compared with the observer-based control strategy and the dynamic output feedback control method, the SOF controller is known to be simpler and also more easily implemented in practice. For these reasons, the problem of designing a SOF controller for continuous-time (and discrete-time) linear systems has been studied in depth and there are different approaches to solve it. We refer the reader to see, for example, the two survey papers (Syrmos et al., 1997) and (Sadabadi and Peaucelle, 2016) and references therein.

Contrary to its simplicity, there was no exact solution or method for the design of the SOF controller and it is still a theoretically difficult problem in control theory. The design problem of a SOF controller is transformed into a solving problem of a Bilinear Matrix Inequality (BMI). In the literature, numerical iterative techniques, heuristic approaches and two-stage methods have been proposed to solve this BMI problem. See, e.g., (Peaucelle and Arzelier, 2001; Agulhari et al., 2012; Li and Gao, 2014; Hilhorst et al., 2015; Sadabadi and Karimi, 2015; Lee et al., 2016; Kohan-Sedgh et al., 2016) and references inside for further methods and references. To overtake the computational problem, several conditions based on LMIs for the design of SOF controllers have been proposed. In (Crusius and Trofino, 1999), LMI-based conditions subject to an equality constraint were developed for linear systems without parametric uncertainties and with polytopic uncertainties. A similar LMI condition with an equality constraint was proposed also in (Chen et al., 2004). To design an LMI condition in (Chen et al., 2004), the equality constraint is satisfied by imposing a structure constraint on the matrix Lyapunov and also on a matrix variable. Another interesting techniques for continuous-time linear systems with polytopic uncertainties can be found in (Dong and Yang, 2013), as well as for nonlinear systems (Zecevic and Siljak, 2004). In addition, some interesting LMI characterizations were also provided in (Apkarian et al., 2001). In (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004; Zecevic and Siljak, 2004), it was assumed that the output or the input matrices are of full rank.

We proposed recently in (Gritli and Belghith, 2018) a new design approach of the SOF controller for continuous-time linear systems that are subject to norm-bounded parametric uncertainties. We have introduced a new technical Lemma which has allowed us to develop new LMI stability conditions and which have proven to be less conservative than those presented in (Crusius and Trofino, 1999). In the present paper, we present another Lemma that will give us relaxed LMI conditions. This Lemma was based on that provided in (Gritli and Belghith, 2018). In (Gritli and Belghith, 2018), the uncertain linear system was adopted to be with a constant input matrix and then only the state and the output matrices are with parametric uncertainties. In this work, the input matrix is also subject to uncertainties. Via a simulation example taken from (Kheloufi et al., 2013; Gritli and Belghith, 2018), We show that the new stability conditions are found to be less restrictive than those established in (Crusius and Trofino, 1999; Apkarian et al., 2001; Gritli and Belghith, 2018).

Notations: Throughout this paper, we use the following notations. The symbol $(\star)$ in large matrices replaces the term that is induced by symmetry. Moreover, $\mathrm{He}\{\boldsymbol{\mathcal{X}}\} = \boldsymbol{\mathcal{X}} + \boldsymbol{\mathcal{X}}^{\mathrm{T}}$. Furthermore, $\boldsymbol{\mathcal{O}}$ and $\boldsymbol{\mathcal{I}}$ are, respectively, the null matrix and the identity matrix with appropriate dimensions.

2. TECHNICAL LEMMAS AND PROBLEM FORMULATION

In this paper, the main objective is to develop new LMI conditions for computing a SOF controller for the robust stabilization of a continuous-linear linear system subject to norm-bounded parametric uncertainties. Before moving on to the main results, the following technical Lemmas are required.

2.1 Technical Lemmas

In this section, we introduce six useful lemmas, which will be used throughout this paper.

Lemma 1. (Young inequality (Boyd et al., 1994)). Given matrices $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{S}}$ with appropriate dimensions, the following inequality holds:

$$\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{S}}^{\mathrm{T}}\boldsymbol{\mathcal{R}} \leq \epsilon\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}} + \epsilon^{-1}\boldsymbol{\mathcal{S}}^{\mathrm{T}}\boldsymbol{\mathcal{S}} \tag{1}$$

for any positive scalar ϵ.

Lemma 2. (Zemouche et al. (2016)). Let $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{S}}$ two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix $\boldsymbol{\mathcal{Q}}$ of appropriate dimension, the following inequality holds:

$$\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{S}}^{\mathrm{T}}\boldsymbol{\mathcal{R}} \leq \frac{1}{2}\left(\boldsymbol{\mathcal{R}} + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{S}}\right)^{\mathrm{T}}\boldsymbol{\mathcal{Q}}^{-1}\left(\boldsymbol{\mathcal{R}} + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{S}}\right) \tag{2}$$

