

# Stability of Switched Differential Repetitive Processes and Iterative Learning Control Design <sup>★</sup>

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**Abstract:** In this paper general stability conditions of differential nonlinear repetitive processes with switching are obtained. The approach is based on the development of a method that uses vector Lyapunov functions and the properties of the counterpart of its divergence. The obtained results are applied to iterative learning control design for switched linear system. An example that demonstrate effectiveness of the new design is given.

*Keywords:* 2D systems, switched systems, differential repetitive processes, stability, vector Lyapunov function, iterative learning control.

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## 1. INTRODUCTION

Switched systems are a class of dynamical systems consisting of a finite number of subsystems and some logical rule of switching between them. Usually, such subsystems are described by a set of coupled differential or difference equations. This class of systems has been intensively studied in the past decades. The interest in these systems has been motivated by the fact that many engineering systems may be modeled using such a framework. Examples of such systems include multiple-model switched systems, hybrid control systems, event-triggered systems and some others. For this class of systems, the stability and stabilization theory contains many interesting and important results. The reader is addressed to the book Liberzon (2003) as a starting point for the literature in this field of investigations and also to the survey papers (Shorten et al., 2007; Lin and Antsaklis, 2009) and to the recent books (Sun and Ge, 2011; Alwan and Liu, 2018).

Repetitive processes are a class of 2D systems that have important role in engineering, especially in theory and applications of iterative learning control (ILC) (Rogers et al., 2007; Hladowski et al., 2010; Paszke et al., 2013; Bolder and Oomen, 2016). Since the pioneering publication by Arimoto et al. (1984) ILC has been an established area of research. The results reported include theory of control law design, experimental verification on laboratory test beds and implementation in industrial applications. The survey papers (Bristow et al., 2006; Ahn et al., 2007) are one starting point for the literature. More recent applications with experimental verification include laser metal deposition, e.g., (Sammons et al., 2019). Also, ILC designs were successfully used in robotic-assisted stroke rehabilitation with supporting clinical trials (Freeman et al., 2012; Meadmore et al., 2014).

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The linear repetitive processes with switching were considered in Bochniak et al. (2006, 2008). These papers were motivated by practical problems of control engineering such as metal rolling, where a metal strip of finite length is shaped by passing through different sets of rolls and the output from one pass forms the input to the next one. In Bochniak et al. (2008), such systems were modelled by linear repetitive processes with switched dynamics. In both papers cited, special switching rules were considered. The final results were control law design algorithms that can be implemented using LMI based computations.

Other 2D systems with switching were considered in Wua et al. (2015); Xu and Zhu (2019); Tian et al. (2019); Yang and Yu (2019). In Wua et al. (2015), the problems of stability analysis and stabilization were investigated for discrete-time 2D switched systems, in form of the linear Fornasini–Marchesini local state-space model. The cases of constrained and restricted switching were considered. In Xu and Zhu (2019), a class of linear discrete-time 2D switched systems in the Fornasini–Marchesini form with multiplicative noise was considered; sufficient conditions of stochastic exponential stability under arbitrary and constrained switching signals were established. In Tian et al. (2019), the stability of 2D switched positive nonlinear systems in the Roesser model with time-varying delays was investigated. In Yang and Yu (2019), the stability analysis and control design problems for the discrete-time 2D switched systems were studied via an event-triggered scheme. The addressed 2D system was represented by the Fornasini–Marchesini local state-space model operating in several modes under a switching signal.

In contrast to the works mentioned above and references therein, this paper considers nonlinear differential repetitive processes with switching. This class of systems was not considered in current literature, despite the fact that such systems play a major role in control engineering practice,

especially in iterative learning control problems. General exponential stability conditions in terms of vector Lyapunov functions are obtained for the cases of constrained and arbitrary switching. Application of these results to iterative learning control design of linear switched system is considered. In contrast with Bochniak et al. (2006, 2008) different switching rule is used. An example of the rotary flexible link model Apkarian et al. (2011) with mode switching is also considered. Both switched and non-switched iterative learning control laws are obtained and compared with each other.

## 2. GENERAL STABILITY CONDITIONS FOR DIFFERENTIAL REPETITIVE PROCESSES WITH SWITCHING

Consider differential nonlinear repetitive processes with a pass length  $T < \infty$  described over  $0 \leq t \leq T$  by the state-space model

$$\dot{x}_{k+1}(t) = f_{1,\sigma(k)}(x_{k+1}(t), y_k(t)), \quad (1)$$

$$y_{k+1}(t) = f_{2,\sigma(k)}(x_{k+1}(t), y_k(t)), \quad k = 0, 1, \dots, \quad (2)$$

where on pass  $k$ ,  $x_k(t) \in \mathbb{R}^{n_x}$  is the state vector,  $y_k(t) \in \mathbb{R}^{n_y}$  is the pass profile vector;  $f_{1,\sigma(k)}$  and  $f_{2,\sigma(k)}$  are nonlinear functions such that  $f_{1,\sigma(k)}(0, 0) = 0$  and  $f_{2,\sigma(k)}(0, 0) = 0$   $k = 0, 1, 2, \dots$ ; in accordance with the standard notation of switched systems theory  $\sigma(k)$  represents a switching signal in the pass direction that is defined as a piecewise constant mapping from the nonnegative integers  $\mathbb{Z}^+$  into a finite index set  $\mathcal{N} = \{0, 1, 2, \dots, N\}$ . Also define the switching times in pass direction as the pass numbers  $k_0, k_1, k_2, \dots$ , on which the pass vector changes mode. In other words, at each pass  $k$  the switching signal specifies the index  $\sigma(k) \in \mathcal{N}$  of the active subsystem, i.e., if  $\sigma(k) = i$  then the subsystem  $i$  is active at the instant  $k$  and model (1), (2) can also be represented as a family of subsystems with given switching rule between them:

$$\begin{aligned} \dot{x}_{k+1}(t) &= f_{1i}(x_{k+1}(t), y_k(t)), \\ y_{k+1}(t) &= f_{2i}(x_{k+1}(t), y_k(t)), \quad i \in \mathcal{N}. \end{aligned}$$

