

# A data-driven immersion technique for linearization of discrete-time nonlinear systems

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**Abstract:** This paper proposes a data-driven immersion approach to obtain linear equivalents or approximations of discrete-time nonlinear systems. Exact linearization can only be achieved for very particular classes of systems. In general cases, we aim to obtain a finite-time linear approximation. Our approach only takes a finite set of trajectories and hence an analytic model is not required. The mismatch between the approximate linear model and the original system is concretely discussed with formal bounds. We also provide a Koopman-operator interpretation of this technique, which shows a link between system immersibility and the Koopman operator theory. Several numerical examples are taken to show the capabilities of the proposed immersion approach. Comparison is also made with other Koopman-based lifting approaches which use radial basis functions and monomials.

*Keywords:* Linearization, immersion, data-driven identification

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## 1. INTRODUCTION

Linearization of nonlinear systems is one of the most well-known research topics in systems and control. The use of linear equivalents or approximations allows us to use standard techniques for linear systems to analyze complex behaviors of nonlinear systems. Classic linearization methods includes the Jacobian linearization method and the feedback linearization method (see, e.g., Section 4.3 & 13 of (Khalil, 2002)).

Another linearization method is the state immersion method, which lifts a nonlinear system into a linear system with a higher dimension, see, e.g., (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988; Menini and Tornambè, 2009). Although the immersion method and feedback linearization can be equivalent in the special case where the immersion is a state diffeomorphism, they are in general different as an immersion does not necessarily preserve the dimension of the system. The immersibility of a nonlinear system is also closely related to the notion of flatness (Fliess et al., 1999). One successful application of state immersion can be found in observer design for nonlinear systems, see, e.g., (Krener and Isidori, 1983). Recently, a new immersion technique has been proposed in (Jungers and Tabuada, 2019) for continuous-time system by the use of polyflows. Although a nilpotency property is required for the exact immersion or linearization, the approximation by polyflows often outperforms the Taylor approximation in practice. The aforementioned immersion techniques usually rely on the Lie derivatives of the trajectories and thus are not directly applicable to systems without an analytic model. In this paper, we will develop

a data-driven immersion technique for discrete-time nonlinear systems and the analytic model of the system is not needed. Inspired by the polyflows approximation in (Jungers and Tabuada, 2019), we establish approximate linearization in a finite time horizon with a discussion on the mismatch. We will also show this immersion technique has a connection with multivariate autoregressive (M-AR) modeling (Harrison et al., 2003).

Operator-theoretic approaches like the Carleman linearization (Kowalski and Steeb, 1991) and the Koopman approach (see, e.g., (Williams et al., 2015) and the references therein) are also promising frameworks to provide an (infinite-dimensional) linear representation of nonlinear systems. For numerical analysis, the infinite-dimensional linear operator is often truncated into finite-dimensional approximations. For instance, finite representations of the Koopman operator can be obtained by using a finite number of observables, which have a few choices: predefined basis functions (see, e.g., (Williams et al., 2015; Mauroy and Goncalves, 2017; Korda and Mezić, 2018)), delay coordinates (see, e.g., ) and neural networks models (Takeishi et al., 2017). However, to the best of our knowledge, connections between Koopman representations and the system immersibility property are not mentioned in the literature. In this paper, we will discuss their connections and provide a Koopman interpretation of the proposed immersion technique. We will also make a comparison with Koopman-based approaches with radial basis functions (Korda and Mezić, 2018) and monomials (Mauroy and Goncalves, 2017).

The rest of the paper is organized as follows. The next section gives the problem statement and a brief review of on the Koopman operator. Section 3 presents the proposed data-driven immersion approach. In Section 4, we will

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discuss some implementations issues. Some simulation results are provided Section 5. Proofs of some theorems and lemmas are not given due to page limitation.

**Notation.** The non-negative integer set is indicated by  $\mathbb{Z}^+$ . For any  $x \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times m}$ ,  $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$  ( $\|x\| = \|x\|_2$  by default),  $\|A\|_p = \sup_{x \neq 0} \|Ax\|_p / \|x\|_p$ , and  $\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2}$ . For any two matrices  $A, B$ ,  $A \otimes B$  denotes the Kronecker product and  $A^{[k]} = \underbrace{A \otimes \dots \otimes A}_{k \text{ times}}$  for  $k \in \mathbb{Z}^+$ .

## 2. PROBLEM STATEMENT AND PRELIMINARIES

### 2.1 Problem statement

We consider the following discrete-time nonlinear system

$$x(t+1) = f(x(t)), \quad y(t) = h(x(t)), \quad t \in \mathbb{Z}^+ \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $y(t) \in \mathbb{R}^m$  is the output,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $h: \mathbb{R}^n \rightarrow \mathbb{R}^m$  are continuous functions. For notational convenience, let

$$\mathbf{h}_M(x) = [(h(x))^T \dots, (h(f^M(x)))^T]^T, \quad \forall M \in \mathbb{Z}^+. \quad (2)$$

where  $f^k(x) = \underbrace{f \circ \dots \circ f}_k(x)$ ,  $f^0(x) = x$ ,  $\forall k \in \mathbb{Z}^+$ . This

paper is concerned with the linearization of the nonlinear system (1) by lifting the dimension. More precisely, we aim to immerse the system (1) into a linear system ( $\tilde{n} > n$ ):

$$\Sigma(A, C): \quad \tilde{x}(t+1) = A\tilde{x}(t), \quad \tilde{y}(t) = C\tilde{x}(t), \quad t \in \mathbb{Z}^+ \quad (3)$$

where  $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ ,  $\tilde{y}(t) \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ , and  $C \in \mathbb{R}^{m \times \tilde{n}}$ .