Lemma 3. (Gritli and Belghith (2018); Chen et al. (2014)). For any matrices $\boldsymbol{\mathcal{M}} \in \mathbb{R}^{r\times s}$ and $\boldsymbol{\mathcal{N}} \in \mathbb{R}^{s\times r}$, and any matrix $\boldsymbol{\mathcal{F}}(t) \in \mathbb{R}^{s\times s}$ satisfying $\boldsymbol{\mathcal{F}}^{\mathrm{T}}(t)\boldsymbol{\mathcal{F}}(t) \leq \boldsymbol{\mathcal{I}}$, and a scalar $\epsilon > 0$, the following inequality holds:

$$\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{F}}(t)\boldsymbol{\mathcal{N}} + \left(\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{F}}(t)\boldsymbol{\mathcal{N}}\right)^{\mathrm{T}} \leq \epsilon^{-1}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}^{\mathrm{T}} + \epsilon\boldsymbol{\mathcal{N}}^{\mathrm{T}}\boldsymbol{\mathcal{N}} \tag{3}$$

Lemma 4. (Gritli and Belghith (2018)). Let $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}^{\mathrm{T}}$, $\boldsymbol{\mathcal{G}} > 0$ (resp. $\boldsymbol{\mathcal{G}} < 0$) and $\boldsymbol{\mathcal{H}}$ be square matrices having the same dimension. The statement (4b) implies (4a).

$$\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}} < 0 \tag{4a}$$

$$\begin{bmatrix} \boldsymbol{\mathcal{Q}} - \mu^{-1}\boldsymbol{\mathcal{G}} & (\star) \\ \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{H}} & \mu^{-1}\boldsymbol{\mathcal{G}} - 2\boldsymbol{\mathcal{I}} \end{bmatrix} < 0 \tag{4b}$$

for all constant $\mu > 0$ (resp. $\mu < 0$).

Lemma 5. Let $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}^{\mathrm{T}}$, $\boldsymbol{\mathcal{G}} > 0$ (resp. $\boldsymbol{\mathcal{G}} < 0$) and $\boldsymbol{\mathcal{H}}$ be square matrices with the same dimension. For all $\mu > 0$ (resp. $\mu < 0$) and η, the statement (5b) implies (5a).

$$\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}} < 0 \tag{5a}$$

$$\begin{bmatrix} \boldsymbol{\mathcal{Q}} - \mu^{-1}\boldsymbol{\mathcal{G}} - \eta\boldsymbol{\mathcal{H}} - \eta\boldsymbol{\mathcal{H}}^{\mathrm{T}} & (\star) & (\star) \\ \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{H}} & \mu^{-1}\boldsymbol{\mathcal{G}} - 2\boldsymbol{\mathcal{I}} & (\star) \\ \dfrac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{G}} \end{bmatrix} < 0 \tag{5b}$$

Proof. Condition (5a) is equal to:

$$\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}} + \mu\boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} - \mu\boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} < 0 \tag{6}$$

for any scalar μ.

As in Lemma 4, we take μ such that $\mu\boldsymbol{\mathcal{G}} > 0$. Using Lemma 4, it is easy to show that the inequality (6) is equivalent to:

$$\begin{bmatrix} \boldsymbol{\mathcal{Q}} - \mu\boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} - \mu^{-1}\boldsymbol{\mathcal{G}} & (\star) \\ \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{H}} & \mu^{-1}\boldsymbol{\mathcal{G}} - 2\boldsymbol{\mathcal{I}} \end{bmatrix} < 0 \tag{7}$$

In addition, since $\mu\boldsymbol{\mathcal{G}} > 0$, Lemma 1 states that for any scalar η, we have:

$$-\mu\boldsymbol{\mathcal{H}}^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{H}} \leq \eta^2\left(\mu\boldsymbol{\mathcal{G}}\right)^{-1} - \eta\boldsymbol{\mathcal{H}} - \eta\boldsymbol{\mathcal{H}}^{\mathrm{T}} \tag{8}$$

Therefore, based on constraint (8), the matrix inequality (7) becomes:

$$\begin{bmatrix} \boldsymbol{\mathcal{Q}} - \mu^{-1}\boldsymbol{\mathcal{G}} + \eta^2\left(\mu\boldsymbol{\mathcal{G}}\right)^{-1} - \eta\boldsymbol{\mathcal{H}} - \eta\boldsymbol{\mathcal{H}}^{\mathrm{T}} & (\star) \\ \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{H}} & \mu^{-1}\boldsymbol{\mathcal{G}} - 2\boldsymbol{\mathcal{I}} \end{bmatrix} < 0 \tag{9}$$

As $\eta^2\left(\mu\boldsymbol{\mathcal{G}}\right)^{-1} = \mu^{-2}\eta^2\left(\mu^{-1}\boldsymbol{\mathcal{G}}\right)^{-1}$, then via the Schur complement (Boyd et al., 1994), the matrix inequality (9) is equivalent to the matrix inequality (5b). This completes then the proof of Lemma 5.