It is assumed that mode switching occurs at the beginning of a corresponding pass. The boundary conditions, that is, the pass state initial vector sequence and the initial pass profile, have the form

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \quad |d_{k+1}|^2 \leq \kappa_d \lambda_d^{k+1}, \\ y_0(t) &= f(t), \quad |f(t)|^2 \leq M_f, \quad 0 \leq t \leq T, \quad k = 0, 1, \dots, \end{aligned} \quad (3)$$

where the entries in  $d_{k+1} \in \mathbb{R}^{n_x}$  are known constants; the entries in  $f(t) \in \mathbb{R}^{n_y}$  are known functions of  $t$ ;  $\kappa_d > 0$  and  $0 < \lambda_d < 1$ ; finally,  $|q|$  denotes the Euclidian norm of a vector  $q$ . It is also assumed that the function  $f_{1i}$  satisfies the Lipschitz condition, i.e.,

$$\begin{aligned} |f_{1i}(x', y') - f_{1i}(x'', y'')| &\leq L(|x' - x''| + |y' - y''|), \\ x', x'' &\in \mathbb{R}^{n_x}, \quad y', y'' \in \mathbb{R}^{n_y}, \quad i \in \mathcal{N}. \end{aligned} \quad (4)$$

Denote by  $N_\sigma(k_f, k_s)$  the number of switchings of  $\sigma$  on an interval  $(k_s, k_f)$  and define the average dwell time as follows.

**Definition 1.** A positive constant  $\kappa_a \in \mathbb{Z}^+$  is called the average dwell time for switching signal in pass direction  $\sigma$  if inequality

$$N_\sigma(k_f, k_s) \leq N_0 + \frac{k_f - k_s}{\kappa_a} \quad (5)$$

holds for  $k_f \geq k_s \geq 0$  and some scalar  $N_0 \geq 0$ .

The inequality (5) means that on average the pass numbers between any two consecutive switchings is no smaller than  $\kappa_a$ .

Following Pakshin et al. (2016, 2018), introduce a concept of exponential stability of the repetitive process (1), (2).

**Definition 2.** The differential repetitive process (1), (2) is said to be exponentially stable if there exist real numbers  $\kappa > 0$ ,  $\lambda > 0$  and  $0 < \zeta < 1$  such that

$$|x_k(t)|^2 + |y_k(t)|^2 \leq \kappa \exp(-\lambda t) \zeta^k \quad (6)$$

for any  $k$  and  $t$ .

For further analysis based on ideas Pakshin et al. (2016, 2018), introduce a vector Lyapunov function of the form

$$V_i(x_{k+1}(t), y_k(t)) = \begin{bmatrix} V_1(x_{k+1}(t)) \\ V_{2i}(y_k(t)) \end{bmatrix}, \quad i \in \mathcal{N}, \quad (7)$$

where  $V_1(x) > 0$ ,  $x \neq 0$ ,  $V_{2i}(y) > 0$ ,  $y \neq 0$ ,  $V_1(0) = 0$  and  $V_{2i}(0) = 0$ . The counterpart of the divergence operator of this function along the trajectories of (1), (2) is given by

$$\mathcal{D}V_i(x_{k+1}(t), y_k(t)) = \frac{dV_1(x_{k+1}(t))}{dt} + \Delta_k V_{2i}(y_k(t)), \quad (8)$$

where

$$\Delta_k V_{2i}(y_k(t)) = V_{2i}(y_{k+1}(t)) - V_{2i}(y_k(t)).$$

The choice of the vector function (7) with the switching-independent entry  $V_1$  is motivated by iterative learning control problems and will be explained in detail below.

**Theorem 1.** Let there exist a vector function (7) and positive scalars  $c_1, c_2, c_3$  and  $c_4$  such that

$$c_1 |x|^2 \leq V_1(x) \leq c_2 |x|^2, \quad (9)$$

$$c_1 |y|^2 \leq V_{2i}(y) \leq c_2 |y|^2, \quad (10)$$

$$\mathcal{D}V_i(x_{k+1}(t), y_k(t)) \leq -c_3(|x_{k+1}(t)|^2 + |y_k(t)|^2), \quad (11)$$

$$\left| \frac{\partial V_1(x)}{\partial x} \right| \leq c_4 |x|, \quad (12)$$

Then the switched repetitive process (1), (2) is exponentially stable for each switching signal in pass direction  $\sigma$  with an average dwell time

$$\kappa_a > \ln \left( \frac{c_1}{c_2} \right) \left( \ln \left( 1 - \frac{c_3}{c_1} \right) \right)^{-1} \quad (13)$$

and an arbitrary positive number  $N_0$ .