### 2.2 The Koopman operator theory

The Koopman operator is able to represent nonlinear dynamics into a (infinite-dimensional) linear framework. Given an infinite-dimensional space  $\mathcal{F}$  of observables  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ , the Koopman operator  $\mathcal{K}: \mathcal{F} \rightarrow \mathcal{F}$  associated with the system (1) is defined as:  $\mathcal{K}g = g \circ f$ ,  $\forall g \in \mathcal{F}$ . Even though the Koopman operator is always linear, computing the infinite-dimensional  $\mathcal{K}$  is often intractable in practice. For numerical analysis,  $\mathcal{K}$  will be restricted to a finite-dimensional subspace, which leads to finite-dimensional approximations of  $\mathcal{K}$ . Given a finite set of observables  $\{g_1, g_2, \dots, g_P\}$  for some  $P \in \mathbb{Z}^+$ , a necessary and sufficient condition is shown in (Takeishi et al., 2017) on the existence of a linear operator  $G \in \mathbb{R}^{P \times P}$  that satisfies  $\mathbf{g}(f(x)) = G\mathbf{g}(x)$ ,  $\forall x \in \mathbb{R}^n$ , where  $\mathbf{g} = [g_1 \ g_2 \ \dots \ g_P]^T$  is the vector of the  $P$  functions. The choice of the observables  $\{g_1, g_2, \dots, g_P\}$  plays a very important role in the finite-dimensional approximation and the optimal choice is often not clear. In this paper, we show that system immersion provides another view of the Koopman operator theory. We will discuss intrinsic connections between Koopman-based lifting techniques and system immersion.

## 3. MAIN RESULTS

### 3.1 Immersibility and linear equivalents

Let us first give the definition of immersibility of discrete-time nonlinear systems, see, e.g., (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988).

*Definition 1.* The system (1) is immersible into  $\Sigma(A, C)$  given in (3) in  $X \subseteq \mathbb{R}^n$  if there exists a map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^{\tilde{n}}$  such that  $h(f^t(x)) = CA^tT(x)$  for  $t \in \mathbb{Z}^+$  and  $x \in X$ . It is globally immersible into  $\Sigma(A, C)$  if  $X = \mathbb{R}^n$ .

When the system (1) is globally immersible into a linear system, it can be considered as the projection of the linear system. We will construct a linear system and establish its equivalence with the system (1). For any  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M := [\gamma_0 \ \gamma_1 \ \dots \ \gamma_M] \in \mathbb{R}^{m \times m(M+1)}$ , we define the following linear system, denoted by  $\Sigma(\boldsymbol{\gamma}_M) := \Sigma(\boldsymbol{\gamma}_M, [I_m \ \mathbf{0}_{m \times Mm}])$ ,

$$\tilde{x}(t+1) = \Gamma(\boldsymbol{\gamma}_M)\tilde{x}(t), \quad \tilde{y}(t) = [I_m \ \mathbf{0}_{m \times Mm}]\tilde{x}(t) \quad (4)$$

where

$$\Gamma(\boldsymbol{\gamma}_M) := \begin{pmatrix} \mathbf{0} & I_m & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_m \\ \gamma_0 & \gamma_1 & \dots & \gamma_{M-1} & \gamma_M \end{pmatrix}. \quad (5)$$

A necessary and sufficient condition of immersibility is given in Theorem 1. Similar arguments can also be found in (Monaco and Normand-Cyrot, 1983; Lee and Marcus, 1988), although the proof is slightly different.

*Theorem 1.* The system (1) is globally immersible into  $\Sigma(A, C)$  defined in (3) if and only if there exist a finite  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M := [\gamma_0 \ \gamma_1 \ \dots \ \gamma_M] \in \mathbb{R}^{m \times m(M+1)}$  such that

$$h(f^{M+1}(x)) = \boldsymbol{\gamma}_M \mathbf{h}_M(x), \quad \forall x \in \mathbb{R}^n. \quad (6)$$

*Proof of Theorem 1: (Sufficiency)* Suppose there exist  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M := [\gamma_0 \ \gamma_1 \ \dots \ \gamma_M]$  such that  $h(f^{M+1}(x)) = \boldsymbol{\gamma}_M \mathbf{h}_M(x)$ ,  $\forall x$ . Hence,  $\mathbf{h}_M(f(x)) = \Gamma(\boldsymbol{\gamma}_M)\mathbf{h}_M(x)$ . Consider the linear system in (4) with the sequence  $\boldsymbol{\gamma}_M$ . We can see that the system (1) is immersible into  $\Sigma(\boldsymbol{\gamma}_M)$  with  $T(x) = \mathbf{h}_M(x)$ . *(Necessity)* Suppose the system (1) is immersible into the system (3). From Definition 1, we know that there exists a function  $T: \mathbb{R}^n \rightarrow \mathbb{R}^{\tilde{n}}$  such that  $h(f^\ell(x)) = CA^\ell T(x)$ ,  $\forall \ell \in \mathbb{Z}^+$ . From the Cayley–Hamilton theorem, there exist  $\{\alpha_i \in \mathbb{R}\}_{i=0}^{\tilde{n}-1}$  such that  $h(f^{\tilde{n}}(x)) = C \sum_{\ell=0}^{\tilde{n}-1} \alpha_\ell A^\ell T(x) = \sum_{\ell=0}^{\tilde{n}-1} \alpha_\ell h(f^\ell(x))$ .  $\square$

The theorem above suggests that the system (1) with the global immersibility property admits a M-AR model (Harrison et al., 2003) in the form of (6). Hence, the problem of finding linear equivalents or approximations is coherently related to the identification problem of M-AR models. The immersibility condition above also leads to other properties.