2.2 Problem formulation

We consider in the present work continuous-time, uncertain, linear systems described by the following equations:

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{\mathcal{A}} + \boldsymbol{\Delta\mathcal{A}}(t)\right)\boldsymbol{x} + \left(\boldsymbol{\mathcal{B}} + \boldsymbol{\Delta\mathcal{B}}(t)\right)\boldsymbol{u} \tag{10a}$$

$$\boldsymbol{y} = \left(\boldsymbol{\mathcal{C}} + \boldsymbol{\Delta\mathcal{C}}(t)\right)\boldsymbol{x} \tag{10b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{y} \in \mathbb{R}^p$ is the output vector, and $\boldsymbol{u} \in \mathbb{R}^m$ is the control input vector. Moreover, in (10), $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ are constant matrices. Furthermore, $\boldsymbol{\Delta\mathcal{A}}(t)$, $\boldsymbol{\Delta\mathcal{B}}(t)$ and $\boldsymbol{\Delta\mathcal{C}}(t)$ are matrices incorporating uncertain parameters.

We assume that the continuous-time linear system in (10) is stabilizable. Furthermore, in the uncertain linear system (10), the uncertainties matrices $\boldsymbol{\Delta\mathcal{A}}(t)$, $\boldsymbol{\Delta\mathcal{B}}(t)$ and $\boldsymbol{\Delta\mathcal{C}}(t)$ are defined like so:

$$\boldsymbol{\Delta\mathcal{A}}(t) = \boldsymbol{\mathcal{M}}_1\boldsymbol{\mathcal{F}}_1(t)\boldsymbol{\mathcal{N}}_1 \tag{11a}$$

$$\boldsymbol{\Delta\mathcal{B}}(t) = \boldsymbol{\mathcal{M}}_2\boldsymbol{\mathcal{F}}_2(t)\boldsymbol{\mathcal{N}}_2 \tag{11b}$$

$$\boldsymbol{\Delta\mathcal{C}}(t) = \boldsymbol{\mathcal{M}}_3\boldsymbol{\mathcal{F}}_3(t)\boldsymbol{\mathcal{N}}_3 \tag{11c}$$

where the matrices $\boldsymbol{\mathcal{M}}_1$, $\boldsymbol{\mathcal{M}}_2$, $\boldsymbol{\mathcal{M}}_3$, $\boldsymbol{\mathcal{N}}_1$, $\boldsymbol{\mathcal{N}}_2$, $\boldsymbol{\mathcal{N}}_3$, $\boldsymbol{\mathcal{F}}_1(t)$, $\boldsymbol{\mathcal{F}}_2(t)$ and $\boldsymbol{\mathcal{F}}_3(t)$ are with appropriate dimensions. In addition, the uncertainties matrices $\boldsymbol{\mathcal{F}}_1(t)$, $\boldsymbol{\mathcal{F}}_2(t)$ and $\boldsymbol{\mathcal{F}}_3(t)$ are assumed to satisfy the following conditions:

$$\boldsymbol{\mathcal{F}}_i^{\mathrm{T}}(t)\,\boldsymbol{\mathcal{F}}_i(t) \leq \boldsymbol{\mathcal{I}}, \qquad \text{for} \qquad i = 1,2,3 \tag{12}$$

The objective of the present work is to design a SOF control law of the following form:

$$\boldsymbol{u} = \boldsymbol{\mathcal{K}}\boldsymbol{y} \tag{13}$$

in order to robustly stabilize the continuous-time uncertain linear system (10) subject to the norm-bounded parametric uncertainties in (11) and under conditions in (12). In expression (13), the matrix $\boldsymbol{\mathcal{K}} \in \mathbb{R}^{m\times p}$ is the SOF feedback gain of the controller to be designed.

By introducing the SOF control law (13), the uncertain continuous linear system (10) in closed loop becomes:

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{\mathcal{A}} + \boldsymbol{\Delta\mathcal{A}}(t) + \left(\boldsymbol{\mathcal{B}} + \boldsymbol{\Delta\mathcal{B}}(t)\right)\boldsymbol{\mathcal{K}}\left(\boldsymbol{\mathcal{C}} + \boldsymbol{\Delta\mathcal{C}}(t)\right)\right)\boldsymbol{x} \tag{14}$$

By considering the Lyapunov function $\mathcal{V}(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{P}}\boldsymbol{x}$, with $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}^{\mathrm{T}}$, the sufficient condition ensuring the asymptotic stabilizability of the closed-loop linear system (14) is defined by the following $\boldsymbol{\mathcal{P}}$-problem:

$$\boldsymbol{\mathcal{P}} > 0 \tag{15a}$$

$$\begin{aligned}&\mathrm{He}\left\{\boldsymbol{\mathcal{P}}\left(\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\right) + \boldsymbol{\mathcal{P}}\left(\boldsymbol{\Delta\mathcal{A}}(t) + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}(t)\right.\right.\\ &\left.\left.+\boldsymbol{\Delta\mathcal{B}}(t)\,\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} + \boldsymbol{\Delta\mathcal{B}}(t)\,\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}(t)\right)\right\} < 0\end{aligned} \tag{15b}$$

The continuous-time uncertain linear system (10) subject to norm-bounded parametric uncertainties, is stabilizable via the SOF control law (13) if and only if there exist two matrices $\boldsymbol{\mathcal{P}} > 0$ and $\boldsymbol{\mathcal{K}}$, of appropriate dimensions, such that the $\boldsymbol{\mathcal{P}}$-problem in (15) is feasible.

