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Ellipsoid bundle and its application to set-membership estimation ⋆

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Abstract: This paper studies set-membership estimation for discrete linear time-varying systems subject to unknown disturbance and noise, which are bounded by ellipsoids. To improve the existing ellipsoid-based set-membership estimation methods, we propose a new set representation tool, called *ellipsoid bundle*, which combines the advantages of ellipsoids and zonotopes for uncertainty set representation and computation. Then, ellipsoidal bundles are used to design a new set-membership estimation method.

Keywords: Ellipsoid, set-membership estimation, zonotope, ellipsoid bundle.

1. INTRODUCTION

State estimation is an important topic in control theory and engineering applications. However, a practical system is usually affected by various model uncertainties, for instance, process disturbances and measurement noises. One of the most important problems in state estimation is how to deal with such uncertainties to obtain reliable estimations. Many existing state estimation methods are designed based on stochastic theory, for instance the well-known Kalman filter(Kalman, 1960). These methods usually assume that uncertainties have known probability distributions. However, in many situations, these probability distributions are unknown. Some disturbances may even not be random. Different from the state estimation methods based on stochastic system theory, set-membership estimation methods only assume that model uncertainties are unknown but bounded. This is a weaker assumption, which can be satisfied in most practical systems. Moreover, set-membership estimation methods can obtain the reachable set of state instead of a single trajectory estimation, which is useful in many applications. Recently, set-membership estimation has received considerable attention and has been applied to various fields such as fault diagnosis (Tang, Wang, and Shen, 2018; Xu, Tan, Wang, Wang, Liang, and Yuan, 2019) and robust control (Canale, Fagiano, and Milanese, 2009; Efimov, Raïssi, and Zolghadri, 2013).

Set-membership estimation aims to construct a compact set enclosing all admissible state values that are consistent with the system model, the input and measurement data. Many geometrical sets, for instance ellipsoids, polytopes and zonotopes, have been used to design set-membership estimation methods. For different choices, an important problem is how to make a tradeoff between estimation accuracy and computation complexity. In the pioneering work of Schweppe (1968), a recursive set-membership estimation method based on ellipsoids was proposed. The ellipsoid-based methods have low computation complexity and have attracted much attention (Fogel and Huang, 1982; Maksarov and Norton, 1996; Chernousko, 2005; Zhang, Wang, Raïssi, and Shen, 2019). However, the estimations by the ellipsoid-based methods are usually conservative since a single ellipsoid is not able to describe a complex reachable set of state. General polytopes can be used to achieve more accurate set-membership estimation (Blanchini and Miani, 2008), but the computation complexity may be very high. Zonotopes, a special kind of polytopes, provide a good tradeoff between estimation accuracy and computation complexity. Recently, set-membership estimation methods based on zonotopes have received considerable attention (Alamo, Bravo, and Camacho, 2005; Combastel, 2015; Tang, Wang, Wang, Raïssi, and Shen, 2019).

The accuracy of existing methods based on ellipsoids is limited by the shape of a single ellipsoid. However, the ellipsoidal estimation sets have smooth boundaries, which is a good property when the results are used for some optimization purposes. Similarly, in the classical curve fitting problem, splines are often preferred to piecewise linear interpolation. In addition, it is more suitable to use ellipsoids in some specific situations, for instance in target tracking problems (Maksarov and Norton, 1996). Considering these, this paper proposes a new set representation tool, called *ellipsoid bundle*, which combines the advantages of ellipsoids and zonotopes. Like zonotopes, ellipsoid bundles are able to describe complex convex sets. Therefore, ellipsoid bundles can be used to design a set-membership estimation method less conservative than the ellipsoid-based method. *Moreover, ellipsoid bundles also have smooth boundaries.* This paper has the following main

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contributions. First, we propose a new set representation tool, ellipsoid bundle, which combines the advantages of ellipsoids and zonotopes for uncertainty set representation and computation. Second, the basic properties of ellipsoid bundles are investigated. Third, a novel set-membership estimation method is proposed based on ellipsoid bundles. Finally, simulation results demonstrate the effectiveness of the proposed set-membership estimation method.

2. PRELIMINARIES

The following notations are standard in this paper. $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ denote the $m \times n$ dimensional and n dimensional Euclidean spaces, respectively. A bold letter represents a set in the rest of this paper. The symbols $\leq$ and $\geq$ are understood element-wise. Given a matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ ($P \preceq 0$) indicates that P is a positive (negative) semidefinite matrix.