*Corollary 1.* Suppose there exist a finite  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M := [\gamma_0 \ \gamma_1 \ \dots \ \gamma_M] \in \mathbb{R}^{m \times m(M+1)}$  such that (6) is satisfied. Then, there exist a map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^{\tilde{n}}$  and an observable pair  $(C, A)$  with the components  $\{T_1(x), \dots, T_{\tilde{n}}(x)\}$  being linearly independent and  $\tilde{n} \leq (M+1)m$  such that  $AT(x) = T(f(x))$  and  $CT(x) = h(x)$  for any  $x \in \mathbb{R}^n$ . In addition, if the system (1) is globally asymptotically stable at the origin,  $A$  is Schur stable.

The following proposition shows the link between the Koopman operator and the immersibility property.

*Proposition 1.* Suppose there exists a finite set of observables  $\mathbf{g}(x) = [g_1(x) \ g_2(x) \ \dots \ g_P(x)]^T$  such that  $\mathbf{g}(f(x)) = G\mathbf{g}(x)$  holds for some  $G \in \mathbb{R}^{P \times P}$  and  $P \in \mathbb{Z}^+$  with  $h(x) =$

$v^T \mathbf{g}(x)$  for some  $v \in \mathbb{R}^{P \times m}$ . For any  $i \in \{1, 2, \dots, m\}$ , let  $M_i$  be the minimal  $M$  that satisfies  $h_i(f^{M+1}(x)) \in \text{span}\{h_i(x), h_i(f(x)), \dots, h_i(f^M(x))\}, \forall x \in \mathbb{R}^n$ . Then, (i) The system (1) is globally immersible into a linear system; and (ii)  $M_i \leq P$  for all  $i \in \{1, 2, \dots, m\}$ .

Proof of Proposition 1: (i) From  $\mathbf{g}(f(x)) = G\mathbf{g}(x)$  and  $h(x) = v^T \mathbf{g}(x)$ ,  $h(f^P(x)) = v^T G^P \mathbf{g}(x)$ . Hence, from the Cayley–Hamilton theorem,  $h(f^P(x))$  can always be expressed by a linear combination of  $\{h(x), \dots, h(f^{P-1}(x))\}$ , which means the immersibility condition in Theorem 1 holds. (ii) This is a direct consequence of (i).  $\square$

Proposition 1 shows that the finite-dimensional Koopman representation implies global immersibility. Conversely, global immersibility also leads to a finite-dimensional Koopman representation by choosing the observables to be  $\{\{h_i(f^k(x))\}_{i=1}^m\}_{k=0}^M$ , where  $M$  satisfies (6).

### 3.2 A family of nonlinear systems with global immersion

For certain families of nonlinear systems, the immersibility property (6) always holds. One special family of polynomial systems is given as follows

$$\xi(t+1) = A_\xi \xi(t) + \varphi(z(t)), \quad z(t+1) = A_z z(t) \quad (7a)$$

$$y(t) = C_\xi \xi(t) + \bar{\varphi}(z(t)) \quad (7b)$$

where  $\xi \in \mathbb{R}^{n_\xi}, z \in \mathbb{R}^{n_z}, y \in \mathbb{R}^m, A_\xi \in \mathbb{R}^{n_\xi \times n_\xi}, A_z \in \mathbb{R}^{n_z \times n_z}, C_\xi \in \mathbb{R}^{m \times n_\xi}$ , and  $\varphi: \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_\xi}$  and  $\bar{\varphi}: \mathbb{R}^{n_z} \rightarrow \mathbb{R}^m$  are polynomial functions of degree  $d \in \mathbb{Z}^+$ , given by,  $\varphi(z) := F_0 + F_1 z^{[1]} + \dots + F_d z^{[d]}$  and  $\bar{\varphi}(z) := \bar{F}_0 + \bar{F}_1 z^{[1]} + \dots + \bar{F}_d z^{[d]}$  with  $z^{[0]} = 1$ , and  $F_i \in \mathbb{R}^{n_\xi \times n_z^i}$  and  $\bar{F}_i \in \mathbb{R}^{m \times n_z^i}$  for  $i \in \{0, 1, \dots, d\}$ . The following lemma is needed.

*Lemma 1.* For any  $A \in \mathbb{R}^{r \times r}$ , and  $r, d \in \mathbb{Z}^+$ , the following properties hold. (i) For any  $z \in \mathbb{R}^r$  and  $k \in \mathbb{Z}^+$ ,  $(Az)^{[d]} = A^{[d]} z^{[d]}$ , and  $(A^k)^{[d]} = (A^{[d]})^k$ . (ii) For any polynomial function  $\psi$  of degree  $d$ , there exist  $M \in \mathbb{Z}^+$  and  $\{\alpha_i \in \mathbb{R}\}_{i=0}^M$  such that  $\psi(A^{M+1}z) = \sum_{i=0}^M \alpha_i \psi(A^i z)$ .