In order to synthesize traceable LMI stability conditions through the $\boldsymbol{\mathcal{P}}$-problem in (15), we will consider first in this paper the uncertainty-free case, for which the linear system (10) without parametric uncertainties, $\boldsymbol{\Delta\mathcal{A}} = \boldsymbol{\mathcal{O}}_{n\times n}$, $\boldsymbol{\Delta\mathcal{B}} = \boldsymbol{\mathcal{O}}_{n\times m}$ and $\boldsymbol{\Delta\mathcal{C}} = \boldsymbol{\mathcal{O}}_{p\times n}$, and then we have the following uncertain linear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{\mathcal{A}}\boldsymbol{x} + \boldsymbol{\mathcal{B}}\boldsymbol{u} \tag{16a}$$

$$\boldsymbol{y} = \boldsymbol{\mathcal{C}}\boldsymbol{x} \tag{16b}$$

Our first objective in this work is to develop LMI stability conditions for the certain linear system (16) under the SOF controller (13). Thus, for the uncertainty-free case, the $\boldsymbol{\mathcal{P}}$-problem in (15) is recast as:

$$\boldsymbol{\mathcal{P}} > 0 \tag{17a}$$

$$\boldsymbol{\mathcal{P}}\left(\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\right) + \left(\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\right)^{\mathrm{T}}\boldsymbol{\mathcal{P}} < 0 \tag{17b}$$

2.3 Some Comments

Notice that the matrix inequality (17b) is a BMI and then the $\boldsymbol{\mathcal{P}}$-problem in (15) forms a BMI setting. Nevertheless, such BMI problem is not numerically exploitable to solve for the two unknown matrices $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{K}}$. Then, transforming such BMI problem into an LMI problem is very difficult due to the presence of the coupling term $\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}$ in (17b). Many researchers have been attempted to overcome this problem and then compute the feedback gain $\boldsymbol{\mathcal{K}}$, but the adopted methods remain quite conservative due to the presence of certain conditions imposed on the structure of the input and output matrices or the presence of an equality constraint (Syrmos et al., 1997; Crusius and Trofino, 1999; Chen et al., 2004; Dong and Yang, 2007, 2013; Lee et al., 2016; Sadabadi and Peaucelle, 2016; Apkarian et al., 2001).

Our main contribution in this article consists in developing new LMI stability conditions by transforming the $\boldsymbol{\mathcal{P}}$-problem in (17) into an LMI condition. Thus, we will compare the obtained results with the existing ones in (Crusius and Trofino, 1999; Apkarian et al., 2001) and also with the previous results in (Gritli and Belghith, 2018). We stress that in (Crusius and Trofino, 1999), the design of the SOF controller was carried out by considering mainly two cases: the case without uncertainty and the case with polytopic uncertainty. The case of norm-bounded parametric uncertainties was not considered and then solved in (Crusius and Trofino, 1999). In (Apkarian et al., 2001), a linear system with polytopic uncertainties and the dynamic output feedback controller were considered. Recently in (Gritli and Belghith, 2018), we developed new LMI stability conditions for the certain linear system (16) and the uncertain one (10) using Lemma 4.

It is worth mentioning that the LMI-based approaches used in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004), either the input matrix $\boldsymbol{\mathcal{B}}$ or the output matrix $\boldsymbol{\mathcal{C}}$ should be of full rank in order to obtain a solution for the stabilization problem. However, if both $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ are not of full rank, then it is not possible to solve the stabilization problem and the LMI-based conditions in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004) are not valid. The feasibility problem of the LMI-based conditions proposed in (Crusius and Trofino, 1999) was discussed in (Gritli and Belghith, 2018). Such rank problem was solved in (Gritli and Belghith, 2018) and no constraint was imposed.

In (Gritli and Belghith, 2018), we considered in system (10) norm-bounded parametric uncertainties only in the state matrix and the output matrix, and then $\boldsymbol{\Delta\mathcal{B}}(t) = \boldsymbol{\mathcal{O}}$. Under this situation, only one case is possible for the linear system (10) in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004). Indeed, as the matrix $\boldsymbol{\mathcal{B}}$ is without uncertainty, then according to the results in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004), the matrix $\boldsymbol{\mathcal{B}}$ should be of full column rank in order to establish some transformations and then linearization of the BMI (15). Then, if the matrix $\boldsymbol{\mathcal{B}}$ is subject to norm-bounded parametric uncertainties, these transformations and methods proposed in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004) are not valid.

