Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Topology-Independent Robust Stability for Networks of Homogeneous MIMO Systems ⋆

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Abstract We study dynamic networks described by a directed graph where the nodes are associated with MIMO systems with transfer-function matrix $F(s)$, representing individual dynamic units, and the arcs are associated with MIMO systems with transfer-function matrix $G(s)$, accounting for the dynamic interactions among the units. In the nominal case, we provide a topology-independent condition for the stability of all possible dynamic networks with a maximum connectivity degree, regardless of their size and interconnection structure. When node and arc transfer-function matrices are affected by norm-bounded homogeneous uncertainties, the robust condition for size- and topology-independent stability depends on the uncertainty magnitude. Both conditions, expressed as constraints for the Nyquist diagram of the poles of the transfer-function matrix $H(s) = F(s)G(s)$, are scalable and can be checked locally to guarantee stability-preserving "plug-and-play" addition of new nodes and arcs.

Keywords: MIMO systems, Networked systems, Robust stability, Uncertain systems

1. INTRODUCTION

Large-scale networks of interacting dynamical systems arise in the most different contexts, from biological systems (Hori et al. 2015) to multi-agent systems where we wish to enforce control (Massioni and Verhaegen 2009; Blanchini et al. 2015; Lin et al. 2016), estimation (Giordano et al. 2016), consensus (Olfati-Saber et al. 2007) or synchronisation (Trentelman et al. 2013). Under which conditions does the stability of the local subsystems guarantee the stability of the whole dynamic network, possibly in the presence of uncertainties?

For interconnections of SISO (single-input and single-output) LTI (linear time invariant) systems, robust stability conditions were provided by Kao et al. (2009) and Jönsson and Kao (2010) for particular topologies; by Lestas and Vinnicombe (2006) using the multivariable Nyquist criterion proposed by Desoer and Wang (1980) and the concept of S-hull; by Hara et al. (2007, 2009, 2014) and Hori et al. (2015) in the generalised frequency variable framework.
Extensions of the above stability conditions to the MIMO (multiple-input and multiple-output) case were provided by Pates and Vinnicombe (2012) with a Nyquist-like approach and by Andersen et al. (2014) and Khong and Rantzer (2014) based on Integral Quadratic Constraints, also used by Pates and Vinnicombe (2017) for control design: these conditions are scalable because, to ensure stability of the whole network, it is enough to satisfy a local condition at each node. The generalised frequency variable technique was recently employed by Hara et al. (2019) to obtain global necessary and sufficient conditions for the robust stability of networks of nominally homogeneous MIMO LTI systems, encompassing also heterogeneous uncertainties for specific topologies.
Dynamic networks with both dynamic nodes and dynamic interconnections (Nepusz and Vicsek 2012) were considered by Blanchini et al. (2017, 2018) in the SISO case: topology-independent robust stability conditions in the frequency domain were obtained for nominally homogeneous node and arc dynamics, with homogeneous or heterogeneous uncertainties.

In this paper, we consider dynamic networks described by directed graphs where both the nodes and the arcs are associated with possibly uncertain MIMO dynamical systems. We assume the dynamics of all the nodes to be identical, and also those of all the arcs. We seek conditions for the *robust* stability of the overall dynamic network, *regardless of its size and interconnection topology*: the only available information about the network is the maximum number of arcs that may enter or leave a node (maximum connectivity degree). We rely on Nyquist-like approaches and on the theorem by Bauer and Fike (1960) to derive the following main results:

- a topology-independent stability condition for nominal dynamic networks (Section 3);
- a topology-independent condition for the robust stability of uncertain dynamic networks subject to norm-bounded homogeneous uncertainties (Section 4).

Both conditions can be checked locally, thus guaranteeing scalability and robustness to online modifications of the dynamic network (such as adding nodes or arcs) in a plug-and-play framework (Bendtsen et al. 2013; Riverso et al. 2013), as long as the maximum connectivity degree is preserved.

Definitions and notation. A directed graph (*digraph*) with N nodes and M arcs is a pair $\mathscr{G} = (\mathscr{N}, \mathscr{A})$ where $\mathscr{N} = \{1, 2, \dots, N\}$ is the set of all nodes and $\mathscr{A}$ is a subset of $\mathscr{N} \times \mathscr{N}$ with cardinality M; if $(i, j) \in \mathscr{A}$, then an arc goes from node i to node j. Two nodes are assumed to be connected by at most one arc. The *degree* $\mathfrak{d}_i$ of node i is the number of arcs that either enter or leave node i. The *maximum connectivity degree* is $\mathfrak{D} = \max_i \mathfrak{d}_i$.

⋆ Work partially supported by the Delft Technology Fellowship grant to GG.

A *path* is a sequence of nodes $i = i_0, i_1, \dots, i_\ell = j$ such that $(i_{h-1}, i_h) \in \mathscr{A}$ for $h = 1, \dots, \ell$. The graph is *strongly connected* if each pair of nodes $i, j \in \mathscr{N}$ is connected by a path. It is *weakly connected* if the associated *undirected graph*, where the presence of arc (i, j) implies the concurrent presence of arc (j, i) (namely, each existing arc can be crossed in both directions), is strongly connected.

In this paper, we consider weakly connected graphs where an additional 'environment' node can be present, which is not explicitly listed. Arcs that either come from or reach the environment node are counted in the degrees $\mathfrak{d}_i$: we can write $\mathfrak{d}_i = \mathfrak{d}_i^{int} + \mathfrak{d}_i^{ext}$, where $\mathfrak{d}_i^{ext}$ is the number of arcs connecting node i with the environment node, while $\mathfrak{d}_i^{int}$ is the number of arcs connecting i with other nodes within the graph.