With Lemma 1, the following theorem can be derived.

*Theorem 2.* The system (7) is globally immersible into a linear system in the form of (3).

Proof of Theorem 2: From Lemma 1, we know that there exist  $M \in \mathbb{Z}^+$ ,  $\{\alpha_i\}_{i=0}^M$  and  $\{\bar{\alpha}_i\}_{i=0}^M$  such that  $\varphi(A_z^{M+1}z) = \sum_{i=0}^M \alpha_i \varphi(A_z^i z)$  and  $\bar{\varphi}(A_z^{M+1}z) = \sum_{i=0}^M \bar{\alpha}_i \bar{\varphi}(A_z^i z)$ . Let  $\tilde{x}_1 = \xi, \tilde{x}_{i+2} = A_\xi \tilde{x}_{i+1} + \varphi((A_z)^i z)$  for  $i = 0, 1, \dots, M$ . It can be verified that

$$\tilde{x}_i(t+1) = \tilde{x}_{i+1}(t), \quad i = 1, 2, \dots, M+1, \quad (8)$$

$$\begin{aligned} \tilde{x}_{M+2}(t+1) &= A_\xi \tilde{x}_{M+2}(t) + \varphi((A_z)^{M+1}z(t)) \\ &= -\alpha_0 A_\xi \tilde{x}_1(t) + \sum_{i=0}^{M-1} (\alpha_i I - \alpha_{i+1} A_\xi) \tilde{x}_{i+2}(t) \\ &\quad + (\alpha_M I + A_\xi) \tilde{x}_{M+2}(t) \end{aligned} \quad (9)$$

Similarly, let  $\tilde{z}_i = \bar{\varphi}((A_z)^{i-1}z)$  for all  $i = 1, 2, \dots, M+2$ . It can be also verified that

$$\tilde{z}_i(t+1) = \tilde{z}_{i+1}(t), \quad i = 1, 2, \dots, M+1, \quad (10)$$

$$\begin{aligned} \tilde{z}_{M+2}(t+1) &= \bar{\varphi}((A_z)^{M+2}z(t)) \\ &= \sum_{i=0}^M \bar{\alpha}_i \bar{\varphi}(A_z^{i+1}z(t)) = \sum_{i=0}^M \bar{\alpha}_i \tilde{z}_{i+2}(t) \end{aligned} \quad (11)$$

Let  $\tilde{x} = [\tilde{x}_1^T \dots \tilde{x}_{M+2}^T]^T$  and  $\tilde{z} = [\tilde{z}_1^T \dots \tilde{z}_{M+2}^T]^T$ . The construction above leads to the following linear system:  $\tilde{x}(t+1) = \tilde{A}_1 \tilde{x}(t), \tilde{z}(t+1) = \tilde{A}_2 \tilde{z}(t)$ , where  $\tilde{A}_1 \in \mathbb{R}^{n_\xi(M+2) \times n_\xi(M+2)}, \tilde{A}_2 \in \mathbb{R}^{n_z(M+2) \times n_z(M+2)}$  are defined in (8) - (9) and (10)-(11) respectively. We can see that  $y(t)$  in (7) can be expressed as  $C_\xi \tilde{x}_1(t) + \tilde{z}_1(t)$ . Hence, the system (7) is immersible to the system (8) - (11).  $\square$

From Theorem 2, the following corollary can be derived.

*Corollary 2.* If the system (1) is globally immersible into a system in the form of

$$\bar{x}(t+1) = \bar{A}\bar{x}(t), \quad \bar{y}(t) = \bar{C}(\bar{x}(t)), \quad t \in \mathbb{Z}^+ \quad (12)$$

where  $\bar{x}(t) \in \mathbb{R}^{\bar{n}}, \bar{y}(t) \in \mathbb{R}^m, \bar{A} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ , and  $\bar{C}: \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^m$  is a polynomial function, it is also globally immersible into a linear system in the form of (3).

Proof of Corollary 2: This is a direct consequence of Theorem 2, because (12) is a special case of (7).  $\square$

This corollary suggests that using a polynomial output in (3) does not necessarily generalize the class of linear equivalents of the system (1).

### 3.3 The lifting method

From the immersibility condition in Theorem 1, we formulate the following problem,  $\forall M \in \mathbb{Z}^+$ ,

$$\Delta_M := \inf_{\gamma_M} \sup_{x \in \mathbb{R}^n} \|h(f^{M+1}(x)) - \gamma_M \mathbf{h}_M(x)\|_\infty \quad (13)$$

where  $\gamma_M \in \mathbb{R}^{m \times m(M+1)}$ . The following lemma is needed.

*Lemma 2.* For any  $M \in \mathbb{Z}^+$ , let  $\Delta_M$  be defined in (13). The sequence  $\{\Delta_M\}_{M=0}^\infty$  is non-increasing.