**Proof.** Consider an interval  $(0, k_f)$  and let  $N_\sigma = N_\sigma(k_f, 0)$  be the number of switchings on this interval. It follows from (11) that there exists  $\bar{c}_3 < c_3$  such that

$$\mathcal{D}V_{\sigma(k)}(x_{k+1}(t), y_k(t)) \leq -\bar{c}_3(x_{k+1}(t) + y_k(t)). \quad (14)$$

Denote  $\zeta = 1 - \frac{\bar{c}_3}{c_2}$  and choose  $\bar{c}_3$  small enough to satisfy the inequality

$$\lambda_d^{\frac{1}{2}} < \zeta < 1, \quad (15)$$

Using (9), (10), (11), rewrite (14) as

$$\begin{aligned} \frac{dV_1(x_{k+1}(t))}{dt} + \lambda V_1(x_{k+1}(t)) \\ + V_{2\sigma(k+1)}(y_{k+1}(t)) - \zeta V_{2\sigma(k)}(y_k(t)) \leq 0, \end{aligned} \quad (16)$$

where  $\lambda = \bar{c}_3/c_2$ .

Solving the inequality (16) with respect to  $V_1(x_{k+1}(t))$  gives

$$V_1(x_{k+1}(t)) \leq V_1(x_{k+1}(0))e^{-\lambda t} - \int_0^t e^{-\lambda(t-s)} [V_{2\sigma(k+1)}(y_{k+1}(s)) - \zeta V_{2\sigma(k)}(y_k(s))] ds. \quad (17)$$

Introducing

$$W_{k+1}(t) := V_1(x_{k+1}(0))e^{-\lambda t} - V_1(x_{k+1}(t)),$$

$$H_{k,\sigma(k)}(t) := \int_0^t e^{-\lambda(t-s)} V_{2,\sigma(k)}(y_k(s)) ds.$$

enables (17) to be rewritten as

$$H_{k+1,\sigma(k+1)}(t) \leq \zeta H_{k,\sigma(k)}(t) + W_{k+1}(t). \quad (18)$$

Let mode  $i$  be switched to mode  $j$  at an instant  $k_n$ . It follows from (10) that

$$V_{2j}(y) \leq \mu V_{2i}(y), \quad i, j \in \mathcal{N}, \quad (19)$$

where  $\mu = \frac{c_2}{c_1} \geq 1$ .

Solving inequality (18) in view of (19) gives

$$H_{n,\sigma(n)}(t) \leq \mu^{N_\sigma} \zeta^n H_{0,\sigma(0)}(t) + \mu^{N_\sigma} \sum_{k=1}^n W_k(t) \zeta^{n-k} \quad (20)$$

or in the previous notations,

$$\begin{aligned} & \sum_{k=1}^n V_1(x_k(t)) \zeta^{n-k} + \int_0^t e^{-\lambda(t-s)} V_2(y_n(s)) ds \\ & \leq \mu^{N_\sigma} \sum_{k=1}^n V_1(x_k(t)) \zeta^{n-k} + \int_0^t e^{-\lambda(t-s)} V_2(y_n(s)) ds \\ & \leq \mu^{N_\sigma} \left( e^{-\lambda t} \sum_{k=1}^n V_1(x_k(0)) \zeta^{n-k} + \zeta^n \int_0^t e^{-\lambda(t-s)} V_2(y_0(s)) ds \right). \end{aligned}$$

The last inequality implies

$$\begin{aligned} & e^{\lambda t} \sum_{k=1}^n V_1(x_k(t)) \zeta^{-k} + \zeta^{-n} \int_0^t e^{\lambda s} V_2(y_n(s)) ds \\ & \leq \mu^{N_\sigma} e^{\lambda t} \sum_{k=1}^n V_1(x_k(t)) \zeta^{-k} + \zeta^{-n} \int_0^t e^{\lambda s} V_2(y_n(s)) ds \\ & \leq \mu^{N_\sigma} \left( \zeta^{-n} \sum_{k=1}^n V_1(x_k(0)) \zeta^{n-k} + e^{\lambda t} \int_0^t e^{-\lambda(t-s)} V_2(y_0(s)) ds \right) \quad (21) \end{aligned}$$

and evaluating of the right-hand side of (21) in view of (3) and (15) yields

$$\begin{aligned} & \mu^{N_\sigma} \left( \zeta^{-n} \sum_{k=1}^n V_1(x_k(0)) \zeta^{n-k} + e^{\lambda t} \int_0^t e^{-\lambda(t-s)} V_2(y_0(s)) ds \right) \\ & \leq \mu^{N_\sigma} \left( \sum_{k=1}^n c_2 \kappa_d \lambda_d^k \zeta^{-k} + \int_0^t c_2 M_f e^{\lambda s} ds \right) \\ & \leq \mu^{N_\sigma} \left( \sum_{k=1}^n c_2 \kappa_d (\lambda_d^{\frac{1}{2}})^k (\lambda_d^{\frac{1}{2}})^k \zeta^{-k} + \frac{c_2 M_f (e^{\lambda t} - 1)}{\lambda} \right) \\ & \leq \mu^{N_\sigma} \left( \sum_{k=1}^n c_2 \kappa_d \zeta^k \zeta^k \zeta^{-k} + \frac{c_2 M_f (e^{\lambda T} - 1)}{\lambda} \right) \\ & \leq \mu^{N_\sigma} \left( \sum_{k=1}^{\infty} c_2 \kappa_d \zeta^k + \frac{c_2 M_f (e^{\lambda T} - 1)}{\lambda} \right) \\ & = \mu^{N_\sigma} \left( \frac{c_2 \kappa_d}{1 - \zeta} + \frac{c_2 M_f (e^{\lambda T} - 1)}{\lambda} \right) = C \mu^{N_\sigma}. \quad (22) \end{aligned}$$