Let $B \in \{-1, 0, 1\}^{N \times M}$ denote the *generalised node-arc incidence matrix* defined as

$$[B]_{ij} = \begin{cases} 1, & \text{if the arc } j \text{ enters node } i, \\ -1, & \text{if the arc } j \text{ leaves node } i, \\ 0, & \text{otherwise,} \end{cases}$$

where arcs leaving or entering the environment node are accounted for, even though such node is not associated with a row of the matrix, and correspond to columns of B with a single nonzero entry. The *generalised* (or grounded) *Laplacian matrix* of the graph is $L = BB^\top$ (Giordano et al. 2016). Its diagonal entries are $L_{ii} = \mathfrak{d}_i$, for $i = 1, \dots, N$, while its off-diagonal entries L_{ij}, $i \neq j$, take the value -1 if there is an arc either from node i to node j or from node j to node i, and 0 otherwise. Matrix L is symmetric, hence its eigenvalues $\{\gamma_k\}_{k=1}^N$ are real, and it can be diagonalised by an orthogonal matrix W: $W^{-1}LW = \operatorname{diag}(\gamma_1, \dots, \gamma_N)$. We denote by $\mathscr{H}$ the space of stable, linear, time invariant and continuous-time transfer functions and by $\mathscr{H}^{q \times m}$ the space of $q \times m$ matrices with entries in $\mathscr{H}$.

The p-th norm of matrix A is $\|A\|_p = \sup_{v \neq 0} \|Av\|_p / \|v\|_p$ and $\otimes$ is the Kronecker product of matrices. If the matrix 2-norm is used, it holds that $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$ and $\|AB\|_2 \leq \|A\|_2 \|B\|_2$. The spectrum of a matrix X is denoted by $\sigma(X)$ and its *condition number* is defined as

$$\mathscr{K}_p(X) = \|X\|_p \|X^{-1}\|_p.$$

If $p = 2$, the subscript is omitted in the matrix norm and the condition number, for simplicity.

Finally, given a set $\mathscr{S}$, the set $\zeta(\mathscr{S})$ is defined as

$$\zeta(\mathscr{S}) = \left\{ s : -\frac{1}{s} \in \mathscr{S} \right\}.$$

2. PROBLEM STATEMENT

Consider a network of N nodes connected by M arcs, represented by the digraph $\mathscr{G}$. The dynamic behaviour of the network is characterised by stable MIMO linear systems associated with both its nodes and its arcs. The *nominal* node and arc dynamics are represented by $F(s) \in \mathscr{H}^{r \times n}$ and $G(s) \in \mathscr{H}^{n \times r}$ respectively. Both dynamics can be subject to homogeneous uncertainties, $\Delta_F(s) \in \mathscr{H}^{r \times n}$ for node dynamics and $\Delta_G(s) \in \mathscr{H}^{n \times r}$ for arc dynamics. The vectors $Y_p(s)$ and $U_q(s)$ represent the output of the p-th node and q-th arc respectively. Note that the environment node has no input, output or dynamics associated with it. Then, the dynamics of node i are given by

$$Y_i(s) = [F(s) + \Delta_F(s)] \sum_{h=1}^{M} [B]_{ih} U_h(s), \tag{1}$$

while the dynamics of arc $h = (i, j) \in \mathscr{A}$ are given by

$$U_h(s) = [G(s) + \Delta_G(s)][Y_i(s) - Y_j(s)]. \tag{2}$$

The output and input vectors can be stacked into $Y(s)$ and $U(s)$ to represent the complete system as follows:

$$Y(s) = [Y_1(s)^\top, \dots, Y_N(s)^\top]^\top, \; U(s) = [U_1(s)^\top, \dots, U_M(s)^\top]^\top.$$

The overall node and arc dynamics can be summarised as

$$Y(s) = [(I_N \otimes F(s)) + (I_N \otimes \Delta_F(s))](B \otimes I_n) U(s)$$
$$U(s) = -[(I_M \otimes G(s)) + (I_M \otimes \Delta_G(s))](B^\top \otimes I_r) Y(s),$$

where I_k denotes the identity matrix of size k.

The characteristic equation for the complete network is

$$\det \Big(I_{Nr} + [(I_N \otimes F(s)) + (I_N \otimes \Delta_F(s))](B \otimes I_n)$$
$$[(I_M \otimes G(s)) + (I_M \otimes \Delta_G(s))](B^\top \otimes I_r) \Big)$$
$$= \det \Big(I_{Nr} + L \otimes (H(s) + \Delta_H(s)) \Big) = 0, \tag{3}$$

where $L = BB^\top$, $H(s) = F(s)G(s) \in \mathscr{H}^{r \times r}$ and

$$\Delta_H(s) = F(s)\Delta_G(s) + \Delta_F(s)G(s) + \Delta_F(s)\Delta_G(s). \tag{4}$$

Without uncertainty, the characteristic equation becomes

$$\det \Big(I_{Nr} + L \otimes H(s) \Big) = 0. \tag{5}$$

We seek topology-independent conditions for the stability of the nominal system (5) and for the robust stability of the homogeneously uncertain system (3)–(4), which hold for all possible network topologies with a given *maximum connectivity degree* $\mathfrak{D}$ and exclusively depend on local information. We assume stability of the nominal local transfer-function matrices.

Assumption 1. The transfer-function matrix $H(s) = F(s)G(s)$ does not have poles in the closed right half plane.

We denote by $\sigma(H(s)) = \{\lambda_1(s), \dots, \lambda_r(s)\}$ the eigenvalues of the transfer-function matrix $H(s)$. The eigenvalues $\{\lambda_i(s)\}_{i=1}^r$ are not rational transfer functions: they are not a quotient of polynomials in s. In general, they are complex functions of the variable s. Therefore, the poles of $\lambda_i(s)$ are not the roots of a polynomial but the set of complex numbers $\tilde{p} \in \mathbb{C}$ such that $\lambda_i^{-1}(\tilde{p}) = 0$. We have the following result.

Theorem 1. Consider the transfer-function matrix $H(s) \in \mathscr{H}^{r \times r}$ and its eigenvalues $\lambda_1(s), \dots, \lambda_r(s)$. Let $\tilde{p} \in \mathbb{C}$ be a pole of the complex function $\lambda_i(s)$, for some $i \in \{1, \dots, r\}$. Then, $\tilde{p}$ is a pole of the transfer-function matrix $H(s)$.

