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Functional reduced order $\mathcal{H}_\infty$ decentralized observer based control for large scale interconnected nonlinear stochastic systems

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Abstract: In this work, the $\mathcal{H}_\infty$ decentralized reduced order observer based control for a class of large scale nonlinear stochastic systems is concerned. In this context we consider subsystems which are interconnected by some nonlinear interconnections under quadratic boundedness and Lipschitz property of the system. The proposed control law is based on the resolution of some LMI.

Keywords: Mean square exponential stability, Interconnected systems, Large scale stochastic system, Brownian motions, Decentralized control, Reduced order observer based control.

1. INTRODUCTION

Numerous works treat the problem of stability and stabilization of large scale linear or nonlinear interconnected systems in the literature, see [Callier et al. 1976, Šiljak 1977, Michel and Miller 1977, Vidyasagar 1980, 1981, Gündeş and Desoer 1990, Šiljak 1991, Davison et al. 2020].

This class of systems are generally composed by many subsystems. They are characterized by a large number of variables, some strong and/or complex interactions between the subsystems variables. This implies a large number of equations and unknowns and some problems in practice to study them.

This type of modeling can be found in industrial processes (power systems), transport networks, economic models, chemical processes, space structure.

It also exists several works which treat the problem of decentralized observer based control design of large scale interconnected systems. Many approaches have been used for the observers design. For example, in Dhbaibi et al. [2009] the authors investigated the problem of $\mathcal{H}_\infty$ decentralized tracking control using a decentralized observer for interconnected nonlinear systems to ensure the asymptotic stability, whereas the $\mathcal{H}_\infty$ criterion has been replaced by a quadratic cost in in Mao and Lin [1990] and Tlili and Benhadj Braiek [2009]. In Gao et al. [2015], the authors propose a dynamic observer based control for large scale nonlinear interconnected systems based on algebraic constraints obtained from estimation error. In Kalsi et al. [2009] a design of decentralized control using a sliding mode observers has been proposed whereas in Zhao et al. [2017] a design of decentralized fault tolerant control scheme based on decentralized control method for a class of large-scale nonlinear systems is given. The problem of decentralized control based on backstepping approach and exploiting the triangular canonical form of the system to guarantee the input-to-state stability of the closed-loop system are investigated in Liu et al. [2007] in the deterministic case without measurement noise, in Liu et al. [2011] in the deterministic case without measurement noise and in Liu et al. [2008] in the stochastic case. A decentralized reduced-order controller is proposed in Bakule and de la Sen [2009] for a class of networked continuous-time complex systems with symmetric nominal interconnections.

So, we note that there are many works concerning the decentralized control for large scale linear or nonlinear interconnected systems in deterministic case; but, in our knowledge, there are less works on the decentralized control for this class of systems in the stochastic case Liu et al. [2008], Hua et al. [2015].

The stochastic description of systems is used when the deterministic approach is not sufficient to model the considered systems. In fact, the stochastic representation can capture all the dynamic behavior of a complex system that is not well given by the deterministic approach. The advantage of SDE (stochastic differential equations) is that they contain a random term which represent the randomness within the systems to model. Thus, the studied systems are composed by two parts: the drift one which represents the dominant action of the system and the diffusion one representing randomness along the dominant behavior. Stochastic modeling has then got a great role during the last years in engineering and sciences. There exist many works about SDE and their simulation like in Has'minskii [1980], Mao [1994], Cyganowski [1996], Mao [1997], Øksendal [2003] and references therein. Stochastic systems are used in various areas of application like system with human operators, economic systems which model some of the uncertainties as stochastically varying lags, mechanical systems subject to random vibrations (e.g. earthquakes), ... (see [Willems and Willems 1976] for example).

This paper is dedicated to the observer-based control of large scale interconnected stochastic systems.

In this paper we deal with reduced order observer based control for large scale stochastic systems which are described by stochastic differential equation (SDE) controlled by noises. These noises are Brownian motions. The considered differential equation corresponds to an Itô process with multiplicative noises. The goal of the control law to be designed is to ensure the mean square exponential stability (MSES) of the obtained closed loop system with an $\mathcal{H}_\infty$ criterion.

The paper is organized as follows. A preliminary of SDE is given in Section 2. The problem to be solved is stated in Section 3. In Section 4, a $\mathcal{H}_\infty$ reduced order decentralized observer-based controller is designed into two steps. The full order case is treated in in Section 5.

Notations. $\mathbb{R}^n$ denote the n-dimensional Euclidean space. $\|A\| = (\sum_{i,j} A_{i,j}^2)^{1/2} = \sqrt{\mathrm{tr}(A^T A)}$ is the Euclidean norm of the matrix A, while $\|x\| = \sqrt{x^T x}$ is the Euclidean norm of the vector x. For matrices A_1, and A_2, $\mathrm{bdiag}(A_1, A_2)$ designates the block diagonal matrix $\left[\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right]$. We denote by $\widehat{L}_2\left([0,\infty);\mathbb{R}^k\right)$ the space of non-anticipatory square-integrable stochastic process $f(.) = (f(t))_{t\in[0,\infty)}$ in $\mathbb{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying

$$\|f\|_{\widehat{L}_2}^2 = \mathbf{E}\left\{\int_0^\infty \|f(t)\|^2 \,\mathrm{d}\,t\right\} < \infty$$

where $\mathbf{E}\{.\}$ is the expectation operator.