for all  $n \leq k_f$  and any  $t$ . Also, it immediately follows from the left-hand side of (21) and (22) that

$$C \mu^{N_\sigma} \geq \mu^{N_\sigma} e^{\lambda t} \sum_{k=1}^n V_1(x_k(t)) \zeta^{-k} \geq \mu^{N_\sigma} e^{\lambda t} \zeta^{-n} |x_n(t)|^2 \quad (23)$$

for all  $n \leq k_f$  and any  $t$ . Hence,

$$|x_n(t)|^2 \leq \frac{C}{c_1} e^{-\lambda t} \zeta^n. \quad (24)$$

Evaluating  $\frac{dV_1(x)}{dt}$  and using the Lipschitz conditions (4) and (12) gives

$$\begin{aligned} \frac{dV_1(x_{k+1}(t))}{dt} & = \frac{\partial V_1(x_{k+1}(t))}{\partial x_{k+1}(t)} f_{1i}(x_{k+1}(t), y_k(t)) \geq \\ & \geq - \left| \frac{\partial V_1(x_{k+1}(t))}{\partial x_{k+1}(t)} \right| |f_{1i}(x_{k+1}(t), y_k(t))| \\ & \geq -c_4 L (|x_{k+1}(t)| + \varepsilon |y_k(t)|) (|x_{k+1}(t)| + |y_k(t)|) \\ & \geq -2c_4 L \left( \frac{\varepsilon + 1}{2\sqrt{\varepsilon}} |x_{k+1}(t)| + \sqrt{\varepsilon} |y_k(t)| \right)^2 \\ & \geq -2c_4 L \left( 2 \left( \frac{\varepsilon + 1}{2\sqrt{\varepsilon}} |x_{k+1}(t)| \right)^2 + 2(\sqrt{\varepsilon} |y_k(t)|)^2 \right) \\ & \geq -\alpha V_1(x_{k+1}(t)) - \beta \varepsilon V_{2i}(y_k(t)), \quad (25) \end{aligned}$$

where  $\alpha = \frac{c_4 L (\varepsilon + 1)^2}{c_1 \varepsilon}$ ,  $\beta = \frac{4c_4 L}{c_1}$ , and  $\varepsilon$  is arbitrary positive scalar. It follows from (16) and (25) that

$$V_{2,\sigma(k+1)}(y_{k+1}(t)) - z_0 V_{2\sigma(k)}(y_k(t)) \leq \alpha V_1(x_{k+1}(t)), \quad (26)$$

where  $z_0 = \zeta + \beta \varepsilon$ . Choosing  $\varepsilon$  small enough so that  $0 < z_0 < 1$ , solving (26) and using (24) give

$$\begin{aligned} V_{2,\sigma(n)}(y_n(t)) & \leq \mu^{N_\sigma} z_0^n V_{2,\sigma(0)}(y_0(t)) \\ & \quad + \frac{\alpha c_2}{c_1} \mu^{N_\sigma} \sum_{k=1}^n z_0^{n-k} \zeta^k e^{-\lambda t}. \quad (27) \end{aligned}$$

It follows from (13) that  $z_\mu = \mu^{\frac{1}{\kappa_a}} < 1$ . Then inequality (27) implies that, for any  $k_f$  and  $t$ ,

$$|y_{k_f}(t)|^2 \leq \frac{c_2 \mu^{N_0} z_\mu^{k_f}}{c_1} |y_0(t)|^2 + \frac{\alpha c_2 \mu^{N_0}}{c_1^2} \sum_{k=1}^{k_f} z_\mu^{k_f-k} \zeta^k e^{-\lambda t} \quad (28)$$

and  $|y_{k_f}(t)|^2 e^{\lambda t} \zeta_0^{-k_f}$  is bounded, where  $z_\mu < \zeta_0 < 1$ . Then taking into account (24) gives

$$|x_k(t)|^2 + |y_k(t)|^2 \leq \bar{C} \exp(-\lambda t) \zeta_0^k$$

for some constant  $\bar{C}$ . This means that the switched repetitive process (1), (2) is exponentially stable and the proof is complete.

The following result is a direct consequence of the theorem proved.

*Corollary 1.* The repetitive process (1), (2) is exponentially stable for an arbitrary switching signal in pass direction  $\sigma$  if there exist a vector function

$$V(x_{k+1}(t), y_k(p)) = [V_1(x_{k+1}(t)) \ V_2(y_k(p))], \quad (29)$$

and positive scalars  $c_1, c_2, c_3$  and  $c_4$  such that

$$c_1 |x|^2 \leq V_1(x) \leq c_2 |x|^2,$$

$$c_1 |y|^2 \leq V_2(y) \leq c_2 |y|^2,$$

$$DV(x_{k+1}(t), y_k(p)) \leq -c_3 (|x_{k+1}(t)|^2 + |y_k(p)|^2),$$

$$\left| \frac{\partial V_1(x)}{\partial x} \right| \leq c_4 |x|.$$

Function (29) is a *common vector Lyapunov function* for all subsystems of the repetitive process (1), (2).