Proof. Since the complex function $\lambda_i(s)$ is an eigenvalue of $H(s)$, it must satisfy the characteristic equation

$$\det \big(\lambda_i(s) I - H(s) \big) = 0. \tag{6}$$

The transfer-function matrix $H(s)$ can be written as

$$H(s) = \frac{1}{d(s)} R(s), \tag{7}$$

where $d(s)$ is the pole polynomial and $R(s)$ is a matrix with polynomial entries. For any $s \in \mathbb{C}$ such that $d(s) \neq 0$, replacing (7) into (6) gives

$$\det \big(\lambda_i(s) d(s) I - R(s) \big) = 0. \tag{8}$$

By contradiction, assume that $\tilde{p}$ is a pole of the complex function $\lambda_i(s)$ but not of the transfer-function matrix $H(s)$. Then, by continuity, $\lim_{s \to \tilde{p}} \{d(s)\} \to d(\tilde{p}) \neq 0$ and $\lim_{s \to \tilde{p}} \{R(s)\} \to R(\tilde{p})$, which is a matrix with finite entries. At the same time, $\lim_{s \to \tilde{p}} \{\lambda_i(s)\} \to \infty$. This in turn implies that $\lim_{s \to \tilde{p}} \Big\{ \det \big(\lambda_i(s) d(s) I - R(s) \big) \Big\} \to \infty$, which contradicts equation (8). Hence, it must be $\lim_{s \to \tilde{p}} \{d(s)\} \to d(\tilde{p}) = 0$, namely, $\tilde{p}$ must be a root of $d(s)$, hence a pole of $H(s)$. □

Remark 1. In view of Theorem 1, if Assumption 1 is satisfied, then the complex functions $\lambda_i(s)$ are stable: there exists some $\varepsilon < 0$ such that, for every pole $\tilde{p}$ of $\lambda_i(s)$, $\mathrm{Re}(\tilde{p}) \leq \varepsilon < 0$. In fact, we can pick ε as the largest real part of all the poles of $H(s)$, which must be strictly negative in view of Assumption 1. This is the condition we will exploit in the following results. Computing the poles of the transfer-function matrix $H(s)$, which are the roots of the denominator polynomial, is much easier than computing the poles of the generic complex functions $\lambda_i(s)$. The results presented in this paper *do not require to analytically compute the functions* $\lambda_i(s)$, nor their poles.

2.1 Technical preliminary results

The considered graphs have peculiar spectral properties. Since $\mathfrak{d}_i = L_{ii} \geq \sum_{j\neq i} |L_{ij}| = \mathfrak{d}_i^{int}$, where the equality is obtained when node i has no connections with the environment node, the symmetric matrix L is column diagonally dominant. By the Gershgorin circle theorem, the (real) eigenvalues of L lie within the union of the circles with radius $\mathfrak{d}_i^{int}$ and center in $(\mathfrak{d}_i, 0)$. Hence, matrix L is positive semidefinite (and, in the presence of at least one arc to/from the environment node, L is non-singular, cf. Giordano et al. 2016). Therefore, all the eigenvalues of L, which are real because L is symmetric, must lie in the circle of radius $\mathfrak{D}$ with center in $(\mathfrak{D}, 0)$, being $\mathfrak{D}$ the maximum diagonal entry of L. Hence,

$$\sigma(L) \subset \{z \in \mathbb{R} : 0 \leq z \leq 2\mathfrak{D}\}. \tag{9}$$

Next, we provide some technical lemmas that are needed to prove our main results in the following sections. In particular, we give a sufficient condition, which is shown to be non-conservative, for the robust stability of the feedback of a generic scalar complex function, where the feedback gain is an eigenvalue of some generalised Laplacian L with maximum connectivity degree $\mathfrak{D}$, in the presence of norm-bounded uncertainties.

Lemma 1. Consider the scalar complex function $h(s) = \bar{h}(s) + \delta_h(s)$, where the nominal function $\bar{h}(s)$ is stable and the uncertainty is bounded as $|\delta_h(s)| \leq \delta_h^{max}(s)$. Consider also the scalar coefficient $\mu \in \sigma(L_\mathfrak{D})$, where $L_\mathfrak{D}$ is the family of generalised Laplacian matrices with maximum connectivity degree $\mathfrak{D}$. Then, the feedback complex function $h_{\text{feed}}(s) = \mu h(s)(1 + \mu h(s))^{-1}$ is robustly stable for all $\mu \in \sigma(L_\mathfrak{D})$ and for all possible realisations of the uncertainty if, for all $\omega \in \mathbb{R}^+$,

$$\min_{\rho \leq -(2\mathfrak{D})^{-1}} |\bar{h}(j\omega) - \rho| > \delta_h^{max}(j\omega). \tag{10}$$

Proof. The Nyquist stability criterion requires $Z = N + P$, where P is the number of unstable poles of the complex function $h(s)$, Z is the number of unstable zeros of the complex function $1 + \mu h(s)$ and N is the number of times the Nyquist diagram encircles the point $-1/\mu$ clockwise. Because the open loop complex function is assumed to be stable ($P = 0$), the closed loop system is stable ($Z = 0$) if and only if $N = 0$, that is, if and only if the Nyquist diagram does not encircle the point $-1/\mu$. Recall that $\sigma(L_\mathfrak{D}) \subset \mathscr{S} = \{z \in \mathbb{R} : 0 \leq z \leq 2\mathfrak{D}\}$. Then, the Nyquist diagram $h(j\omega)$ cannot encircle the point $-1/\mu$ if, for all $\omega \in \mathbb{R}^+$, it has an empty intersection with the set

$$\zeta(\mathscr{S}) = \left\{ z \in \mathbb{R} : z \leq \frac{-1}{2\mathfrak{D}} \right\}. \tag{11}$$