Definition 1. Given two sets, $\mathbf{X}_1 \subset \mathbb{R}^n$ and $\mathbf{X}_2 \subset \mathbb{R}^n$, their Minkowski sum is defined as

$$\mathbf{X}_1 \oplus \mathbf{X}_2 = \{x \in \mathbb{R}^n : x = x_1 + x_2,\, x_1 \in \mathbf{X}_1,\, x_2 \in \mathbf{X}_2\}.$$

Definition 2. An m-order zonotope is an affine transformation of the hypercube $\mathbf{B}^m = [-1, 1]^m$, i.e.

$$\boldsymbol{\mathcal{Z}} = \langle p, H\rangle = \{x \in \mathbb{R}^n : x = p + Hz,\, z \in \mathbf{B}^m\},$$

where $p \in \mathbb{R}^n$ is the center of $\boldsymbol{\mathcal{Z}}$ and $H \in \mathbb{R}^{n\times m}$ defines its shape and size.

Definition 3. A non-degenerate ellipsoid is defined as

$$\{x \in \mathbb{R}^n : (x - c)^T P^{-1}(x - c) \leq 1\}, \tag{1}$$

where $c \in \mathbb{R}^n$ is the center of the ellipsoid and $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix which determines its shape and size.

This definition, though widely adopted in the literature, does not cover degenerate ellipsoids, for which the matrix P would be singular. The following definition (Durieu, Walter, and Polyak, 2001) covering both degenerate and non-degenerate ellipsoids will be adopted in the remaining part of this paper.

$$\{x \in \mathbb{R}^n : x = c + Lz,\, z \in \mathbb{R}^m,\, z^T z \leq 1\}, \tag{2}$$

where $c \in \mathbb{R}^n$ and $L \in \mathbb{R}^{n\times m}$.

For non-degenerate ellipsoids (with non-singular P), these two definitions are equivalent with $P = LL^T$. Moreover, any matrix $L \in R^{n\times m}$ such that LL^T is equal to the same P defines the same set with (2). The parametrization of an ellipsoid by L is thus clearly redundant. To avoid this redundancy, in what follows, an ellipsoid will be denoted by $\mathbf{E}(c, P)$, with c representing its center and $P = LL^T$.

Definition 4. The support function of a convex set $\mathbf{S} \subset \mathbb{R}^n$ in terms of a vector $l \in \mathbb{R}^n$ is defined as

$$\rho_{\mathbf{S}}(l) = \max_{s\in\mathbf{S}} l^T s.$$

Lemma 1. (Schweppe, 1968) Given a non-degenerate ellipsoid $\mathbf{E}(c, P)$ defined as (1), its support function in terms of $l \in \mathbb{R}^n$ is

$$\rho_{\mathbf{E}(c,P)}(l) = l^T c + \sqrt{l^T P l}.$$

Although Lemma 1 was proposed for non-degenerate ellipsoids in Schweppe (1968), it can be easily extended to the ellipsoids defined as (2).

Lemma 2. Give an ellipsoid $\mathbf{E}(c, P)$ defined as (2) with $P = LL^T$, its support function in terms of $l \in \mathbb{R}^n$ is

$$\rho_{\mathbf{E}(c,P)}(l) = l^T c + \sqrt{l^T P l}.$$

Proof. According to the definition of $\mathbf{E}(c, P)$ in (2), we have

$$\begin{aligned}\rho_{\mathbf{E}(c,P)}(l) &= \max_{x\in\mathbf{E}(c,P)} l^T x \\ &= \max_{z^T z\leq 1} (l^T c + l^T L z) \\ &= l^T c + \max_{z^T z\leq 1} (l^T L) z \\ &= l^T c + \rho_{\mathbf{E}(0,I)}(L^T l)\end{aligned} \tag{3}$$

Since $\mathbf{E}(0, I)$ is a non-degenerate ellipsoid, then according to Lemma 1 and (3), we have

$$\begin{aligned}\rho_{\mathbf{E}(c,P)} &= l^T c + \sqrt{l^T L L^T l} \\ &= l^T c + \sqrt{l^T P l}.\end{aligned}$$

□

Lemma 3. (Schweppe, 1968) Given two convex sets, $\mathbf{S}_1 \subset \mathbb{R}^n$, $\mathbf{S}_2 \subset \mathbb{R}^n$, and a vector $l \in \mathbb{R}^n$, we have

$$\rho_{\mathbf{S}_1\oplus\mathbf{S}_2}(l) = \rho_{\mathbf{S}_1}(l) + \rho_{\mathbf{S}_2}(l).$$

3. PROBLEM FORMULATION

Consider the following discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + w_k \\ y_k = C_k x_k + v_k \end{cases} \tag{4}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ are the vectors of state, input and measurement output, respectively. $w_k \in \mathbb{R}^{n_x}$ and $v_k \in \mathbb{R}^{n_y}$ are the process disturbance and measurement noise. $A_k \in \mathbb{R}^{n_x\times n_x}$, $B_k \in \mathbb{R}^{n_x\times n_u}$ and $C_k \in \mathbb{R}^{n_y\times n_x}$ are known matrices.