From Lemma 2, the basic idea is to increase  $M$  until  $\Delta_M$  is sufficiently small. Then, with the solution  $\gamma_M$ , we can construct  $\Sigma(\gamma_M)$  as defined in (4). Suppose the system (1) is globally immersible to some linear system, from Theorem 1, there always exists a  $M$  such that  $\Delta_M = 0$ . The relation between the Koopman operator and system immersibility, as stated in Proposition 1, allows us to establish a Koopman interpretation of the formulation (13). Consider  $\{\{h_i(f^k(x))\}_{i=1}^m\}_{k=0}^M$  as the observables, with the Koopman operator  $\mathcal{K}$ , we have

$$\mathcal{K} \mathbf{h}_M(x) = \mathbf{h}_M(f(x)). \quad (14)$$

Such observables are called delay coordinates in (Susuki, 2015; Arbabi and Mezic, 2017). With the solution  $\gamma_M$  from (13),  $\Gamma(\gamma_M)$  defined in (5) can be considered as a finite-dimensional approximation of the Koopman operator.

In general, Problem (13) is nonconvex and computing its exact solution is difficult. For this reason, we aim to obtain a numerical solution by using a finite set of initial states. Given  $N$  initial states, denoted by  $\Omega_N := \{x^1, \dots, x^N\}$ , a numerical approximation of Problem (13) can be given by the least squares regression

$$\min_{\gamma_M} \sum_{x \in \Omega_N} \|h(f^{M+1}(x)) - \gamma_M \mathbf{h}_M(x)\|^2. \quad (15)$$

This formulation does not require explicit knowledge of  $f(x)$  and  $h(x)$  and hence it can be solved under a data-driven framework. In fact, Problem (15) is a simple least square approximation problem and can be easily solved

with the classical pseudo inverse technique. For completeness, we give an explicit expression of the solution. Let

$$\Theta_N = [h(f^{M+1}(x^1)) \ \dots \ h(f^{M+1}(x^N))] \quad (16)$$

$$\Pi_N = [\mathbf{h}_M(x^1) \ \dots \ \mathbf{h}_M(x^N)]. \quad (17)$$

Let the solution of Problem (15) be denoted by  $\hat{\boldsymbol{\gamma}}_M := [\hat{\gamma}_0, \hat{\gamma}_1, \dots, \hat{\gamma}_M]$ . Suppose  $\Pi_N$  is full row rank, the solution  $\hat{\boldsymbol{\gamma}}_M$  can be uniquely be expressed as

$$\hat{\boldsymbol{\gamma}}_M = \Theta_N \Pi_N^+ \quad (18)$$

where  $+$  denotes the pseudo inverse. The optimum of Problem (13) is approximated by

$$\hat{\delta}(M, \Omega_N) = \max_{x \in \Omega_N} \|h(f^{M+1}(x)) - \hat{\boldsymbol{\gamma}}_M \mathbf{h}_M(x)\|_\infty \quad (19)$$

The lifting method is summarized in Algorithm 1. When  $\{\{h_i(f^k(x))\}_{i=1}^m\}_{k=0}^M$  are not linearly independent,  $\Pi_N$  will not be full row rank. In this case, we can first remove the redundant functions before using the lifting algorithm. The details are not given due to page limitation.

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**Algorithm 1** Data-driven lifting for linearization

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**Input:**  $\Omega_N$  and some given tolerance  $\delta > 0$

**Output:**  $M$ ,  $\hat{\boldsymbol{\gamma}}_M$  and  $\hat{\delta}(M, \Omega_N)$

*Initialization:* Set  $M = 0$ ;

- 1: For any  $x \in \Omega_N$ , generate the trajectory  $\mathbf{h}_{M+1}(x)$ ;
  - 2: Compute  $\Theta_N$  and  $\Pi_N$  from (16) and (17) respectively;
  - 3: Obtain  $\hat{\boldsymbol{\gamma}}_M$  and  $\hat{\delta}(M, \Omega_N)$  from (18) and (19) respectively. If  $\hat{\delta}(M, \Omega_N) \leq \delta$ , terminate; otherwise, set  $M \leftarrow M + 1$  and go to Step 1.
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The algorithm above is similar to the Krylov method proposed in Rowley et al. (2009), where the  $(M + 1)^{th}$  iterate is approximated by a linear combination of the previous iterates, i.e.,  $\boldsymbol{\gamma}_M$  in (15) becomes a vector in  $\mathbb{R}^{M+1}$  instead of a matrix in  $\mathbb{R}^{m \times (M+1)m}$ . This also shows the connection to the Koopman operator framework. If the global immersibility property holds, the algorithm above converges in finite time, as stated in the following theorem.

*Theorem 3.* Given  $N$  initial states  $\Omega_N$ , Algorithm 1 converges for any  $\delta > 0$ . Suppose the system (1) is globally immersible to some linear system in the form of (3). Then, there exists  $M \in \mathbb{Z}^+$  such that  $\hat{\delta}(M, \Omega_N) = 0$ , where  $\hat{\delta}(M, \Omega_N)$  is defined in (19). In addition, if  $\Pi_N$ , defined in (17), is full row rank, the system (1) is globally immersible to  $\Sigma(\hat{\boldsymbol{\gamma}}_M)$ , where  $\hat{\boldsymbol{\gamma}}_M$  is given in (18).