This condition can be interpreted geometrically with the help of Figure 1 to get the equivalent condition that the distance of $\bar{h}(j\omega)$ from the set $\zeta(\mathscr{S})$ must be larger than the uncertainty, $\min_{\rho \leq -(2\mathfrak{D})^{-1}} |\bar{h}(j\omega) - \rho| > |\delta_h(j\omega)|$, for any possible uncertainty $|\delta_h(j\omega)| \leq \delta_h^{max}(j\omega)$, thus leading to (10). □

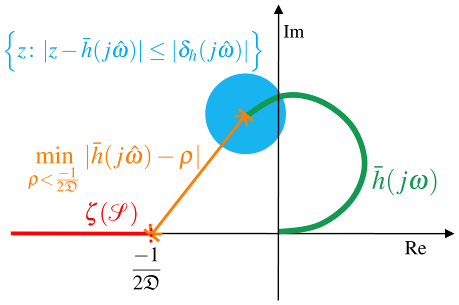

Figure 1. **Visualisation of the robust stability condition in Lemma 1.** Nyquist diagram of the nominal transfer function $\bar{h}(j\omega)$ (green), uncertainty disk centred at $\bar{h}(j\hat{\omega})$ representing $\delta_h(j\hat{\omega})$ at the frequency $\hat{\omega}$ (cyan), set $\zeta(\mathscr{S})$ (red) and distance of $\bar{h}(j\hat{\omega})$ from the set $\zeta(\mathscr{S})$ (orange). For all $\omega \in \mathbb{R}^+$, the distance of $\bar{h}(j\omega)$ from the set $\zeta(\mathscr{S})$ needs to be larger than the uncertainty radius $\delta_h^{max}(j\omega) \geq |\delta_h(j\omega)|$.

The condition (10) in Lemma 1 is not conservative, as stated in the following result, whose proof is omitted for space reasons.

Lemma 2. Consider the scalar complex function $h(s) = \bar{h}(s) + \delta_h(s)$, where the nominal function $\bar{h}(s)$ is stable and the uncertainty is bounded as $|\delta_h(s)| \leq \delta_h^{max}(s)$. If

$$\min_{\rho \leq -(2\mathfrak{D})^{-1}} |\bar{h}(j\omega) - \rho| < \delta_h^{max}(j\omega), \tag{12}$$

for some $\omega \in \mathbb{R}^+$, then there exists a graph with maximum connectivity degree $\mathfrak{D}$ and generalised Laplacian matrix $L_\mathfrak{D}$ such that the complex function $h_{\text{feed}}(s) = \mu h(s)(1 + \mu h(s))^{-1}$, where $\mu \in \sigma(L_\mathfrak{D})$, is unstable for some uncertainty realisation.

3. TOPOLOGY-INDEPENDENT NOMINAL STABILITY

We provide a topology-independent stability condition for the interconnection of identical MIMO systems at the nodes through identical MIMO arc dynamics.

Theorem 2. Given the system (5) under Assumption 1, stability is ensured for *all* networks with maximum connectivity degree $\mathfrak{D}$ if, for all $i \in \{1, \ldots, r\}$ and $\omega \in \mathbb{R}^+$,

$$\min_{\rho \leq -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho| > 0. \tag{13}$$

Proof. The generalised Laplacian can be diagonalised as $W^{-1}LW = \mathrm{diag}\{\gamma_k\}_{k=1}^N$ and the transfer function matrix can be triangularised as $V(s)^{-1}H(s)V(s) = \Lambda$, where Λ is a triangular matrix carrying on the diagonal the eigenvalues $\{\lambda_k(s)\}_{k=1}^r$ of $H(s)$. Then, we can pre-multiply the characteristic polynomial $p(s) = \det\left(I_{Nr} + L \otimes H(s)\right)$ by $\det((W \otimes V(s))^{-1})$ and post-multiply it by $\det(W \otimes V(s))$ to get

$$p(s) = \prod_{i=1}^{r} \prod_{k=1}^{N} (1 + \gamma_k \lambda_i(s)). \tag{14}$$

The characteristic polynomial (14) is stable if and only if each polynomial $1 + \gamma_k \lambda_i(s)$ is stable.

Assumption 1 and Theorem 1 guarantee that the complex functions $\lambda_i(s)$ are stable; see Remark 1. Therefore, we can apply Lemma 1 where $\bar{h}(s) = \lambda_i(s)$ and $\delta_h^{max} \equiv 0$ to conclude that the polynomials $1 + \gamma_k \lambda_i(s)$ are stable if, for all i and all $\omega \in \mathbb{R}^+$, (13) holds. □

Condition (13) can be checked locally, regardless of the network size and topology, and guarantees the stability of the whole dynamic network also if new nodes and arcs are added or removed, as long as the maximum connectivity degree is $\mathfrak{D}$. This ensures robustness to online modifications, and scalability.

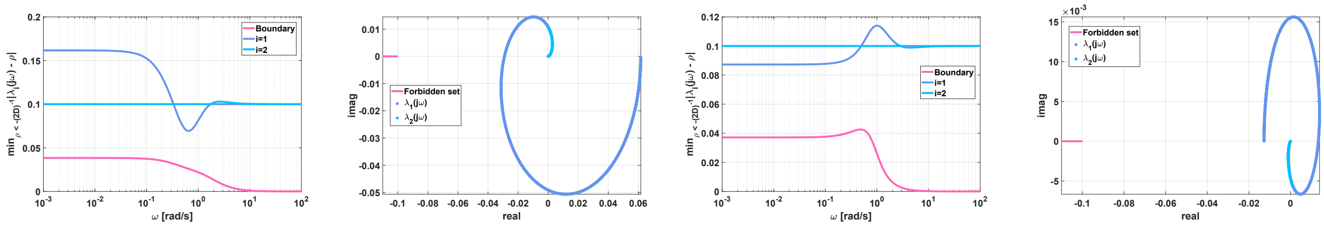

Figure 2. **Visualisations of the stability conditions** (13) **in Theorem 2 and** (15) **in Theorem 4, for the systems in Examples 1 (first and second panel) and 2 (third and fourth panel). First and third panel:** The two blue lines are $\min_{\rho \le -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho|$ for $i = 1, 2$, with $\mathfrak{D} = 5$. Nominal topology-independent stability requires that they are strictly above zero, while robust topology-independent stability requires that they are strictly above the bound $\mathscr{K}(V(j\omega))\Delta_H^{max}(j\omega)$ shown in magenta. **Second and fourth panel:** Nominal stability is guaranteed since the blue curves $\lambda_1(j\omega)$ and $\lambda_2(j\omega)$ never intersect the forbidden set $\mathscr{S} = \{z \in \mathbb{R} \colon z \le -0.1\}$, shown in magenta.