The process disturbance and measurement noise are assumed to be unknown but bounded by known ellipsoids.

$$w_k \in \mathbf{Q}_k = \mathbf{E}(0, Q_k), \quad v_k \in \mathbf{R}_k = \mathbf{E}(0, R_k), \tag{5}$$

where $Q_k \in \mathbb{R}^{n_x\times n_x}$ and $R_k \in \mathbb{R}^{n_y\times n_y}$ are known positive definite matrices. These ellipsoidal assumptions are particularly suitable in target tracking problems, where uncertainties are naturally related to the Euclidean distance.

In addition, the initial state is also assumed to be unknown but bounded by an ellipsoid.

$$x_0 \in \mathbf{P}_0 = \mathbf{E}(c_0, P_0), \tag{6}$$

where $c_0 \in \mathbb{R}^{n_x}$ and $P_0 \in \mathbb{R}^{n_x\times n_x}$ is a positive definite matrix.

In this paper, we aim to obtain a convex set $\mathbf{X}_k \subset \mathbb{R}^{n_x}$ such that

$$x_k \in \mathbf{X}_k, \quad \forall k \geq 0.$$

In order to increase the tightness of this convex set, we need improve the existing ellipsoid-based methods. In this paper, we will propose a new set representation tool and design a set-membership estimation method based on the proposed tool.

4. ELLIPSOID BUNDLE

The ellipsoid-based set-membership estimation methods have low computation complexity. However, the estimations obtained by the ellipsoid-based methods may be very conservative since a single ellipsoid has limitation to describe a complex set. Compared to ellipsoids, zonotopes (a special kind of polytopes) are able to describe complex sets with finite parameters. Zonotopes provide a way to make a good tradeoff between estimation accuracy and computation complexity, but they have non-smooth boundaries and are less suitable in some applications, like in target tracking.

In order to overcome the drawbacks of the ellipsoid-based methods, we define a new form of sets which combine some characteristics of ellipsoids and zonotopes. To this end, we need first review the relevant knowledge of zonotopes.

In fact, an m-order zonotope $\boldsymbol{\mathcal{Z}} = \langle p, H\rangle$ as in Definition 2 is also the Minkowski sum of the vector p and m centered line segments as follows.

$$\boldsymbol{\mathcal{Z}} = p \oplus \bigoplus_{i=1}^{m} \mathbf{L}_i, \tag{7}$$

where

$$\mathbf{L}_i = \{l \in \mathbb{R}^n : l = a_i h_i,\, a_i \in [-1, 1]\}$$

and h_i is the i-th column of H.

Motivated by (7), we define a new form of sets, called ellipsoid bundles, which are defined as follows.

Definition 5. An m-order *ellipsoid bundle* $\boldsymbol{\mathcal{E}}(c, H) \subset \mathbf{R}^n$ is the Minkowski sum of the vector c and m centered ellipsoids as follows.

$$\boldsymbol{\mathcal{E}} = c \oplus \bigoplus_{i=1}^{m} \mathbf{E}(0, P_i),$$

where $c \in \mathbb{R}_n$ and $H = [P_1, \ldots, P_m]$, $P_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$, are positive semidefinite matrices.

Figure 1 presents a 2-order ellipsoid bundle in 3-dimensional space.

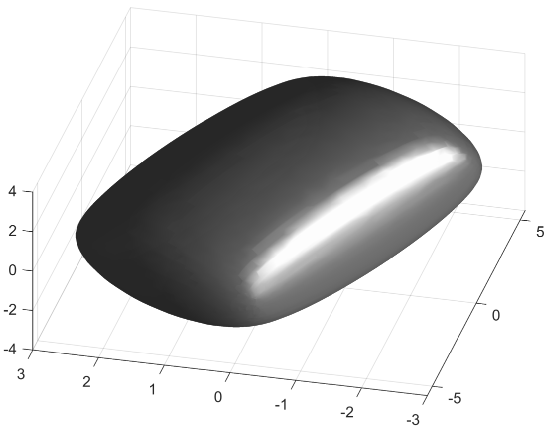

Fig. 1. A 2-order ellipsoid bundle in 3-dimensional space.

Similar to zonotopes, the Minkowski sum and linear transformation of ellipsoid bundles can be implemented by simple operations.

Property 1 (Minkowski sum). Given two ellipsoid bundles, $\boldsymbol{\mathcal{E}}(c_1, H_1)$ and $\boldsymbol{\mathcal{E}}(c_2, H_2)$, their Minkowski sum satisfies

$$\boldsymbol{\mathcal{E}}_1 \oplus \boldsymbol{\mathcal{E}}_2 = \boldsymbol{\mathcal{E}}(\overline{c}, \overline{H}),$$

where $\overline{c} = c_1 + c_2$ and $\overline{H} = [H_1,\ H_2]$.