### 3. ITERATIVE LEARNING CONTROL DESIGN FOR SWITCHED LINEAR SYSTEM

Consider a linear discrete-time system in repetitive mode described by the linear state-space model

$$\begin{aligned} \dot{x}_k(t) &= A(i)x_k(t) + B(i)u_k(t), \\ y_k(t) &= Cx_k(t), \quad i \in \mathcal{N}, \quad k = 0, 1, \dots \end{aligned} \quad (30)$$

where the nonnegative integer  $k$  denotes the pass number;  $u_k(t) \in \mathbb{R}^{n_u}, x_k(t) \in \mathbb{R}^{n_x}$  and  $y_k(t) \in \mathbb{R}^{n_y}$  are the input, state and output vectors, respectively, at instant  $0 \leq t \leq T$ , where  $T$  is a pass length;  $i \in \mathcal{N}$  indicates the mode of this system that can be changed (switched) in pass direction. Let  $y_{ref}(t)$  be a supplied reference vector over  $0 \leq t \leq T$ . Then  $e_k(t) = y_{ref}(t) - y_k(t)$  is the error on pass  $k$  and the problem is to construct a sequence of pass inputs  $\{u_k\}$  such that the performance achieved is gradually improving with each successive pass. This can be refined to the following convergence conditions on the input and error:

$$\lim_{k \rightarrow \infty} |e_k(t)| = 0, \quad \lim_{k \rightarrow \infty} |u_k(t) - u_\infty(t)| = 0, \quad (31)$$

where  $u_\infty(t)$  is termed the learned control.

Assume that the boundary conditions have form

$$\begin{aligned} y_0(t) &= Cx_0(t), \quad 0 \leq t \leq T, \\ x_k(0) &= x_0, \quad k \geq 0, \quad y_k(0) = y_{ref}(0) \end{aligned} \quad (32)$$

A common ILC law is to select the input on the current pass as that used on the previous pass plus a correction. In this paper, the ILC law on pass  $k+1$  is of the form

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p), \quad (33)$$

where  $\Delta u_{k+1}(p)$  is the correction term to be designed.

For writing the ILC dynamics in the differential repetitive process form introduce, for analysis purposes only, the vector

$$\xi_{k+1}(t) = x_{k+1}(t) - x_k(t). \quad (34)$$

Assume that  $CB \neq 0$ . Then the controlled dynamics can be written as

$$\begin{aligned} \dot{\xi}_{k+1}(t) &= A(i)\xi_{k+1}(t) + B(i)\Delta u_{k+1}(t), \\ \epsilon_{k+1}(t) &= -CA(i)\eta_{k+1}(t) + \epsilon_k(t) - CB(i)\Delta u_{k+1}(t), \end{aligned} \quad (35)$$

where  $\epsilon_k(t) = \dot{e}_k(t)$ . System (35) is a linear differential repetitive process in standard form. If the control correction term ensures exponential stability of (35), then under the boundary conditions (32) the ILC law (33) converges in the sense of conditions (31). Choose the correction law in the form

$$\Delta u_{k+1}(t) = K_1 \xi_{k+1}(t) + K_2(i) \epsilon_k(t). \quad (36)$$

The main role of the first term in (36) is to stabilize the dynamics, while the second term has a dominant effect on the pass-to-pass error convergence. Due to this fact, this term is chosen as mode dependent. Introducing the notations

$$\begin{aligned} A_{11}(i) &= A(i), \quad A_{12}(i) = 0, \quad A_{21}(i) = -CA(i), \\ A_{22}(i) &= I, \quad B_1(i) = B(i), \quad B_2(i) = -CB(i), \\ K(i) &= [K_1 \ K_2(i)], \quad A_{cjl}(i) = A_{jl}(i) + B_j K_l(i) \end{aligned}$$

rewrite system (35) with the correction law (36) in the form

$$\begin{aligned} \dot{\xi}_{k+1}(t) &= A_{c11} \xi_{k+1}(t) + A_{c12} \epsilon_k(t), \\ \epsilon_{k+1}(t) &= A_{c21} \xi_{k+1}(t) + A_{c22} \epsilon_k(t), \end{aligned} \quad (37)$$

To find the conditions of exponential stability of (37) (or what is the same, of (35), (36)), consider a candidate vector Lyapunov function (7) with  $V_1(\xi_{k+1}(p)) = \xi_{k+1}^T(p) P_1 \xi_{k+1}(p)$ ,  $V_2(\epsilon_k) = \epsilon_k^T(p) P_2 \epsilon_k(p)$ , where  $P_1 \succ 0$  and  $P_2(i) \succ 0$ ,  $i \in \mathcal{N}$ . Calculating the divergence of (7) along the trajectories of (35), (36) gives

$$\begin{aligned} DV(\xi_{k+1}(t), \epsilon_k(t)) &= \eta_k^T(t) (\bar{A}_c^T(i) I^{(1,0)} P(i) \\ &+ P(i) I^{(1,0)} \bar{A}_c(i) + \bar{A}_c^T(i) I^{(0,1)} P(i) \bar{A}_c(i) \\ &- I^{(0,1)} P(i)) \eta_k(t), \end{aligned} \quad (38)$$

where  $\bar{A}_c(i) = \begin{bmatrix} A_{c11}(i) & A_{c12}(i) \\ A_{c21}(i) & A_{c22}(i) \end{bmatrix}$ ,  $I^{(1,0)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ ,  $I^{(0,1)} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ ,  $\eta_k(t) = [\xi_{k+1}^T(t), \epsilon_k^T(t)]^T$ ,  $P(i) = \text{diag}[P_1 \ P_2(i)]$ .