Remark 2. In view of Lemma 2, condition (13) is essentially non-conservative. Indeed, if the condition is violated and $\min_{\rho \le -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho|$ is zero at some ω, then there is zero stability margin, since infinitesimal uncertainties bounded by $\varepsilon > 0$, *no matter how small*, can lead to instability.

Example 1. Consider an arbitrary network where the node and arc nominal transfer-function matrices are

$$F(s) = \frac{1}{s^2 + 3.412s + 2.871} \begin{bmatrix} 0.1307s - 0.08404 \\ -0.1105s + 0.06774 \end{bmatrix},$$

$$G(s) = \frac{1}{s^2 + 1.805s + 0.4837} \begin{bmatrix} 0.1485s - 0.753 \\ 0.4924s + 0.329 \end{bmatrix}^\top,$$

while the maximum connectivity degree is $\mathfrak{D} = 5$. Numerically evaluating the eigenvalues $\lambda_1(s)$ and $\lambda_2(s)$ of $H(s)$ for $s = j\omega$ is enough to check that the graphical stability condition (13) in Theorem 2 is satisfied: the first panel in Figure 2 shows that both the blue lines representing $\min_{\rho \le -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho|$ for $i = 1, 2$ are strictly above zero, while the third panel in Figure 2 shows that both $\lambda_1(j\omega)$ and $\lambda_2(j\omega)$ never intersect the forbidden set $\mathscr{S} = \{z \in \mathbb{R} \colon z \le -0.1\}$.

Example 2. Consider an arbitrary network where the node and arc nominal transfer-function matrices are

$$F(s) = \frac{1}{s^2 + 3.62s + 3.523} \begin{bmatrix} -0.06235s + 0.1982 \\ 0.09975s + 0.1692 \end{bmatrix},$$

$$G(s) = \frac{1}{s^2 + 1.213s + 0.7026} \begin{bmatrix} -0.3224s - 0.2307 \\ 0.2809s + 0.08381 \end{bmatrix}^\top,$$

while the maximum connectivity degree is $\mathfrak{D} = 5$. The second and fourth panel in Figure 2 show that the graphical stability condition (13) in Theorem 2 is satisfied.

Example 3. Consider the node and arc nominal transfer-function matrices

$$F(s) = \frac{1}{s^2 + 3.338s + 2.613} \begin{bmatrix} 0.08071s + 2.308 \\ 0.6187s + 1.78 \end{bmatrix},$$

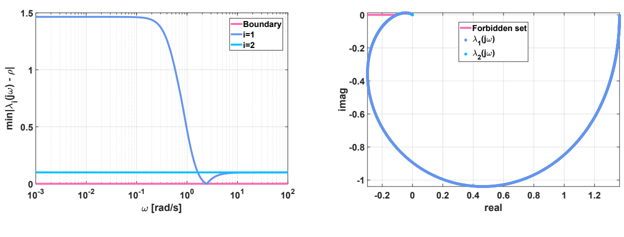

Figure 3. **Alternative visualisations of the stability condition** (13) **in Theorem 2, for the system in Example 3. Left**: the two blue lines are $\min_{\rho \le -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho|$ for $i = 1, 2$, with $\mathfrak{D} = 5$. The condition for nominal topology-independent stability is that they are strictly above zero, which is violated. **Right**: for nominal stability, the curves $\lambda_1(j\omega)$ and $\lambda_2(j\omega)$ should never intersect the forbidden set $\mathscr{S} = \{z \in \mathbb{R} \colon z \le -0.1\}$; the condition is violated.

$$G(s) = \frac{1}{s^2 + 0.9124s + 0.2505} \begin{bmatrix} 0.8022s + 0.2204 \\ 0.1696s + 0.2161 \end{bmatrix}^\top.$$

We wish to check whether stability is guaranteed for all networks with maximum connectivity degree $\mathfrak{D} = 5$. The graphical stability condition (13) in Theorem 2 is violated: in Figure 3 left, the blue line representing $\min_{\rho \le -(2\mathfrak{D})^{-1}} |\lambda_1(j\omega) - \rho|$ is zero for $\omega \approx 2.5$, while Figure 3 right shows that $\lambda_1(j\omega)$ intersects the forbidden set $\mathscr{S} = \{z \in \mathbb{R} \colon z \le -0.1\}$. There exists at least one dynamic network with the given node and arc nominal transfer-function matrices and with maximum connectivity degree 5 that is not stable: take, for instance, the network with 5 nodes and 9 arcs (8 internal and 1 external) whose topology is described by the generalised Laplacian matrix

$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

It can be seen that the overall interconnected system is unstable, because it has positive-real-part poles.

4. TOPOLOGY-INDEPENDENT ROBUST STABILITY WITH HOMOGENEOUS UNCERTAINTIES

Also in the presence of homogeneous uncertainties, affecting both node and arc MIMO dynamics, we can provide a topology-independent condition for robust stability, which relies on the eigenvalue decomposition of the uncertain system.

We need a bound for the eigenvalues of uncertain matrices, which is provided by the Bauer-Fike theorem.

Theorem 3. (Bauer and Fike 1960). Consider the two matrices $A, M \in \mathbb{R}^{n \times n}$, with A diagonalisable, that is, $V^{-1}AV = \text{diag}(\lambda_1, ..., \lambda_n)$ for some $V \in \mathbb{C}^{n \times n}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. For every (complex) eigenvalue β of $A + M$, there exists an index $i \in \{1, \dots, n\}$ such that $|\beta - \lambda_i| \le \mathscr{K}_p(V)\|M\|_p$.