In the present paper, we will develop enhanced LMI stability conditions using Lemma 5 for the robust stabilization of the uncertain linear system (10) under the SOF controller (13). We will consider first the uncertainty-free case and then conditions in (17). After that, we will extend the obtained results for the case of norm-bounded parametric uncertainties using the conditions in (15).

3. ENHANCED LMI CONDITIONS

In this part, we present our main results and contributions for the design of new LMI conditions and then the robust

SOF controller (13) allowing the robust stabilization of the continuous-time linear system (10) under the SOF controller (13). We will use Lemma 5 to design these LMI conditions. Thus, we begin by developing LMI stability conditions in the uncertainty-free case, and then the linear system (16) using the stability condition in the $\boldsymbol{\mathcal{P}}$-problem (15).

3.1 Uncertainty-Free Case

Let us consider the linear system (16) with the controller (13) and its stability conditions (17), for which (17b) is a BMI. As indicated above, the problem is due to the presence of the coupling term $\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}$ in (17b). To solve this problem, we have to decouple $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{K}}$ from each other. In what follows, we present our main results to solve this problem in the uncertainty-free case using mainly Lemma 5.

Theorem 6. If, for two constants $\mu > 0$ and η fixed *a priori*, there exist two matrices $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{K}}$ of adequate dimensions, solutions of the following LMI:

$$\begin{bmatrix} \mathrm{He}\{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}} - \eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\} - \mu^{-1}\boldsymbol{\mathcal{P}} & (\star) & (\star) \\ \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} & \mu^{-1}\boldsymbol{\mathcal{P}} - 2\boldsymbol{\mathcal{I}} & (\star) \\ \frac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{P}} \end{bmatrix} < 0 \tag{18}$$

Therefore, the SOF control law (13) stabilizes the certain continuous-time linear system (16).

Proof. Let us consider Lemma 5 and making the following change of variables: $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}$. Then, by means of Lemma 5, we obtain LMI (18). We stress that the BMI (15b) with the condition (15a) hold if the LMI (18) is feasible. This ends the proof of Theorem 6.

3.2 Norm-Bounded Uncertainty Case

We present now an improved version of the LMI stability condition developed in (Gritli and Belghith, 2018) for the continuous-time linear system (10) subject to norm-bounded parametric uncertainties. This new condition is constructed via the LMI (18) in Theorem 6.

Theorem 7. Consider the uncertain continuous-time linear system (10) with norm-bounded parametric uncertainties $\boldsymbol{\Delta\mathcal{A}}$, $\boldsymbol{\Delta\mathcal{B}}$ and $\boldsymbol{\Delta\mathcal{C}}$ satisfying (11a)-(11c) and (12). If, for two scalars $\mu > 0$ and η fixed *a priori*, there exist matrices $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{K}}$ of adequate dimensions, and scalars ϵ_1, ϵ_2 and ϵ_3, solutions of the following LMI:

$$\begin{bmatrix} \sum_{11} & (\star) & (\star) & (\star) & \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{M}}_1 & (\star) & -\eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{M}}_3 \\ \sum_{21} & \sum_{22} & (\star) & (\star) & \boldsymbol{\mathcal{O}} & (\star) & \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{M}}_3 \\ \frac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{P}} & \boldsymbol{\mathcal{O}} & (\star) & \boldsymbol{\mathcal{O}} & \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & -2\boldsymbol{\mathcal{Q}} & \boldsymbol{\mathcal{O}} & (\star) & \boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{M}}_3 \\ (\star) & (\star) & (\star) & (\star) & -\epsilon_1\boldsymbol{\mathcal{I}} & (\star) & \boldsymbol{\mathcal{O}} \\ \boldsymbol{\mathcal{N}}_2\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{N}}_2\boldsymbol{\mathcal{Q}} & \boldsymbol{\mathcal{O}} & -\epsilon_2\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\epsilon_3\boldsymbol{\mathcal{I}} \end{bmatrix} < 0 \tag{19}$$

with $\sum_{11} = -\mu^{-1}\boldsymbol{\mathcal{P}} + \mathrm{He}\{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}} - \eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\} + \epsilon_1\boldsymbol{\mathcal{N}}_1^{\mathrm{T}}\boldsymbol{\mathcal{N}}_1 + \epsilon_2\eta^2\boldsymbol{\mathcal{M}}_2\boldsymbol{\mathcal{M}}_2^{\mathrm{T}} + \epsilon_3\boldsymbol{\mathcal{N}}_3^{\mathrm{T}}\boldsymbol{\mathcal{N}}_3$, $\sum_{21} = \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} - \epsilon_2\mu\eta\boldsymbol{\mathcal{M}}_2\boldsymbol{\mathcal{M}}_2^{\mathrm{T}}$ and $\sum_{22} = \mu^{-1}\boldsymbol{\mathcal{P}} - 2\boldsymbol{\mathcal{I}} + \epsilon_2\mu^2\boldsymbol{\mathcal{M}}_2\boldsymbol{\mathcal{M}}_2^{\mathrm{T}}$.