Let the matrices  $P(i) = \text{diag}[P_1 \ P_2(i)] \succ 0$  and  $K$  be a solution to

$$\begin{aligned} \bar{A}_c^T(i) I^{(1,0)} P(i) + P(i) I^{(1,0)} \bar{A}_c(i) \\ + \bar{A}_c^T(i) I^{(0,1)} P(i) \bar{A}_c(i) - I^{(0,1)} P(i) \\ + Q + K^T R K \preceq 0, \quad i \in \mathcal{N}, \end{aligned} \quad (39)$$

where  $Q = \text{diag}[Q_1, Q_2] \succ 0$  and  $R \succ 0$  are weight matrices to be selected. Then it follows from Theorem 1 that system (37) is exponentially stable for each switching signal in pass direction  $\sigma$  with an average dwell time (13) with  $c_1 = \min\{\lambda_{\min}(P_1), \min_{i \in \mathcal{N}}(\lambda_{\min}(P_2(i)))\}$ ,  $c_2 = \max\{\lambda_{\max}(P_1), \max_{i \in \mathcal{N}}(\lambda_{\max}(P_2(i)))\}$ ,  $c_3 = \lambda(Q + K^T R K)$ . Using Schur's complement formula, (39) is reduced to the following LMI with respect to  $X =$

$\text{diag}[X_1 \ X_2]$ , where  $X_1 = P_1^{-1}$  and  $X_2 = P_2^{-1}$  and  $Y_1 = KX_1$ ,  $Y_2 = KX_2$ :

$$\begin{bmatrix} (A_{11}(i)X_1 + B_1(i)Y_1) & (A_{12}(i)X_2(i) + B_1(i)Y_2(i)) & 0 & 0 \\ +(A_{11}(i)X_1 + B_1(i)Y_1)^T & -X_2(i) & 0 & 0 \\ (A_{12}(i)X_2(i) + B_1(i)Y_2(i))^T & 0 & 0 & -X_1 \\ 0 & 0 & 0 & 0 \\ (A_{21}(i)X_1 + B_2(i)Y_1) & (A_{22}(i)X_2(i) + B_2(i)Y_2(i)) & 0 & 0 \\ X_1 & 0 & 0 & 0 \\ 0 & X_2(i) & 0 & 0 \\ Y_1 & Y_2(i) & 0 & 0 \\ (A_{21}(i)X_1 + B_2(i)Y_1)^T & X_1 & 0 & Y_1^T \\ (A_{22}(i)X_2(i) + B_2(i)Y_2(i))^T & 0 & X_2(i) & Y_2(i)^T \\ 0 & 0 & 0 & 0 \\ -X_2(i) & 0 & 0 & 0 \\ 0 & -Q_1^{-1} & 0 & 0 \\ 0 & 0 & -Q_2^{-1} & 0 \\ 0 & 0 & 0 & -R^{-1} \end{bmatrix} \leq 0, \quad (40)$$

$X_1 \succ 0, X_2(i) \succ 0, i \in \mathcal{N}.$

If this LMI is feasible, then  $K_1 = Y_1 X_1^{-1}$  and  $K_2(i) = Y_2(i) X_2^{-1}(i)$ .

#### 4. EXAMPLE

Consider a flexible link manipulator that moves loads of two various masses in horizontal plane. The change in the mass of the moved loads can be considered as a switching of the mode. The dynamics of this manipulator can be described by the following state space model Apkarian et al. (2011)

$$\begin{aligned} \dot{x}(t) &= A(i)x(t) + Bu(t), \quad i = 1, 2, \\ y(t) &= Cx(t), \end{aligned} \quad (41)$$

where  $x = [\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}]^T$ ;  $\theta$  is the servo angle;  $\alpha$  is the flexible link angle.  $B = [0 \ 0 \ \frac{1}{J_{eq}} \ -\frac{1}{J_{eq}}]^T$ ,  $C = [1 \ 0 \ 0 \ 0]$  and

$$A(i) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_s}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} & 0 \\ 0 & -\frac{K_s(J_l(i) + J_{eq})}{J_l(i)J_{eq}} & \frac{B_{eq}}{J_{eq}} & 0 \end{bmatrix}, \quad i = 1, 2,$$

$K_s$  is the stiffness of the flexible link;  $J_{eq}$  is the moment of inertia of the servo,  $B_{eq}$  is the viscous friction coefficient of the servo; finally,  $J_l(i)$  is the moment of inertia of the flexible link. The results shown in the remainder of this paper correspond to the case  $K_s = 1.3 \text{ N m/rad}$ ,  $J_{eq} = 2.08 \cdot 10^{-3} \text{ kg m}^2$ ,  $B_{eq} = 0.004 \text{ N m/(rad/s)}$ ,  $J_l(1) = 3.8 \cdot 10^{-3} \text{ kg m}^2$ ,  $J_l(2) = 8.5 \cdot 10^{-3} \text{ kg m}^2$ . It is assumed that all state variables are available for the controller.