Here we will only use the 2-norm and condition number $\mathscr{K}_2$, and omit the subscripts for clarity.

To be able to apply the Bauer-Fike theorem, we need to assume the diagonalisability of $H(s)$.

Assumption 2. The transfer-function matrix $H(s) = F(s)G(s)$, with eigenvalues $\sigma(H(s)) = \{\lambda_1(s), \dots, \lambda_r(s)\}$, can be diagonalised by the change-of-basis matrix $V(s)$, so that

$$V(s)^{-1}H(s)V(s) = \text{diag}(\lambda_1(s), ..., \lambda_r(s)).$$

Remark 3. For the important classes of MISO and SIMO systems, Assumption 2 is automatically satisfied. In fact, if $F(s)$ is a row vector and $G(s)$ is a column vector, then $H(s)$ is a scalar function; if $F(s)$ is a column vector and $G(s)$ is a row vector, then $H(s)$ is a rank-one matrix, hence it is diagonalisable.

We can now state and prove the main result of this section.

Theorem 4. Given the system (3)–(4) under Assumptions 1 and 2, stability is ensured for *all* networks with maximum connectivity degree $\mathfrak{D}$ and for all uncertainties with $\|\Delta_H(j\omega)\| \leq \Delta_H^{max}(j\omega)$ if, for all $i \in \{1,\dots,r\}$ and $\omega \in \mathbb{R}^+$,

$$\min_{\rho \leq -(2\mathfrak{D})^{-1}} |\lambda_i(j\omega) - \rho| > \mathscr{K}(V(j\omega))\Delta_H^{max}(j\omega). \tag{15}$$

Proof. Recalling that $W^{-1}LW$ is a diagonal matrix carrying the eigenvalues $\{\gamma_k\}_{k=1}^N$ in the diagonal, pre-multiplying the characteristic polynomial $p(s) = \det\left(I_{Nr} + L \otimes (H(s) + \Delta_H(s))\right)$ by $\det((W \otimes I_r)^{-1})$ and post-multiplying it by $\det(W \otimes I_r)$ yields

$$p(s) = \prod_{k=1}^N \det\left(I_r + \gamma_k\big(H(s) + \Delta_H(s)\big)\right).$$

Let $\{\beta_q(s)\}_{q=1}^r$ denote the eigenvalues of $H(s) + \Delta_H(s)$. Then the characteristic polynomial can be written as

$$p(s) = \prod_{q=1}^r \prod_{k=1}^N (1 + \gamma_k \beta_q(s)), \tag{16}$$

which is stable if and only if each polynomial $1 + \gamma_k \beta_q(s)$ is stable. In view of Assumption 2, we can apply the Bauer-Fike theorem; hence, there exists an index $i \in \{1,\dots r\}$ such that $\beta_q(s) = \lambda_i(s) + \delta_{\lambda_i}(s)$, where

$$|\beta_q(s) - \lambda_i(s)| = |\delta_{\lambda_i}(s)| \leq \mathscr{K}(V(s))\|\Delta_H(s)\| \leq \mathscr{K}(V(s))\Delta_H^{max}(s).$$

Therefore, if all the uncertain complex functions $\lambda_i(s) + \delta_{\lambda_i}(s)$ are stable for all possible scalar feedback gains $\gamma_k \in \sigma(L)$, where L is the generalised Laplacian matrix of any network with maximum connectivity degree $\mathfrak{D}$, then the characteristic polynomial $p(s)$ is stable. Since Assumption 1 and Theorem 1 guarantee that the nominal complex functions $\lambda_i(s)$ are stable (Remark 1), the proof can be concluded by applying Lemma 1, where $\bar{h}(s) = \lambda_i(s)$, $\delta_h(s) = \delta_{\lambda_i}(s)$ and $\delta_h^{max}(s) = \mathscr{K}(V(s))\Delta_H^{max}(s)$, for all $i \in \{1,\dots r\}$. □

The graphical condition (15) for robust stability is scalable, because it can be checked locally and it is size- and topology-independent. It allows stability-preserving plug-and-play modifications (Bendtsen et al. 2013; Riverso et al. 2013) to the network as long as the maximum connectivity degree is $\mathfrak{D}$, which can be checked only by the newly added nodes or arcs.

Some conservativeness is introduced by the Bauer-Fike theorem, on which the proof of the result relies.

Let us now consider a suitable upper bound $\phi(j\omega)$ for the spectral radius of $H(j\omega)$,

$$|\lambda_i(j\omega)| \leq \phi(j\omega) \quad \text{for all } i, \quad \text{for } \omega \in \mathbb{R}_+, \tag{17}$$

and a suitable upper bound $\xi(j\omega)$ for the condition number $\mathscr{K}(V(j\omega))$,

$$\mathscr{K}(V(j\omega)) \leq \xi(j\omega) \quad \text{for } \omega \in \mathbb{R}_+. \tag{18}$$

Bounds of these types have been well studied in the literature; for instance, Cheng (2014) presents a comprehensive survey on bounds for condition numbers.

Then, a more conservative sufficient condition, which however allows to assess robust topology-independent stability without the need of computing the eigenvalues and eigenvectors of $H(s)$, is the following.