2. PRELIMINARIES ON SDE

We consider the following class of stochastic differential equation (SDE)

$$\mathrm{d}\,x = f(x)\,\mathrm{d}\,t + g(x)\,\mathrm{d}\,w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $w \in \mathbb{R}^d$ is a multi-dimensional independent Brownian motion.

To guarantee the existence and the uniqueness of the solution x of the SDE (1), the functions $f(x)$ and $g(x)$ satisfy the following relations $\forall\, x \in \mathbb{R}^n$, $\forall\, y \in \mathbb{R}^n$ (see [Mao 1997])

$$\|f(x)\|^2 + \|g(x)\|^2 \leqslant k_1(1+\|x\|^2), \tag{2a}$$

$$\|f(x) - f(y)\| \vee \|g(x) - g(y)\| \leqslant k_2\,\|x-y\|\,, \tag{2b}$$

where k_1 and k_2 are given strictly positive reals.

The function $f(x)$ is Lebesgue integrable and the function $g(x)$ is Lebesgue square-integrable as it is needed for Itô calculus [Mao 1997].

To study the MSES stability we use the following definition.

Definition 1. The equilibrium of SDE (1) is said to be MSES if

$$\limsup_{t\to+\infty} \frac{1}{t}\ln(\mathbf{E}(\|x(t,t_0,x_0)\|^2)) < 0. \tag{3}$$

Relation (3) stands that there exist $M > 0$ and $\alpha > 0$ such that

$$\mathbf{E}\Big(\|x(t,t_0,x_0)\|^2\Big) \leqslant M\,\|x_0\|^2\, e^{-\alpha(t-t_0)}$$

for all $x_0 \in \mathbb{R}^n$ and $t \geqslant t_0 \geqslant 0$.

The Lyapunov function $V(x)$ with the two following Itô stochastic differential operators associated with the SDE (1)

$$\mathrm{d}\,V(x) = \mathfrak{L}V(x)\,\mathrm{d}\,t + \mathfrak{B}V(x)\,\mathrm{d}\,w, \tag{4a}$$

$$\mathfrak{L}V(x) = \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\,\mathrm{tr}\left(g^T(x)\frac{\partial^2 V(x)}{\partial x^2}g(x)\right), \tag{4b}$$

$$\mathfrak{B}V(x) = \frac{\partial V(x)}{\partial x}g(x). \tag{4c}$$

To ensure the MSES stability we use the following lemma which gives sufficient conditions on a Lyapunov function candidate.

The following lemma can be used to study the stability of a SDE for $t_0 = 0$ [Mao 1997, Hu and Mao 2008].

Lemma 2. [Mao 1997] Assume that there exist a Lyapunov function $V(x)$ which is twice continuously differentiable on x, and $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that

$$c_1\,\|x\|^2 \leqslant V(x) \leqslant c_2\,\|x\|^2\,, \tag{5}$$

$$\mathfrak{L}(V(x)) \leqslant -c_3 V(x) \quad \forall\, x \in \mathbb{R}^n, \tag{6}$$

then the equilibrium point of the SDE (1) is mean-square exponentially stable, i.e.

$$\mathbf{E}\left\{\|x(t)\|^2\right\} \leqslant \frac{c_2}{c_1}\,\|x_0\|^2 e^{-c_3 t} \quad \forall\, t \geqslant 0, \quad \forall\, x_0 \in \mathbb{R}^n. \tag{7}$$

In this paper we focus our attention in the case where the dimension n of the state $x(t)$ is large, specially when the stochastic system is an interconnected one.

The following lemma will be used in the sequel.

Lemma 3. [Petersen 1987] Let three matrices $A \in \mathbb{R}^{n\times q}$, $B \in \mathbb{R}^{p\times n}$ et $C \in \mathbb{R}^{q\times p}$ with $C^T C \leqslant I_p$, then for all real $\mu > 0$, then

$$2x^T ACBy \leqslant \mu x^T AA^T x + \frac{1}{\mu}y^T B^T By \tag{8}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.

3. PROBLEM STATEMENT

We consider the following SDE

$$\mathrm{d}\,x_i = (A_{t_i}x_i + B_{v_i}v_i + B_i u_i + h_i(t,x))\,\mathrm{d}\,t + \sum_{i=1}^{N} A_{w_i}x_i\,\mathrm{d}\,w_i \tag{9a}$$

$$y_i = C_i x_i \tag{9b}$$

$$z_i = C_{z_i}x_i + D_i v_i \tag{9c}$$

where $i = 1,\dots,N$, $x_i \in \mathbb{R}^{n_i}$ is the state vector, $u_i \in \mathbb{R}^{m_i}$ is the control input, $v_i \in \mathbb{R}^{q_i}$ is the perturbation vector with bounded energy, $w_i \in \mathbb{R}^{d_i}$ is a multi-dimensional independent Brownian motion, $z_i \in \mathbb{R}^{k_i}$ is the controlled output and $y_i \in \mathbb{R}^{p_i}$ is the measured output. A_i, C_i, B_i, B_{v_i}, C_{1i} and D_i are constant matrices, $h_i(t,x)$ designs the nonlinear interconnection function of i^{th} subsystem where $x^T = [x_1^T,\dots,x_N^T] \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$.