**Proof of Theorem 3:** The convergence is a direct consequence of Lemma 2. Suppose the system (1) is globally immersible to some linear system, from Theorem 1, there exist  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M$  such that  $h(f^{M+1}(x)) = \boldsymbol{\gamma}_M \mathbf{h}_M(x)$  for any  $x \in \mathbb{R}^n$ . Hence,  $\boldsymbol{\gamma}_M$  is a solution to Problem (15) and the optimum is exactly 0. As a result,  $\hat{\delta}(M, \Omega_N) = 0$ . If  $\Pi_N$  is full rank,  $\boldsymbol{\gamma}_M$  is the unique the solution.  $\square$

Although the linear system  $\Sigma(\hat{\boldsymbol{\gamma}}_M)$  obtained from the lifting method in Algorithm 1 can be exactly equivalent in some cases, as shown in Theorem 3, it can only be considered as an approximate linearized system in general. We will formally discuss about the mismatch in the next subsection.

### 3.4 Approximate immersion and mismatch

We then discuss approximate immersibility, which is formally defined below.

*Definition 2.* Given  $X \subseteq \mathbb{R}^n$ ,  $t_f \in \mathbb{Z}^+$  and  $\epsilon > 0$ , the system (1) is  $(\epsilon, t_f)$ -immersible into  $\Sigma(A, C)$  in (3) with  $\tilde{n} \leq mt_f$  in  $X$  if there exists a map  $T : X \rightarrow \mathbb{R}^{\tilde{n}}$  such that  $\max_{0 \leq t \leq t_f} \|h(f^t(x)) - CA^t T(x)\|_\infty \leq \epsilon, \forall x \in X$ . The corresponding linear system is called a  $(\epsilon, t_f)$ -linearized system of the system (1).

A bound on mismatch between the linearized system and system (1) is provided below.

*Lemma 3.* Given  $X \subseteq \mathbb{R}^n$ ,  $\delta > 0$  and  $t_f \in \mathbb{Z}^+$ , suppose there exist  $M \in \mathbb{Z}^+$  and  $\boldsymbol{\gamma}_M \in \mathbb{R}^{m \times m(M+1)}$  such that  $\max_{0 \leq k \leq t_f} \|h(f^{M+1+k}(x)) - \boldsymbol{\gamma}_M \mathbf{h}_M(f^k(x))\|_\infty \leq \delta, \forall x \in X$ . Let

$$\bar{\epsilon}(\delta, \boldsymbol{\gamma}_M; t_f) := \sum_{\ell=0}^{t_f-1} \|[I_m \ \mathbf{0}]\Gamma(\boldsymbol{\gamma}_M)^\ell \begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}\|_\infty \delta. \quad (20)$$

Then, for any  $\bar{t} \in \{0, 1, \dots, t_f\}$ , the system (1) is  $(\bar{\epsilon}(\delta, \boldsymbol{\gamma}_M; \bar{t}), \bar{t})$ -immersible into  $\Sigma(\boldsymbol{\gamma}_M)$  in  $X$ .

To compute the bound on the mismatch in Lemma 3, for generally nonlinear systems, we have to solve a problem of infinite number of linear constraints. For this reason, we use the scenario approach (Calafiore, 2010). Suppose the initial conditions  $\Omega_N$  are randomly and uniformly sampled from  $X$  with the corresponding trajectories for the given  $t_f$ , the sampled problem is given by

$$\max_{0 \leq k \leq t_f, x \in \Omega_N} \|h(f^{M+1+k}(x)) - \boldsymbol{\gamma}_M \mathbf{h}_M(f^k(x))\|_\infty \quad (21)$$

This problem can be converted into a linear optimization problem with  $1 + m^2(M+1)$  variables and  $2mN(t_f+1)$  linear inequality constraints. Let  $(\delta^*(\Omega_N), \boldsymbol{\gamma}_M^*(\Omega_N))$  denote the optimum of the sampled problem (21). Adapted from Theorem 3.3 in (Calafiore, 2010), the following theorem can be achieved.

*Theorem 4.* Given  $X \subseteq \mathbb{R}^n$  and  $t_f \in \mathbb{Z}^+$ , let  $\Omega_N$  be the set of  $N$  initial conditions that are randomly and uniformly sampled from  $X$  and  $(\delta^*(\Omega_N), \boldsymbol{\gamma}_M^*(\Omega_N))$  be the solution of Problem (21). For any  $c \in (0, 1)$ ,  $d \in \mathbb{Z}^+$ , let  $\Phi(c; d, N) = \sum_{j=0}^d \binom{N}{j} c^j (1-c)^{N-j}$ . Then, given  $c \in (0, 1)$ , with probability no smaller than  $1 - \Phi(c; t_f m^2 + 1, N)$ , for any  $\bar{t} \leq t_f$ , the system (1) is  $(\bar{\epsilon}(\delta^*(\Omega_N), \boldsymbol{\gamma}_M^*(\Omega_N); \bar{t}), \bar{t})$ -immersible into  $\Sigma(\boldsymbol{\gamma}_M^*(\Omega_N))$  in no smaller than  $1 - c$  of the region in  $X$

As shown in (Calafiore, 2010),  $\Phi(c; t_f m^2 + 1, N)$  can be computed by the regularized incomplete beta function  $\mathcal{I}(c; a, b)$ . The overall procedure for determining the approximate linearized system is summarized as below.