Property 2 (Linear transformation). Given an m-order ellipsoid bundle $\boldsymbol{\mathcal{E}}(c, H)$ and a matrix $M \in \mathbb{R}^{q\times n}$, we have

$$M\boldsymbol{\mathcal{E}}(c, H) = \boldsymbol{\mathcal{E}}\big(Mc, \mathscr{L}(M, H)\big),$$

where

$$\mathscr{L}(M, H) = \left[\tilde{P}_1,\ \ldots,\ \tilde{P}_m\right], \quad \tilde{P}_i = MP_iM^T,\, i = 1, \ldots, m.$$

Property 1 and Property 2 are direct corollaries of the definition of Minkowski sum and the linear transformation of ellipsoids.

When applying ellipsoid bundles to set-membership estimation, the order of the estimated ellipsoid bundles will increase linearly with the time k, like in zonotope-based estimators. To limit the computational burden, we need to design a reduction method to enclose a high-order ellipsoid bundle by a lower one, again like in the case of zonotopes. To this end, we first introduce two necessary lemmas.

Lemma 4. (Durieu, Walter, and Polyak, 2001) Given the ellipsoid bundle $\boldsymbol{\mathcal{E}}(c, H)$, we can obtain a parameterized ellipsoid $\mathbf{E}(c, P_\alpha)$ enclosing $\boldsymbol{\mathcal{E}}(c, H)$, where P_α satisfies

$$P_\alpha = \sum_{i=1}^{m} \frac{1}{\alpha_i} P_i,$$

where $\alpha_i > 0$ for all $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Lemma 5. (Durieu, Walter, and Polyak, 2001) In the family of $\mathbf{E}(c, P_\alpha)$, the minimal-trace ellipsoid containing $\boldsymbol{\mathcal{E}}(c, H)$ satisfies

$$\alpha_i = \frac{\sqrt{\mathrm{tr}(P_i)}}{\sum_{i=1}^{m}\sqrt{\mathrm{tr}(P_i)}},$$

where $\mathrm{tr}(P_i)$ denotes the trace of P_i.

Based on Lemma 4 and 5, we propose a reduction method, which can be described by the following theorem.

Theorem 1. Given an m-order ellipsoid bundle $\boldsymbol{\mathcal{E}}(c, H) \subset \mathbb{R}^n$ and a fix integer $s(0 < s < m)$. Reordering $\mathbf{E}(0, P_i)$ in decreasing trace of P_i yields $\boldsymbol{\mathcal{E}}(c, \widetilde{H})$, where $\widetilde{H} = \left[P_{\sigma(1)},\ \ldots,\ P_{\sigma(m)}\right]$ and $\sigma(i)$ is a permutation of $i = 1, \ldots, m$. Then, we can obtain an s-order ellipsoid bundle $\boldsymbol{\mathcal{E}}\big(c, \mathscr{R}_s(H)\big)$ such that $\boldsymbol{\mathcal{E}}(c, H) \subseteq \boldsymbol{\mathcal{E}}\big(c, \mathscr{R}_s(H)\big)$, where

$$\mathscr{R}_s(H) = \left[P_{\sigma(1)},\ \ldots,\ P_{\sigma(s-1)},\ \widetilde{P}\right],$$
$$\widetilde{P} = \sum_{i=s}^{m}\sqrt{\mathrm{tr}(P_{\sigma(i)})}\sum_{j=s}^{m}\frac{P_{\sigma(j)}}{\sqrt{\mathrm{tr}(P_{\sigma(j)})}}.$$

Proof. From Definition 5, the reordering of P_i in H does not change the scope of $\boldsymbol{\mathcal{E}}(c, H)$. Therefore, $\boldsymbol{\mathcal{E}}(c, \widetilde{H}) = \boldsymbol{\mathcal{E}}(c, H)$.

The reordered ellipsoid bundle $\boldsymbol{\mathcal{E}}(c, \widetilde{H})$ can be rewritten as

$$\boldsymbol{\mathcal{E}}(c, \widetilde{H}) = \boldsymbol{\mathcal{E}}(c, H_1) \oplus \boldsymbol{\mathcal{E}}(0, H_2),$$

where

$$H_1 = \left[P_{\sigma(1)},\ \ldots,\ P_{\sigma(s-1)}\right],$$
$$H_2 = \left[P_{\sigma(s)},\ \ldots, P_{\sigma(m)}\right].$$