The required reference trajectory is the servo angle  $\theta(t)$  chosen to simulate a pick-and-place process of duration 3 s. This trajectory is given by

$$y_{ref}(t) = \frac{\pi t^2}{6} - \frac{\pi t^3}{27}. \quad (42)$$

In the case under consideration,  $CB = 0$  and the model from the previous section cannot be used for ILC design. Under the boundary conditions

$$y_{ref}(0) = Cx_k(0), \quad \dot{x}_k(0) = C_1 x_k(0), \quad k = 0, 1, \dots, \quad (43)$$

where  $C_1 = [0 \ 0 \ 1 \ 0]$ , it is possible to use the second derivative of  $y_{ref}(t)$ . Denote  $\varepsilon_k(t) = \ddot{e}_k(t)$  and write a counterpart of (35) as

$$\begin{aligned} \dot{\xi}_{k+1}(t) &= A(i)\xi_{k+1}(t) + B(i)\Delta u_{k+1}(t), \\ \varepsilon_{k+1}(t) &= -C_1 A(i)\xi_{k+1}(t) + \varepsilon_k(t) \\ &\quad - C_1 B(i)\Delta u_{k+1}(t), \end{aligned} \quad (44)$$

where all the notations are defined above. Applying the correction law

$$\Delta u_{k+1}(t) = K_1 \xi_{k+1}(t) + K_2 \varepsilon_k(t) \quad (45)$$

to (44) gives (37) with  $C$  replaced by  $C_1$ ; if (44) with the correction law (45) is exponentially stable, then  $e_k \rightarrow 0$  as  $k \rightarrow \infty$  under the boundary conditions (43). The ILC law that corresponds to (45) has the form

$$\begin{aligned} u_{k+1}(t) &= (I - K_2(i)C_1 B(i))u_k(t) + K_1(x_{k+1}(t) \\ &\quad - x_k(t)) + K_2(i)(\ddot{y}_{ref}(t) - C_1 A(i)x_k(t)), \end{aligned} \quad (46)$$

where the matrices  $K_1$  and  $K_2(i)$  are given by the solution to the LMI (40) with  $C$  replaced by  $C_1$ . Minimizing the objective function  $-(\text{tr}X_1 + \text{tr}X_2)$  subject to the LMI constraints (40) with  $Q = \text{diag}[1 \ 1 \ 1 \ 1 \ 10^5]$ ,  $R = 0.01$  gives

$$\begin{aligned} K_1 &= [0 \ -1.3 \ 3.9 \cdot 10^{-3} \ 2.0 \cdot 10^{-3}], \\ K_2(1) &= 2.8 \cdot 10^{-3}, \quad K_2(2) = 4.6 \cdot 10^{-4}. \end{aligned} \quad (47)$$

To measure the performance of this ILC law, introduce the mean square error

$$E(k) = \sqrt{\frac{1}{T} \int_0^T |e_k(t)|^2 dt}. \quad (48)$$

Assume that the load changes from mode 1 to mode 2 on pass 10, then from mode 2 to mode 1 on pass 20 and finally from mode 1 to mode 2 on pass 30, assume also that the required accuracy in terms of  $E(k)$  is  $E^* = 2^\circ$ . The progression of  $E(k)$  for this case is shown in Fig. 1. A monotonic pass-to-pass error convergence takes place and  $E(k) \leq E^*$  for all  $k \geq 3$ .

Using the common vector Lyapunov function gives a control law without switching; the gain matrices obtained by minimizing the objective function  $-(\text{tr}X_1 + \text{tr}X_2)$  subject to the LMI constraints (40) with  $C = C_1$  and  $X_2(i) = X_2$ ,  $Y_2(i) = Y_2$  are as follows:

$$\begin{aligned} K_1 &= [0 \ -1.33 \ 3.6 \cdot 10^{-3} \ 6.3 \cdot 10^{-3}], \\ K_2 &= 2.1 \cdot 10^{-3}. \end{aligned} \quad (49)$$

For the same switching rule, the progression of  $E(k)$  is shown in Fig. 2. In this case a short-term loss of accuracy occurs by changing modes. So, the required accuracy does not achieved in this case and the problem cannot be solved on the basis of a common Lyapunov function.

#### 5. CONCLUSIONS

Further research in this field is connected with solution of stability and stabilization problems for other classes of nonlinear 2D systems with switching: continuous-discrete and continuous Roesser models, Fornasini–Marchesini models and others. Iterative learning control algorithms with different forms of switching also seem to be an interesting direction of investigations.

#### REFERENCES

Ahn, H.S., Chen, Y.Q., and Moore, K.L. (2007). Iterative learning control: Brief survey and categorization. *IEEE*

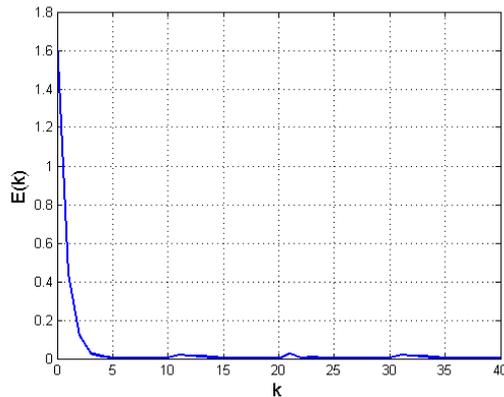


Fig. 1. Progression of  $E(k)$  with plant switching on the passes 10, 20 and 30 for switched ILC gain matrices (47).