## 4. IMPLEMENTATION ISSUES

### 4.1 Regularization

To improve numerical stability, we can consider the regularized version of Problem (15):

$$\min_{\boldsymbol{\gamma}_M} \sum_{i=1}^N \|h(f^{M+1}(x^i)) - \boldsymbol{\gamma}_M \mathbf{h}_M(x^i)\|^2 + \rho \|\boldsymbol{\gamma}_M\|_F^2, \quad (22)$$

**Algorithm 2** The data-driven method for computing an approximate linearized system and mismatch bounds

**Input:**  $t_f, \beta, c, M,$  and  $X$

**Output:**  $\gamma_M^*(\Omega_N)$  and  $\{\bar{\epsilon}(\delta^*(\Omega_N), \gamma_M^*(\Omega_N); \bar{t})\}_{\bar{t}=0}^{t_f}$

- 1: *Initialization:* Determine  $N$  such that  $\mathcal{I}(c; t_f m^2 + 2, N - t_f m^2 - 1) \geq 1 - \beta$ . Randomly and uniformly sample  $N$  points  $\Omega_N$  in  $X$  and generate  $\{h_{2t_f}(x)\}_{x \in \Omega_N}$ .
- 2: Solve Problem (21) and obtain  $(\delta^*(\Omega_N), \gamma_M^*(\Omega_N))$ ;
- 3: Compute  $\bar{\epsilon}(\delta^*(\Omega_N), \gamma_M^*(\Omega_N); \bar{t})$  from (20) using  $(\delta^*(\Omega_N), \gamma_M^*(\Omega_N))$  for any  $\bar{t} \in \{0, 1, \dots, t_f\}$ ;

where  $\rho > 0$  is some given parameter. The solution of the regularized problem becomes  $\Theta_N \Pi_N^T (\Pi_N \Pi_N^T + \rho I)^{-1}$ .

#### 4.2 Speed-up strategies for computing mismatch bounds

Although Problem (21) is a linear optimization problem, as  $N$  increases, the number of constraints grows and it will take increasing amount of memory to solve the problem. To speed up the computations in Algorithm 2, a standard strategy is to approximate the solution of Problem (21) with the least squares solution by solving  $\min_{\gamma_M} \sum_{x \in \Omega_N} \sum_{k=0}^{t_f} \|h(f^{M+1+k}(x)) - \gamma_M h_M(f^k(x))\|^2$ . Note that the solution of this problem is different from the solution of Problem (15) as it takes the trajectories from 0 to  $t_f + M + 1$  while Problem (15) only take the trajectories from 0 to  $M + 1$ . Another speed-up strategy for solving Problem (21) is to use distributed consensus algorithms, see, e.g., (Notarstefano and Bullo, 2011).

#### 4.3 Model reduction

For some nonlinear systems, the dimension of the linearized system can be very high, especially when there are limit cycles and chaotic behaviors. Hence, it is necessary to reduce the model of the linearized system. From the perspective of Koopman operator theory, instead of using the full observables in (14), we take fewer observables with linear transformation. The details are not given due to page limitation.

## 5. SIMULATION RESULTS

*Example 1.* (The LaSalle example). We consider the following nonlinear system (LaSalle, 1986):  $x_1^+ = -\frac{x_2}{1+x_1^2}, x_2^+ = \frac{0.9x_1}{1+x_2^2}, y = [x_1, x_2]^T$ . This example is globally asymptotically stable at the origin. We consider the linearization in the region  $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$ . With Algorithm 1, we uniformly randomly generate 1000 samples and shows the values of  $\hat{\delta}(M, \Omega_N)$  for different values of  $M$  in Figure 1 for 50 realizations of  $\Omega_N$ . Although there is no formal guarantee, the linearized system is already a good approximation to the original nonlinear system when  $M = 3$ . We then use Algorithm 2 to obtain a linearized system with the probabilistic guarantee derived in Theorem 4. Let  $\beta = 0.05, c = 0.01,$  and  $t_f = 50$ . We take  $N$  to be  $5 \times 10^4$  to make sure that  $\mathcal{I}(c; t_f m^2 + 2, N - t_f m^2 - 1) \geq 1 - \beta$ . The comparison between the original system and the solution from Algorithm 2 is made in Figure 2 with the mismatch bounds, which are denoted by the vertical grey lines.

*Example 2.* (Two-dimensional logistic map). Consider the two-dimensional logistic map in (López-Ruiz and Pérez-Garcia, 1991):  $x_1^+ = (3x_1 + 1)x_2(1 - x_2), x_2^+ = (3x_2 +$

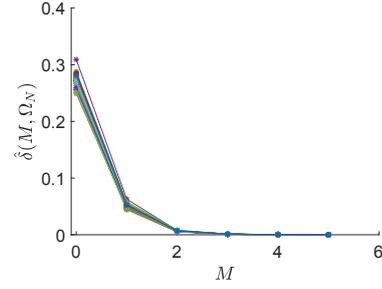


Fig. 1. The values of  $\hat{\delta}(M, \Omega_N)$  for different values of  $M$  for the LaSalle example.