According to Lemma 4 and 5, $\boldsymbol{\mathcal{E}}(0, H_2) \subseteq \boldsymbol{\mathcal{E}}(0, \widetilde{P})$. Then, by using Property 1 (Minkowski sum), we have

$$\begin{aligned}\boldsymbol{\mathcal{E}}(c, \widetilde{H}) &\subseteq \boldsymbol{\mathcal{E}}(c, H_1) \oplus \boldsymbol{\mathcal{E}}(0, \widetilde{P})\\ &= \boldsymbol{\mathcal{E}}(c, \left[H_1,\ \widetilde{P}\right])\\ &= \boldsymbol{\mathcal{E}}\big(c, \mathscr{R}_s(H)\big).\end{aligned}$$

It follows that $\boldsymbol{\mathcal{E}}(c, H) \subseteq \boldsymbol{\mathcal{E}}\big(c, \mathscr{R}_s(H)\big)$. □

5. SET-MEMBERSHIP ESTIMATION BASED ON ELLIPSOID BUNDLES

Based on the ellipsoid bundles described in Section 4, we propose a new set-membership estimation method for system (4). The proposed method obtains the set-membership estimation of state by combining a single state trajectory estimator with the reachable set estimation of estimation errors based on ellipsoid bundles.

The single state trajectory estimator has the following observer structure:

$$\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + L_k(y_k - C_k\hat{x}_k), \tag{8}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the state estimation and $L_k \in \mathbb{R}^{n_x\times n_y}$ is the observer gain matrix.

Define the estimation error as $e_k = x_k - \hat{x}_k$. Subtract (8) from (4), then we obtain the following error system.

$$e_{k+1} = (A_k - L_kC_k)e_k + w_k - L_kv_k. \tag{9}$$

Based on the reachable set analysis of error system (9), we can obtain the set-membership estimation of x_k in the form of ellipsoid bundles. The proposed method can be described by the following theorem.

Theorem 2. For system (4), x_k can be bounded by the ellipsoid bundle $\boldsymbol{\mathcal{E}}(\hat{x}_k, H_k)$, where $H_k \in \mathbb{R}^{n_x\times(n_x\times m)}$ satisfies

$$H_{k+1} = \left[\mathscr{L}(A_k - L_kC_k, \widetilde{H}_k),\ Q_k,\ L_kR_kL_k^T\right], \tag{10}$$

where

$$\widetilde{H}_k = \begin{cases} H_k, & m \le s, \\ \mathscr{R}_s(H_k), & m > s \end{cases} \tag{11}$$

and

$$H_0 = P_0, \quad \hat{x}_0 = c_0 \tag{12}$$

and s is a fixed positive integer.

Proof. From (6) and (12), we have

$$x_0 \in \boldsymbol{\mathcal{E}}(\hat{x}_0, H_0).$$

When $k \ge 0$, if $x_k \in \boldsymbol{\mathcal{E}}(\hat{x}_k, H_k)$, then according to Theorem 1 and (11), we have $\boldsymbol{\mathcal{E}}(\hat{x}_k, H_k) \subseteq \boldsymbol{\mathcal{E}}(\hat{x}_k, \widetilde{H}_k)$. Therefore,

$$x_k \in \boldsymbol{\mathcal{E}}(\hat{x}_k, \widetilde{H}_k).$$

Since $e_k = x_k - \hat{x}_k$, it follows that

$$e_k \in \boldsymbol{\mathcal{E}}(0, \widetilde{H}_k).$$

Then, from (5) and (9), we have

$$\begin{aligned} e_{k+1} &\in (A_k - L_kC_k)\boldsymbol{\mathcal{E}}(0, \widetilde{H}_k) \oplus \mathbf{Q}_k \oplus L_k\mathbf{R}_k \\ &= (A_k - L_kC_k)\boldsymbol{\mathcal{E}}(0, \widetilde{H}_k) \oplus \boldsymbol{\mathcal{E}}(0, Q_k) \oplus L_k\boldsymbol{\mathcal{E}}(0, R_k). \end{aligned} \tag{13}$$

According to Property 1 and 2, (13) implies $e_{k+1} \in \boldsymbol{\mathcal{E}}(0, H_{k+1})$. Then, we have

$$x_{k+1} = \hat{x}_{k+1} + e_{k+1} \in \boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1}).$$

By induction, $x_k \in \boldsymbol{\mathcal{E}}(\hat{x}_k, H_k)$ holds for all $k \ge 0$. □

In order to increase estimation accuracy, we need design L_k such that the size of $\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})$ is minimized. Similar to zonotopes, we propose the following size criterion for the ellipsoid bundle $\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})$.

$$J_{k+1}\big(\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})\big) = \sum_{i=1}^{p+2} \mathrm{tr}(P_i^{k+1}). \tag{14}$$

where $H_{k+1} = \left[P_1^{k+1},\ \ldots,\ P_{p+2}^{k+1}\right]$ and $p = \min\{m, s\}$. $J_{k+1}\big(\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})\big)$ is the sum of the squared half-axes of all the ellipsoids $\mathbf{E}(0, P_i^{k+1})$, $i = 1, \ldots, p+2$, which determine the shape and size of the ellipsoid bundle $\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})$.