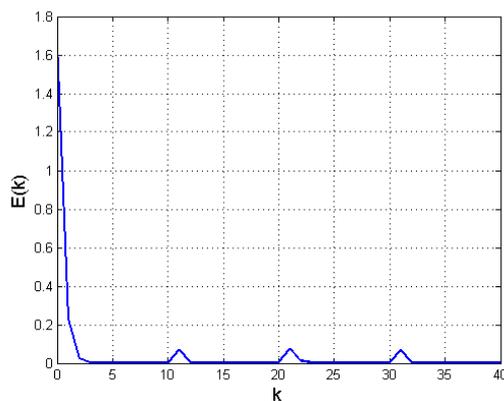


Fig. 2. Progression of  $E(k)$  with plant switching on the passes 10, 20 and 30 for constant ILC gain matrices (49).

*Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews*, 37(6), 1099–1121.

Alwan, M.S. and Liu, X. (2018). *Theory of Hybrid Systems: Deterministic and Stochastic*. Springer Nature Singapore Pte Ltd. and Higher Education Press, Beijing.

Apkarian, J., Karam, P., and Lévis, M. (2011). *Workbook on Flexible Link Experiment for Matlab®/Simulink® Users*. Quanser.

Arimoto, S., Kawamura, S., and Miyazaki, F. (1984). Bettering operation of robots by learning. *Journal of Robotic Systems*, 1(2), 123–140.

Bochniak, J., Galkowski, K., and Rogers, E. (2008). Multi-machine operations modelled and controlled as switched linear repetitive processes. *International Journal of Control*, 81, 1549–1567.

Bochniak, J., Galkowski, K., Rogers, E., Mehdi, D., Bachelier, O., and Kummert, A. (2006). Stabilization of discrete linear repetitive processes with switched dynamics. *Multidim. Syst. Sign. Process.*, 17, 271–295.

Bolder, J. and Oomen, T. (2016). Iterative learning control: A 2D system approach. *Automatica*, 71, 247–253.

Bristow, D.A., Tharayil, M., and Alleyne, A. (2006). A survey of iterative learning control. *IEEE Control Systems Magazine*, 26(3), 96–114.

Freeman, C.T., Rogers, E., Hughes, A.M., Burrbridge, J.H., and Meadmore, K.L. (2012). Iterative learning control

in health care: electrical stimulation and robotic-assisted upper-limb stroke rehabilitation. *IEEE Control Systems Magazine*, 47, 70–80.

Hladowski, L., Galkowski, K., Cai, Z., Rogers, E., Freeman, C.T., and Lewin, P.L. (2010). Experimentally supported 2D systems based iterative learning control law design for error convergence and performance. *Control Engineering Practice*, 18, 339–348.

Liberzon, D. (2003). *Switching in Systems and Control*. Birkhäuser, Boston, MA.

Lin, H. and Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54, 308–321.

Meadmore, K.L., Exell, T.A., Hallewell, E., Hughes, A.M., Freeman, C.T., Kutlu, M., Benson, V., Rogers, E., and Burrbridge, J.H. (2014). The application of precisely controlled functional electrical stimulation to the shoulder, elbow and wrist for upper limb stroke rehabilitation: a feasibility study. *Journal of NeuroEngineering and Rehabilitation*, 11:105.

Pakshin, P., Emelianova, J., Galkowski, K., and Rogers, E. (2018). Stabilization of two-dimensional nonlinear systems described by Fornasini–Marchesini and Roesser models. *SIAM J. Control Optim.*, 56, 3848–3866.

Pakshin, P.V., Emelianova, J., Emelianov, M., Galkowski, K., and Rogers, E. (2016). Dissipativity and stabilization of nonlinear repetitive processes. *Systems & Control Letters*, 91, 14–20.

Paszke, W., Rogers, E., Galkowski, K., and Cai, Z. (2013). Robust finite frequency range iterative learning control design with experimental verification. *Control Engineering Practice*, 23, 1310–1320.

Rogers, E., Galkowski, K., and Owens, D.H. (2007). *Control Systems Theory and Applications for Linear Repetitive Processes*, volume 349 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, Germany.

Sammons, P.M., Gegel, M.L., Bristow, D.A., and Landers, R.G. (2019). Repetitive process control of additive manufacturing with application to laser metal deposition. *IEEE Transactions on Control Systems Technology*, 27(2).

Shorten, R., Wirth, F., Mason, O., Wulff, K., and King, C. (2007). Stability criteria for switched and hybrid systems. *SIAM Review*, 49, 545–592.

Sun, Z. and Ge, S.S. (2011). *Stability Theory of Switched Dynamical Systems*. Springer-Verlag, London.

Tian, D., Liu, S., and Wang, W. (2019). Global exponential stability of 2D switched positive nonlinear systems described by the Roesser model. *Int. J. Robust Nonlinear Control*, 29, 2272–2282.

Wua, L., Yang, R., Shi, P., and Sue, X. (2015). Stability analysis and stabilization of 2-D switched systems under arbitrary and restricted switchings. *Automatica*, 59, 206–215.

Xu, L. and Zhu, Q. (2019). Stability analysis of 2-D switched systems with multiplicative noise under arbitrary and restricted switching signals. *International Journal of Systems Science*, 50, 191–202.

Yang, R. and Yu, Y. (2019). Event-triggered control of discrete-time 2-D switched Fornasini–Marchesini systems. *European Journal of Control*, 48, 42–51.