Corollary 1. Consider the system (3)–(4) under Assumptions 1 and 2 and define $\mathscr{C}$ as

$$\mathscr{C} = \sup_{\omega \in \mathbb{R}^+} \{\phi(j\omega)\},$$

where $\phi(j\omega)$ is the bound in (17), and $\mathscr{M}$ as

$$\mathscr{M} = \sup_{\omega \in \mathbb{R}^+} \{\xi(j\omega)\Delta_H^{max}(j\omega)\},$$

where $\xi(j\omega)$ is the bound in (18) and $\|\Delta_H(j\omega)\| \leq \Delta_H^{max}(j\omega)$. Then, stability is ensured for *all* networks with maximum connectivity degree $\mathfrak{D}$ if

$$\mathscr{C} + \mathscr{M} < (2\mathfrak{D})^{-1}. \tag{19}$$

Proof. Using the same decomposition as in the proof of Theorem 4, the eigenvalues of matrix $H(s) + \Delta_H(s)$ can be written as $\beta_q(j\omega) = \lambda_i(j\omega) + \delta_{\lambda_i}(j\omega)$ for some $i \in \{1,\dots,r\}$, where

$$|\delta_{\lambda_i}(j\omega)| < \mathscr{K}(V(j\omega))\|\Delta_H(j\omega)\| \leq \xi(j\omega)\Delta_H^{max}(j\omega).$$

Hence

$$|\beta_q(j\omega)| \leq |\lambda_i(j\omega)| + |\delta_{\lambda_i}(j\omega)| \leq \mathscr{C} + \mathscr{M}, \quad \forall \omega \in \mathbb{R}^+.$$

If

$$\mathscr{C} + \mathscr{M} < \min_{z \in \zeta(\mathscr{S})} |z| = \frac{1}{2\mathfrak{D}}, \tag{20}$$

where $\mathscr{S} = \{z \in \mathbb{R} : 0 \leq z \leq 2\mathfrak{D}\}$, then the distance between $\lambda_i(j\omega) + \delta_{\lambda_i}(j\omega)$ and the set $\zeta(\mathscr{S})$ is larger than zero for all i and for all possible bounded realisations of the uncertainty, which ensures topology-independent robust stability of all networks with maximum connectivity degree $\mathfrak{D}$. □

Corollary 2. Consider the system (3)–(4) under Assumptions 1 and 2, let $\mathscr{C}$ and $\mathscr{M}$ be defined as in Corollary 1 and define

$$\mathscr{T} = \frac{1}{2(\mathscr{C} + \mathscr{M})}. \tag{21}$$

Then, stability is ensured for *all* networks where each node i satisfies $\mathfrak{d}_i < \mathscr{T}$.

Proof. With $\mathscr{T}$ as defined in (21), the inequality (19) becomes $\mathfrak{D} < \mathscr{T}$ and is of course satisfied if and only if $\mathfrak{d}_i < \mathscr{T}$ for all $i = 1,\dots,N$, since $\mathfrak{D} = \max_{i=1,\dots,N}\{\mathfrak{d}_i\}$. □

Corollary 2 gives *fully local* sufficient conditions for robust stability, which are *independent of the network size and topology* and do not even rely on the shared knowledge of the maximum connectivity degree. As long as each node satisfies the local condition, new arcs and nodes can be added, removed or modified, and the overall networked system remains stable. Furthermore two separate stable dynamic networks can be connected and, as long as all the connecting nodes satisfy the local condition, the resulting dynamic network is stable.

Remark 4. The conditions in Theorems 2 and 4 allow us to verify if, *given* a maximum connectivity degree $\mathfrak{D}$, we have topology-independent stability for all networks with node dynamics $F(s)$ and arc dynamics $G(s)$, possibly in the presence of homogeneous bounded uncertainties. The conditions in Corollaries 1 and 2 can be alternatively interpreted as providing the largest $\mathfrak{D}$ such that all networks with maximum connectivity degree $\mathfrak{D}$, node dynamics $F(s)$ and arc dynamics $G(s)$ are guaranteed to be (robustly) stable.

Example 4. In an arbitrary network, assume that all node and arc dynamics are nominally as in Example 1 and are affected by suitably bounded, but unknown, homogeneous uncertainties. The bound $\mathscr{K}(V(j\omega))\Delta_H^{max}(j\omega)$ is reported in magenta in the first panel of Figure 2, which shows that the graphical robust stability condition (15) in Theorem 4 is satisfied for all networks with maximum connectivity degree $\mathfrak{D} = 5$. The

condition is satisfied up to $\mathfrak{D} = 8$, but violated for $\mathfrak{D} \geq 9$.
For the given transfer functions, we have the bounds

$$\mathscr{C} = 0.0616 \text{ and } \mathscr{M} = 0.0384.$$

Therefore, by Corollary 2, topology-independent stability is guaranteed for all networks where the connectivity degree is $\mathfrak{d}_i < 5$ for each $i \in \{1, \dots, N\}$. This shows that the condition in Corollary 2 is more conservative.
In the absence of uncertainties, $\mathscr{M} = 0$, Corollary 2 allows for a maximum connectivity degree of 8: each node could be connected to 8 other nodes and the network would remain stable. This is conservative, because the condition in Theorem 2 is satisfied for values of $\mathfrak{D}$ up to 17.

Example 5. Consider an arbitrary network with nominal node and arc dynamics as in Example 2, affected by unknown, homogeneous uncertainties that are suitably bounded. The bound $\mathscr{K}(V(j\omega))\Delta_H^{max}(j\omega)$ is reported in magenta in the second panel of Figure 2, which shows that the graphical robust stability condition (15) in Theorem 4 is satisfied for all networks with maximum connectivity degree $\mathfrak{D} = 5$. The condition is satisfied up to $\mathfrak{D} = 9$, but violated for $\mathfrak{D} \geq 10$.
Since, for the given transfer functions, we have the bounds

$$\mathscr{C} = 0.016 \text{ and } \mathscr{M} = 0.0426.$$

Corollary 2 guarantees topology-independent stability for all networks, *regardless of the number of nodes and of the network topology*, if the connectivity degree is $\mathfrak{d}_i < 8.5$ for each $i \in \{1, \dots, N\}$. The increased conservativeness of the condition in Corollary 2 is reduced in this example.
If there were no uncertainties ($\mathscr{M} = 0$), the maximum connectivity degree according to Corollary 2 would become 31. However, the condition in Theorem 2 is satisfied for values of $\mathfrak{D}$ that are even more than 40.