The above size criterion is similar to the F-radius of zonotopes, which can provide an efficient way to design the observer gain in real time. To minimize the size criterion J_{k+1}, we propose the following theorem.

Theorem 3. For the ellipsoid bundle $\boldsymbol{\mathcal{E}}(\hat{x}_{k+1}, H_{k+1})$ obtained from (10), denote $\widetilde{H}_k = \left[\widetilde{P}_1^k,\ \ldots,\ \widetilde{P}_p^k\right]$, the optimal L_k minimizing the size criterion J_{k+1} satisfies

$$\begin{aligned} \Theta_k &= \sum_{i=1}^{p} \widetilde{P}_i^k, \\ V_k &= C_k\Theta_kC_k^T + R_k, \\ L_k &= A_k\Theta_kC_k^TV_k^{-1}. \end{aligned} \tag{15}$$

Proof. From (10), we have

$$H_{k+1} = \left[P_1^{k+1},\ \ldots,\ P_{p+2}^{k+1}\right],$$

where

$$P_i^{k+1} = \begin{cases} (A_k - L_kC_k)\widetilde{P}_i^k(A_k - L_kC_k)^T, & 1 \le i \le p; \\ Q_k, & i = p+1; \\ L_kR_kL_k^T, & i = p+2. \end{cases}$$

Then, from (14), we have

$$\begin{aligned} J_{k+1} = &\sum_{i=1}^{p} \mathrm{tr}\big((A_k - L_kC_k)\widetilde{P}_i^k(A_k - L_kC_k)^T\big) \\ &+ \mathrm{tr}(Q_k) + \mathrm{tr}(L_kR_kL_k^T). \end{aligned}$$

We can obtain the derivative of J_{k+1} for L_k as follows.

$$\frac{\partial J_{k+1}}{\partial L_k} = 2L_kV_k - 2A_kP_kC_k^T. \tag{16}$$

Let (16) equal zero, we obtain (15). Therefore, the L_k obtained from (15) is a stationary point. From (15), we have $V_k \succeq R_k$. Since R_k is positive definite, V_k is invertible. Therefore, the L_k obtained from (15) is the unique stationary point. In addition, J_{k+1} is a quadratic function of L_k and is non-negative for any L_k, then J_{k+1} is a convex function of L_k. Therefore, the L_k obtained from (15) is the optimal solution minimizing the size criterion J_{k+1}. □

Remark 1. The assumptions (5) and (6) can be relaxed as follows.

$$w_k \in \boldsymbol{\mathcal{E}}(0, W_k), \quad v_k \in \boldsymbol{\mathcal{E}}(0, V_k), \quad x_0 \in \boldsymbol{\mathcal{E}}(c_0, H_0),$$

which can be used to describe more complex sets of uncertainties. In this case, the relevant adjustments of the proposed method are trivial, i.e. P_0 to H_0, Q_k to W_k, $L_kR_kL_k^T$ to $\mathcal{L}(L_k, V_k)$ and R_k to $\sum_{i=1}^{n_v} R_k^i$, where n_v is the order of $\boldsymbol{\mathcal{E}}(0, V_k)$ and $V_k = \left[R_k^1\ \ldots\ R_k^{n_v}\right]$.

In fact, an m-order zonotope $\boldsymbol{\mathcal{Z}} = \langle p, H\rangle$ can be represented by an m-order ellipsoid bundle $\boldsymbol{\mathcal{E}}(p, M)$, where $M = \left[h_1h_1^T\ \ldots\ h_mh_m^T\right]$ and h_i is the i-th column of H. This is because that the line segment $\mathbf{L}_i = \{l \in \mathbb{R}^n : l = a_ih_i, a_i \in [-1, 1]\}$ is actually the degenerate ellipsoid $\mathbf{E}(0, h_ih_i^T)$. In this case, zonotopes can be seen as a special kind of ellipsoid bundles. The proposed method can also apply to the case with zonotope-bounded uncertainties. Therefore, the proposed method has wide range of applications.

In some practical applications of set-membership estimation for instance fault diagnosis, for the sake of simplicity, the interval estimation including the upper and lower bounds of the estimated state is usually derived from the estimation set to serve as thresholds. For the proposed method based on ellipsoid bundles, we propose the following theorem to obtain the interval estimation of state.