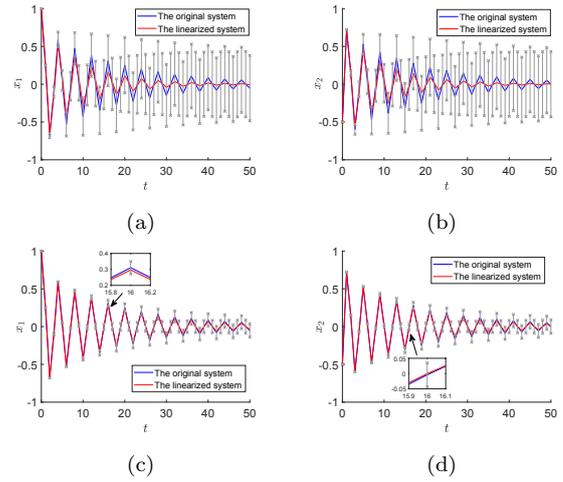


Fig. 2. Comparison and mismatch between the original system and the linearized system for the LaSalle example: (a) & (b)  $M = 1$ ; (c) & (d)  $M = 3$

$1)x_1(1 - x_1)$  with  $y = [x_1, x_2]^T$ . We consider the linearization in the region  $X = \{x \in \mathbb{R}^2 : x \geq 0, \|x\|_\infty \leq 1\}$ . As shown in (López-Ruiz and Pérez-Garcia, 1991), there are two limit cycles in  $X$ . We use the same setting in the previous example except that  $t_f = 100$ . Again, we take  $N$  to be  $5 \times 10^4$  and generate  $N$  trajectories. As mentioned in Section 4.2, we can use the least squares solution to speed up the linearization procedure. The comparison of the trajectories starting from two initial states  $[0.1 \ 0.8]^T$  and  $[0.7 \ 0.4]^T$  is given in Figure 3 for two different choices of  $M$ . As we can see, the linearized system is already a good approximation to the original system when  $M = 30$ .

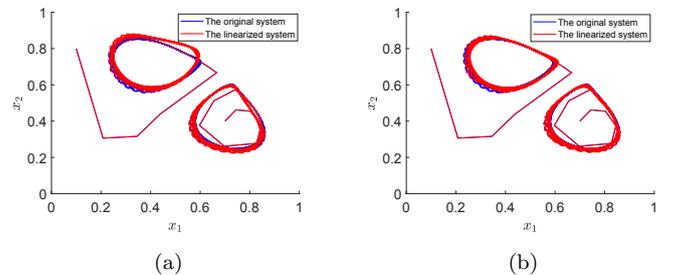


Fig. 3. Trajectories for the logistic map example: (a)  $M = 25$ ; (b)  $M = 30$ .

*Example 3.* (Van der Pol oscillator). In the rest of this section, we will make comparison with other Koopman-based approaches. Consider the uncontrolled Van der Pol

oscillator (Korda and Mezić, 2018):  $\dot{x}_1 = 2x_2$ ,  $\dot{x}_2 = -0.8x_1 + 2x_2 - 10x_1^2x_2$  with  $y = [x_1, x_2]^T$ . We discretize the continuous system using the Runge-Kutta four method with discretization period  $T_s = 0.1$ . Consider the same setting in (Korda and Mezić, 2018), we simulate 200 trajectories over 1000 sampling periods and the initial conditions are generated randomly and uniformly in the unit box  $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$ . We take 100 radial basis functions in (Korda and Mezić, 2018) in the form of  $g(x) = \|x - x_0\|^2 \log(\|x - x_0\|)$ , where  $x_0$  is randomly selected with the uniform distribution on  $X$ . The state itself is also contained in the basis functions. Hence, the dimension of the lifted state-space is 102 in total. Then, we consider the Koopman approach with monomials (Mauroy and Goncalves, 2017). For a fair comparison, we use monomials with the maximal degree being 13, given by  $\{x_1^{s_1}x_2^{s_2} : s_1 + s_2 \leq 13, s_1, s_2 \in \mathbb{Z}^+\}$ . The number of monomials is  $15!/(13!2!) = 105$ . For the proposed approach, we will take  $M = 30$  and the dimension is 62, which is much smaller. We solve the regularized problem in Section 4.1 with  $\rho = 10^{-6}$ . Consider the two initial conditions used in (Korda and Mezić, 2018):  $x(0) = [0.5 \ 0.5]^T$  and  $x(0) = [-0.1 \ -0.5]^T$ , the trajectories (over 1000 sampling periods) of the different approaches are shown in Figure 4. We can see that the proposed approach is able to get a better approximation with a lower dimension.

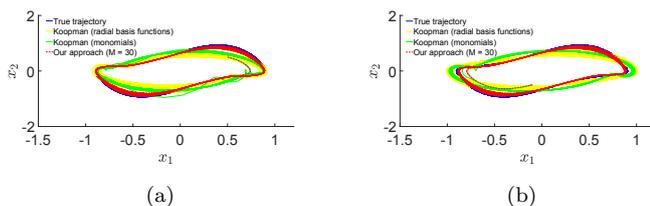


Fig. 4. Comparison of different approaches for the Van der Pol oscillator: (a)  $x(0) = [0.5 \ 0.5]^T$ ; (b)  $x(0) = [-0.1 \ -0.5]^T$

## 6. CONCLUSIONS

In this paper, we have presented a data-driven immersion technique for the linearization of discrete-time nonlinear systems without an analytic model. A necessary and sufficient condition is shown for exact linearization of certain nonlinear systems. For arbitrary nonlinear systems, we consider approximate immersion with the bounds on system mismatch. The proposed technique also has a Koopman interpretation by considering the outputs as the observables. Finally, we have demonstrated the performance of the proposed technique on several examples with a fixed point and limit cycles. A comparison simulation is made between the proposed approach and Koopman-based approaches that use radial basis functions and monomials. From the comparison, we have shown that the proposed approach is able to achieve a better approximation with a smaller lifted dimension.

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