5. CONCLUSIONS AND FUTURE WORK

We have investigated the stability of homogeneous dynamic networks where both the nodes and the arcs have MIMO dynamics, described by the uncertain transfer-function matrices $F + \Delta_F$ and $G + \Delta_G$ respectively. The transfer-function matrices F and G are assumed to be stable and the uncertainties bounded. In the nominal case, we have provided a stability condition that is topology-independent and exclusively relies on the knowledge of the maximum connectivity degree of the network. The condition constraints the Nyquist diagram of the poles of the transfer-function matrix $H = FG$. In the presence of homogeneous uncertainties, the topology-independent condition for robust stability depends on the magnitude of the uncertainty and relies on the bound for the eigenvalues of uncertain matrices given by the Bauer-Fike theorem. An advantage of the obtained conditions, which guarantee robust stability regardless of the network size and topology, is that they can be checked locally to ensure stability of the network also when nodes and arcs are added or removed online (Bendtsen et al. 2013; Riverso et al. 2013). Future research aims at providing robust stability conditions when possibly heterogeneous uncertainties affect both node and arc dynamics. In this case, it is currently unclear whether topology-independent conditions can be achieved or the network interconnection needs to be taken into account when assessing robust stability of the overall dynamic network.

REFERENCES

Andersen, M., Pakazad, S., Hansson, A., and Rantzer, A. (2014). Robust stability analysis of sparsely interconnected uncertain systems. *IEEE Trans. Autom. Control*, 59(8), 2151–2156.

Bauer, F.L. and Fike, C.T. (1960). Norms and exclusion theorems. *Numerische Mathematik*, 2(1), 137–141.

Bendtsen, J., Trangbaek, K., and Stoustrup J. (2013). Plug-and-play control – Modifying control systems online. *IEEE Trans. Control Systems Tech.*, 21 (1), 79–93.

Blanchini, F., Casagrande, D., Giordano, G., and Viaro, U. (2017). Topology-independent robust stability of homogeneous dynamic networks. *IFAC-PapersOnLine*, 50(1), 1736–1741.

Blanchini, F., Casagrande, D., Giordano, G., and Viaro, U. (2018). A bounded complementary sensitivity function ensures topology-independent stability of homogeneous dynamical networks. *IEEE Trans. Autom. Control*, 63(4), 1140–1146.

Blanchini, F., Franco, E., and Giordano, G. (2015). Network-decentralized control strategies for stabilization. *IEEE Trans. Autom. Control*, 60 (2), pp. 491–496.

Cheng, G. (2014). Note on some upper bounds for the condition number. *J. Math. Inequalities*, 8(2), 6.

Desoer, C. and Wang, Y.T. (1980). On the generalized Nyquist stability criterion. *IEEE Trans. Autom. Control*, 25(2), 187–196.

Giordano, G., Blanchini, F., Franco, E., Mardanlou, V., and Montessoro, P.L. (2016). The smallest eigenvalue of the generalized Laplacian matrix, with application to network-decentralized estimation for homogeneous systems. *IEEE Trans. Netw. Science Eng.*, 3(4), 312–324.

Hara, S., Iwasaki, T., and Hori, Y. (2019). Robust stability analysis for LTI systems with generalized frequency variables and its application to gene regulatory networks. *Automatica*, 105, 96–106.

Hara, S., Hayakawa, T., and Sugata, H. (2007). Stability analysis of linear systems with generalized frequency variables and its applications to formation control. In *IEEE Conf. Decision Control*, 1459–1466.

Hara, S., Hayakawa, T., and Sugata, H. (2009). LTI systems with generalized frequency variables: A unified framework for homogeneous multi-agent dynamical systems. *SICE J. Control, Meas., and Syst. Integration*, 2(5).

Hara, S., Tanaka, H., and Iwasaki, T. (2014). Stability analysis of systems with generalized frequency variables. *IEEE Trans. Autom. Control*, 59(2), 313–326.

Hori, Y., Miyazako, H., Kumagai, S., and Hara, S. (2015). Coordinated spatial pattern formation in biomolecular communication networks. *IEEE Trans. Molecular, Biological and Multi-Scale Comm.*, 1(2), 111–121.

Jönsson, U. T., and Kao, C-Y. (2010). A scalable robust stability criterion for systems with heterogeneous LTI components. *IEEE Trans. Autom. Control*, 55(10), 2219–2234.

Kao, C-Y., Jönsson, U., and Fujioka, H. (2009). Characterization of robust stability of a class of interconnected systems. *Automatica*, 45, 217–224.

Khong, S. and Rantzer, A. (2014). Scalable stability conditions for heterogeneous networks via integral quadratic constraints. In *European Control Conf.*, 2863–2867.

Lestas, I. and Vinnicombe, G. (2006). Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems. *IEEE Trans. Autom. Control*, 51(10), 1613–1625.

Lin, Z., Wang, L., Han, Z. and Fu, M. (2016). A graph Laplacian approach to coordinate-free formation stabilization for directed networks *IEEE Trans. Autom. Control*, 61(5), 1269–1280.

Massioni, P., and Verhaegen, M. (2009). Distributed control for identical dynamically coupled systems: A decomposition approach. *IEEE Trans. Autom. Control*, 54(1), 124–135.

Nepusz, T. and Vicsek, T. (2012). Controlling edge dynamics in complex networks. *Nature Physics*, 8, 568–573.

Olfati-Saber, R., Fax, J. A., and Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings IEEE*, 95(1), 215–233.

Pates, R. and Vinnicombe, G. (2012). Stability certificates for networks of heterogeneous linear systems. In *IEEE Conf. Decision Control*, 6915–6920.

Pates, R. and Vinnicombe, G. (2017). Scalable design of heterogeneous networks. *IEEE Trans. Autom. Control*, 62(5), 2318–2333.

Riverso, S., Farina, M., and Ferrari-Trecate, G. (2013). Plug-and-play decentralized model predictive control for linear systems. *IEEE Trans. Autom. Control*, 58 (10), 2608–2614.

Trentelman, H. L., Takaba, K., and Monshizadeh, N. (2013). Robust synchronization of uncertain linear multi-agent systems. *IEEE Trans. Autom. Control*, 58(6), 1511–1523.