Theorem 4. For system (4), we can obtain an interval vector $[\underline{x}_k, \overline{x}_k]$ such that $\underline{x}_k \le x_k \le \overline{x}_k$, where $\underline{x}_k$ and $\overline{x}_k$ satisfy

$$\begin{cases} \overline{x}_k = \hat{x}_k + \overline{e}_k \\ \underline{x}_k = \hat{x}_k + \underline{e}_k \end{cases}$$

where $\underline{e}_k = -\overline{e}_k$ and

$$\overline{e}_k(j) = \sum_{i=1}^{m} \sqrt{P_i^k(j,j)}, \quad j = 1, \ldots, n_x.$$

Proof. According to Theorem 2, $x_k \in \boldsymbol{\mathcal{E}}(\hat{x}_k, H_k)$. Since $e_k = x_k - \hat{x}_k$, we have $e_k \in \boldsymbol{\mathcal{E}}(0, H_k)$.

From Definition 5, we have

$$\boldsymbol{\mathcal{E}}(0, H_k) = \bigoplus_{i=1}^{m} \mathbf{E}(0, P_i^k). \quad (17)$$

Define $l_j \in \mathbb{R}^{n_x}$ as the vector with its j-th element equal to 1 and the others equal to 0, where $j = 1, \ldots, n_x$. For example, $l_1 = [1\ 0\ \ldots\ 0]^T$.

According to Lemma 2, we have

$$\rho_{\mathbf{E}(0,P_i^k)}(l_j) = \sqrt{l_j^T P_i^k l_j} = \sqrt{P_i^k(j,j)},$$

where $P_i^k(j,j)$ is the j-th diagonal element of P_i^k.

Then, from Lemma 3 and (17), we have

$$\rho_{\boldsymbol{\mathcal{E}}(0,H_k)}(l_j) = \sum_{i=1}^{m} \rho_{\mathbf{E}(0,P_i^k)}(l_j) = \sum_{i=1}^{m} \sqrt{P_i^k(j,j)} = \overline{e}_k(j).$$

Since $e_k \in \boldsymbol{\mathcal{E}}(0, H_k)$, then according to Definition 4, we have

$$e_k(j) = l_j^T e_k \le \rho_{\boldsymbol{\mathcal{E}}(0,H_k)}(l_j), \quad j = 1, \ldots, n_x.$$

It follows that

$$e_k \le \overline{e}_k. \quad (18)$$

Note that $\boldsymbol{\mathcal{E}}(0, H_k)$ is centrosymmetric, which implies that $-e_k \in \boldsymbol{\mathcal{E}}(0, H_k)$. In the same way, we can obtain $-e_k \le \overline{e}_k$. It follows that

$$e_k \ge -\overline{e}_k = \underline{e}_k. \quad (19)$$

Since $x_k = \hat{x}_k + e_k$, then from (18) and (19), we have

$$\hat{x}_k + \underline{e}_k \le x_k \le \hat{x}_k + \overline{e}_k.$$

□

6. SIMULATION RESULTS

In this section, a numerical example in the form of (4) is used to demonstrate the performance of the proposed method. The example has the following parameters.

$$A_k = \begin{bmatrix} 0.3 + 0.2\sin(0.5k) & -0.7 - 0.2\sin(0.25k) \\ 0.6 & -0.5 + 0.3\cos(0.5k) \end{bmatrix},$$
$$B_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_k = [1 + 0.2\sin(0.2k)\ \ 0].$$

In the simulation, the input is set as $u_k = 0.4\sin(0.5k)$.

Since ellipsoid bundles have many similar characteristics to zonotopes, we compare the proposed method with the Zonotopic Kalman Filter (ZKF) in Combastel (2015). However, zonotope-based methods including ZKF assume that disturbances and noises are bounded by certain zonotopes, usually boxes, which is different from the proposed method. To make comparisons, we consider the following two cases.

Case 1. In the simulation, w_k, v_k and x_0 are set bounded by ellipsoids as in (5) and (6) with

$$Q_k = 0.0025I_2, \quad R_k = 0.0025I_2, \quad P_0 = 0.04I_2.$$

For ZKF, which requires zonotopes bounding w_k, c_k and x_0, we use the smallest boxes enclosing the corresponding assumed ball-shaped uncertainty sets:

$$\mathbf{Q}_k \subset \mathbf{W}_k = \langle 0, 0.05I_2 \rangle, \quad \mathbf{R}_k \subset \mathbf{V}_k = \langle 0, 0.05I_2 \rangle,$$
$$\mathbf{P}_0 \subset \mathbf{X}_0 = \langle 0, 0.2I_2 \rangle.$$

In the simulation, the reduction order of the proposed method is set as $s = 5$ and that of ZKF is set as 10. The simulation results are shown in Figure 2, which shows that the proposed method can obtain more accurate estimation results. Therefore, the proposed method is more suitable for the case with ellipsoid-bounded disturbances and noises.