Then, the SOF control law (13) robustly stabilizes the uncertain linear system (10).

Proof. Let us consider the LMI (18) in Theorem 6. Then, by considering the uncertainties matrices $\boldsymbol{\Delta\mathcal{A}}$, $\boldsymbol{\Delta\mathcal{B}}$ and $\boldsymbol{\Delta\mathcal{C}}$, the LMI (18) becomes:

$$\begin{bmatrix} \Delta_{11}^1 + \Delta_{11}^2 & (\star) & (\star) \\ \Delta_{21}^1 + \Delta_{21}^2 & \mu^{-1}\boldsymbol{\mathcal{P}} - 2\boldsymbol{\mathcal{I}} & (\star) \\ \frac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} \mathrm{He}\{-\eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}\} & (\star) & (\star) \\ \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} < 0 \tag{20}$$

with $\Delta_{11}^1 = -\mu^{-1}\boldsymbol{\mathcal{P}} + \mathrm{He}\{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}} - \eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\}$, $\Delta_{11}^2 = \mathrm{He}\{\boldsymbol{\mathcal{P}}\boldsymbol{\Delta\mathcal{A}} - \eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} - \eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}\}$, $\Delta_{21}^1 = \boldsymbol{\mathcal{I}} + \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}$, and $\Delta_{21}^2 = \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} + \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}$.

The last right matrix in the left-hand side of the expression (20) is rewritten like so:

$$\begin{bmatrix} \mathrm{He}\{-\eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}\} & (\star) & (\star) \\ \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} = \mathrm{He}\left\{ \begin{bmatrix} -\eta\boldsymbol{\Delta\mathcal{B}} \\ \mu\boldsymbol{\Delta\mathcal{B}} \\ \boldsymbol{\mathcal{O}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} \right\} \tag{21}$$

Relying on Lemma 2, we obtain, for all symmetric matrix $\boldsymbol{\mathcal{Q}} > 0$, the following inequality:

$$\mathrm{He}\left\{ \begin{bmatrix} -\eta\boldsymbol{\Delta\mathcal{B}} \\ \mu\boldsymbol{\Delta\mathcal{B}} \\ \boldsymbol{\mathcal{O}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} \right\} \leq \frac{1}{2} \begin{bmatrix} (\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}})^{\mathrm{T}} - \eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{O}} \end{bmatrix} \boldsymbol{\mathcal{Q}}^{-1} \begin{bmatrix} (\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}})^{\mathrm{T}} - \eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{O}} \end{bmatrix}^{\mathrm{T}} \tag{22}$$

Accordingly, by means of the Schur lemma, the matrix inequality (20) is equivalent to:

$$\begin{bmatrix} \Delta_{11}^1 + \Delta_{11}^2 & (\star) & (\star) & (\star) \\ \Delta_{21}^1 + \Delta_{21}^2 & \mu^{-1}\boldsymbol{\mathcal{P}} - 2\boldsymbol{\mathcal{I}} & (\star) & (\star) \\ \frac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{P}} & (\star) \\ \boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} - (\eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}})^{\mathrm{T}} & (\mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}})^{\mathrm{T}} & \boldsymbol{\mathcal{O}} & -2\boldsymbol{\mathcal{Q}} \end{bmatrix} < 0 \tag{23}$$

Rearranging the matrix inequality (23) as follows:

$$\begin{bmatrix} \Delta_{11}^1 & (\star) & (\star) & (\star) \\ \Delta_{21}^1 & \mu^{-1}\boldsymbol{\mathcal{P}} - 2\boldsymbol{\mathcal{I}} & (\star) & (\star) \\ \frac{\eta}{\mu}\boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{O}} & -\mu^{-1}\boldsymbol{\mathcal{P}} & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & -2\boldsymbol{\mathcal{Q}} \end{bmatrix} + \begin{bmatrix} \mathrm{He}\{\boldsymbol{\mathcal{P}}\boldsymbol{\Delta\mathcal{A}}\} & (\star) & (\star) & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & (\star) & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} + \begin{bmatrix} \mathrm{He}\{-\eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}}\} & (\star) & (\star) & -\eta\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{O}} & (\star) & \mu\boldsymbol{\Delta\mathcal{B}}\boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \\ (\star) & (\star) & (\star) & \boldsymbol{\mathcal{O}} \end{bmatrix} + \begin{bmatrix} \mathrm{He}\{-\eta\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}}\} & (\star) & (\star) & (\star) \\ \mu\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & (\star) & (\star) \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & (\star) \\ \boldsymbol{\mathcal{K}}\boldsymbol{\Delta\mathcal{C}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{O}} \end{bmatrix} < 0 \tag{24}$$