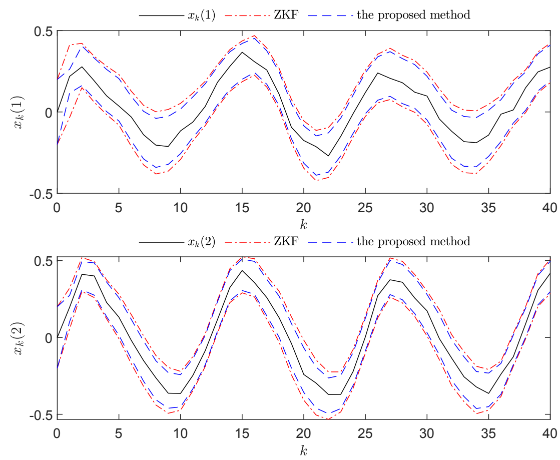

Fig. 2. The interval estimation of x_k in case 1.

Case 2. The assumed ball-shaped uncertainty sets for w_k, v_k and x_0 in Case 1 put ZKF at a disadvantage w.r.t. the proposed method. To make a fair comparison, in this case, w_k, v_k and x_0 are randomly generated within the following zonotopes.

$$w_k \in \mathbf{W}_k = \langle 0, 0.05I_2 \rangle, \quad v_k \in \mathbf{V}_k = \langle 0, 0.05I_2 \rangle,$$
$$x_0 \in \mathbf{X}_0 = \langle 0, 0.2I_2 \rangle.$$

The proposed method requires ellipsoid bundles bounding w_k, c_k and x_0. If we use the smallest ellipsoids enclosing the corresponding assumed square-shaped uncertainty sets, the proposed method will be clearly put at a disadvantage. Note that $\mathbf{W}_k$, $\mathbf{V}_k$ and $\mathbf{X}_0$ can also be represented by ellipsoid bundles as follows.

$$\mathbf{W}_k = \boldsymbol{\mathcal{E}}(0, W_k), \quad \mathbf{V}_k = \boldsymbol{\mathcal{E}}(0, V_k), \quad \mathbf{X}_0 = \boldsymbol{\mathcal{E}}(0, H_0),$$

where

$$W_k = \begin{bmatrix} 0.0025 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0025 \end{bmatrix}, \quad V_k = \begin{bmatrix} 0.0025 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0025 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.04 \end{bmatrix}.$$

The simulation results are shown in Figure 3 and 4. They show that the estimation results obtained by ZKF and the proposed method are close, but the ones by the proposed method are more accurate. Therefore, the proposed method can also deal with the zonotope-bounded uncertainties and can even obtain more accurate estimation results than the zonotope-based method.

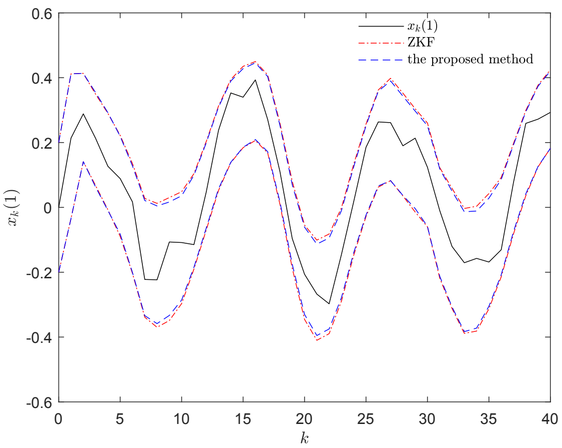

Fig. 3. The interval estimation of $x_k(1)$ in case 2.

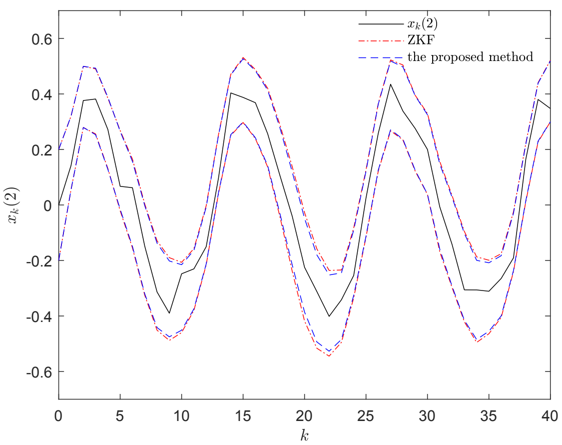

Fig. 4. The interval estimation of $x_k(2)$ in case 2.

7. CONCLUSION

In this paper, we propose a new set representation tool, ellipsoid bundle, which combines certain advantages of ellipsoids and zonotopes. The basic properties of ellipsoid bundles are investigated. Then, a novel set-membership estimation method is proposed based on ellipsoid bundles. Both ellipsoid-bounded uncertainties and zonotope-based ones are considered. Simulation results have demonstrated the effectiveness of the proposed set-membership estimation method.

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