By substituting expressions of uncertainty matrices defined in (11) with conditions in (12), and relying on the Young inequality in Lemma 1, we derive the following relations:

$$\begin{bmatrix} \mathrm{He}\{\mathcal{P}\Delta\mathcal{A}\} & (\star) & (\star) & (\star) \\ \mathcal{O} & \mathcal{O} & (\star) & (\star) \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & (\star) \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \leq$$

$$\epsilon_1^{-1} \begin{bmatrix} \mathcal{P}\mathcal{M}_1 \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{P}\mathcal{M}_1 \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix}^{\mathrm{T}} + \epsilon_1 \begin{bmatrix} \mathcal{N}_1^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{N}_1^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix}^{\mathrm{T}} \tag{25}$$

$$\begin{bmatrix} \mathrm{He}\{-\eta\Delta\mathcal{B}\mathcal{K}\mathcal{C}\} & (\star) & (\star) & -\eta\Delta\mathcal{B}\mathcal{Q} \\ \mu\Delta\mathcal{B}\mathcal{K}\mathcal{C} & \mathcal{O} & (\star) & \Delta\mathcal{B}\mathcal{Q} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ (\star) & (\star) & (\star) & \mathcal{O} \end{bmatrix} \leq$$

$$\epsilon_2 \begin{bmatrix} -\eta\mathcal{M}_2 \\ \mu\mathcal{M}_2 \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix} \begin{bmatrix} -\eta\mathcal{M}_2 \\ \mu\mathcal{M}_2 \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix}^{\mathrm{T}} + \epsilon_2^{-1} \begin{bmatrix} (\mathcal{N}_2\mathcal{K}\mathcal{C})^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ (\mathcal{N}_2\mathcal{Q})^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} (\mathcal{N}_2\mathcal{K}\mathcal{C})^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ (\mathcal{N}_2\mathcal{Q})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{26}$$

$$\begin{bmatrix} \mathrm{He}\{-\eta\mathcal{B}\mathcal{K}\Delta\mathcal{C}\} & (\star) & (\star) & (\star) \\ \mu\mathcal{B}\mathcal{K}\Delta\mathcal{C} & \mathcal{O} & (\star) & (\star) \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & (\star) \\ \mathcal{K}\Delta\mathcal{C} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \leq$$

$$\epsilon_3^{-1} \begin{bmatrix} -\eta\mathcal{B}\mathcal{K}\mathcal{M}_3 \\ \mu\mathcal{B}\mathcal{K}\mathcal{M}_3 \\ \mathcal{O} \\ \mathcal{K}\mathcal{M}_3 \end{bmatrix} \begin{bmatrix} -\eta\mathcal{B}\mathcal{K}\mathcal{M}_3 \\ \mu\mathcal{B}\mathcal{K}\mathcal{M}_3 \\ \mathcal{O} \\ \mathcal{K}\mathcal{M}_3 \end{bmatrix}^{\mathrm{T}} + \epsilon_3 \begin{bmatrix} \mathcal{N}_3^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{N}_3^{\mathrm{T}} \\ \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix}^{\mathrm{T}} \tag{27}$$

Finally, by substituting these previous inequalities in the matrix inequality (24), and relying on Schur complement Lemma, we obtain the LMI (19). Hence, the proof of Theorem 7 is complete.

4. NUMERICAL RESULTS

In this section, we provide some numerical results showing the superiority of the new established LMI conditions (19) in Theorem 7 for the design of the SOF controller (13) to robustly stabilize the uncertain linear system (10). The proposed LMI-based methodology is compared with respect to the results in (Crusius and Trofino, 1999), (Apkarian et al., 2001), and (Gritli and Belghith, 2018) are presented.

4.1 Conservatism Evaluation

❶ *Case* $\Delta\mathcal{B}(t) = \mathcal{O}$

As noted previously, the methods proposed in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004) consider the case $\Delta\mathcal{B}(t) = \mathcal{O}$. Then, in order to establish comparisons with such methods, we need to take $\Delta\mathcal{B}(t) = \mathcal{O}$ in the uncertain linear system (10).

We take the same example in (Kheloufi et al., 2013; Gritli and Belghith, 2018). The system has the following parameters:

$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & -2 & -5 \end{bmatrix}$, $\Delta\mathcal{A} = \begin{bmatrix} 0 & 0 & a(t) \\ 0 & b(t) & 0 \\ c(t) & 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathcal{C} = [\,1\ 0\ 1\,]$, $\Delta\mathcal{C} = [\,0\ d(t)\ 0\,]$, where

$$a(t) \leq \alpha \qquad b(t) \leq \beta \qquad c(t) \leq \gamma \qquad d(t) \leq \delta \tag{28}$$

The two uncertainties matrices $\Delta\mathcal{A}(t)$ and $\Delta\mathcal{C}(t)$ can be rewritten under the form (11) with:

$\mathcal{F}_1(t) = \begin{bmatrix} \frac{a(t)}{\alpha} & 0 & 0 \\ 0 & \frac{b(t)}{\beta} & 0 \\ 0 & 0 & \frac{c(t)}{\gamma} \end{bmatrix}$, $\mathcal{N}_1 = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & \beta & 0 \\ \gamma & 0 & 0 \end{bmatrix}$, $\mathcal{F}_3(t) = \frac{d(t)}{\delta}$, $\mathcal{N}_3 = [\,0\ \delta\ 0\,]$, $\mathcal{M}_1 = \mathcal{I}$ and $\mathcal{M}_3 = 1$.

As in (Gritli and Belghith, 2018), we take $\alpha = \beta = \gamma = \delta$. Under this situation, we test in Table 1 the feasibility of the LMI-based problem established in (Crusius and Trofino, 1999) augmented with the parametric uncertainties (see LMI (41) and equality constraint (22) in (Gritli and Belghith, 2018)), the LMI condition (33) designed in (Gritli and Belghith, 2018), and the LMI condition (19) developed in the present work. We used also some results in (Apkarian et al., 2001) to design also an LMI stability condition (In fact, this LMI condition is not developed in this paper for the lack of space). Thus, the objective is to look for the maximum value of α, $\alpha_{\max}$, that should be tolerated by each method.

Table 1 shows the superiority of the proposed enhanced LMI conditions compared with the results by (Crusius and Trofino, 1999), (Apkarian et al., 2001), and (Gritli and Belghith, 2018).

❷ *Case* $\Delta\mathcal{B}(t) \neq \mathcal{O}$

In this second case, we consider the presence of norm-bounded parametric uncertainties in the input matrix in the system (10), and then we have $\Delta\mathcal{B}(t) \neq \mathcal{O}$. Then, we take $\Delta\mathcal{B}(t) = \begin{bmatrix} d(t) & 0 \\ 0 & 0 \\ 0 & e(t) \end{bmatrix}$, where

$$d(t) \leq \xi \qquad\qquad e(t) \leq \zeta$$

The uncertainty matrix $\Delta\mathcal{B}(t)$ can be rewritten under the form (11b) and condition (12) with:

$$\mathcal{F}_2(t) = \begin{bmatrix} \frac{d(t)}{\xi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{e(t)}{\zeta} \end{bmatrix}, \quad \mathcal{M}_2 = \mathcal{I}, \quad \mathcal{N}_2 = \begin{bmatrix} \xi & 0 \\ 0 & 0 \\ 0 & \zeta \end{bmatrix} \tag{29}$$

For simplicity, we take also $\alpha = \beta = \gamma = \delta = \xi = \zeta$. We recall that the proposed methods in (Crusius and Trofino, 1999; Apkarian et al., 2001; Chen et al., 2004) are not valid in the present case, i.e. when $\Delta\mathcal{B}(t) \neq \mathcal{O}$.

As previously, we tested the feasibility of the LMI (19). We found that this LMI is feasible for all $\alpha_{\max} \leq 0.3697$. This result is obtained for $\mu = 0.02$ and $\eta = -0.0001$.

4.2 Simulation

Using the previous numerical results, we take $\alpha_{\max} = 0.3697$. For this value, the SOF gain is computed to be $\mathcal{K} = [\,-3.4713\ \ -0.8541\,]^{\mathrm{T}}$. We plotted in Fig. 1 the temporal evolution of the three states, i.e. x_1, x_2 and x_3, of the

Table 1. Superiority of our relaxed LMIs for the design of the SOF gain

Method	Crusius and Trofino (1999)	Apkarian et al. (2001)	Gritli and Belghith (2018)	LMI (19)
			$\mu = 0.2150$	$\mu = 0.33$ & $\eta = -1/27$
$\alpha_{\max}$	0.4146	0.7559	0.7668	0.8271

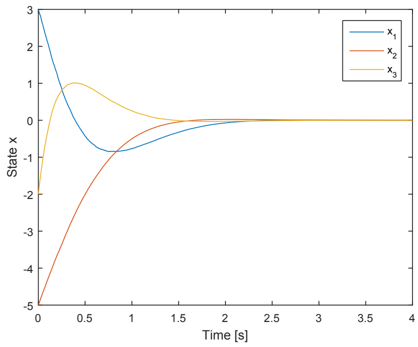

Fig. 1. Temporal evolution of the three states x_1, x_2 and x_3 in the presence of norm-bounded parametric uncertainties with $\alpha_{\max} = 0.3697$.

uncertain linear system (10) under the SOF controller (13). It is obvious that the three states of the uncertain system are robustly stabilized at the zero-equilibrium point.

5. CONCLUSION

In this paper, a SOF controller for continuous-time linear systems under norm-bounded parametric uncertainties is proposed and synthesized by solving LMI conditions easily tractable by numerical softwares. Based on a new lemma and convenient matrix decompositions, we established a relaxed LMI condition, where two cases are studied: first, the uncertainty-free case, and then by considering the norm-bounded uncertainties. It is shown also that the new LMI conditions, at least via a single simulation example, are less conservative than those available in the literature, namely the results in (Crusius and Trofino, 1999; Apkarian et al., 2001) and (Gritli and Belghith, 2018).

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