As in many works like Zhu and Pagilla [2007], Stanković and Šiljak [2009], Zečević and Šiljak [2010], the functions $h_i(t,x)$ are piecewise continuous vector functions in both arguments and satisfy in their domains of continuity the following quadratic inequalities

$$h_i(t,x)^T h_i(t,x) \leqslant \alpha_i^2 x^T H_i^T H_i x \qquad i,\dots,N \tag{10}$$

where α_i are interconnection bounds and $H_i \in \mathbb{R}^{\ell_i \times n}$ are constant bounding matrices.

We can write the interconnected system in the compact form as follows

$$\mathrm{d}\, x = (A_t x + B_v v + Bu + h(t,x))\,\mathrm{d}\, t + A_w x \,\mathrm{d}\, w \tag{11a}$$
$$y = Cx \tag{11b}$$
$$z = C_z x + Dv \tag{11c}$$

where $u^T = [u_1^T,\dots,u_N^T]$, $v^T = [v_1^T,\dots,v_N^T]$, $y^T = [y_1^T,\dots,y_N^T]$, $w^T = [w_1^T,\dots,w_N^T]$, $A_t = \mathrm{bdiag}(A_{t_1},\dots,A_{t_N})$, $C = \mathrm{bdiag}(C_1,\dots,C_N)$, $B = \mathrm{bdiag}(B_1,\dots,B_N)$, $B_v = \mathrm{bdiag}(B_{v_1},\dots,B_{v_N})$, $A_w = \mathrm{bdiag}(A_{w_1},\dots,A_{w_N})$, $C_z = \mathrm{bdiag}(C_{z_1},\dots,C_{z_N})$, $D = \mathrm{bdiag}(D_1,\dots,D_N)$ and $h^T(t,x) = [h_1^T(t,x),\dots,h_N^T(t,x)]$ is the global nonlinear interconnection function. Without loss of generality, we have $m_i \leqslant n_i$.

Using (10), the global interconnection is written as follows

$$h(t,x)^T h(t,x) \leqslant x^T H^T \Phi^{-1} H x \tag{12}$$

where $\Phi = \mathrm{bdiag}(\Phi_1,\dots,\Phi_N)$, $\Phi_i = \alpha_i^{-2} I_{\ell_i}$ and $H^T = [H_1^T,\dots,H_N^T]$.

We define $n = \sum_{i=1}^N n_i$, $m = \sum_{i=1}^N m_i$, $q = \sum_{i=1}^N q_i$, $p = \sum_{i=1}^N p_i$, $d = \sum_{i=1}^N d_i$, $k = \sum_{i=1}^N k_i$ and $\ell = \sum_{i=1}^N \ell_i$.

We consider the following decentralized functional reduced order observer described by, for $i = 1,\dots,N$

$$\mathrm{d}\,\eta_i = M_i \eta_i \,\mathrm{d}\, t + J_i y_i \,\mathrm{d}\, t + G_i u_i \,\mathrm{d}\, t \tag{13a}$$
$$u_i = \eta_i + E_i y_i \tag{13b}$$

where E_i, M_i, J_i and G_i are gain matrices to determine. $\eta_i \in \mathbb{R}^{m_i}$ is the state of the observer (13) The nonlinear function $h(t,x)$ is not considered in the synthesis of the gain observer, so the functional observer structure is totally decentralized.

According to the notations used in (11), we define $\eta^T = [\eta_1^T,\dots,\eta_N^T]$, $M = \mathrm{bdiag}(M_1,\dots,M_N)$, $J = \mathrm{bdiag}(J_1,\dots,J_N)$, $G = \mathrm{bdiag}(G_1,\dots,G_N)$ and $E = \mathrm{bdiag}(E_1,\dots,E_N)$.

Notice that, unlike the literature on decentralized observer based control, the control law u_i is directly estimated by the decentralized functional observer (13) which is of minimal order since $\dim(\eta_i) = \dim(u_i)$.

Using Definition 4, the problem to be treated is stated in Problem 5.

Definition 4. [Zasadzinski et al. 2007] The system (11) is said to be stabilizable based on a decentralized functional reduced order observer (13) if there exist a gain matrix $L = \mathrm{bdiag}(L_1,\dots,L_N)$, N functional reduced order observers given by (13) and a control law $u = Lx$ such that

- **(i)** $\lim_{t\to\infty} \mathbf{E}\,\|u - Lx\|^2 = 0$ if $v = 0$,
- **(ii)** the closed-loop system given by (11) and (13) is MSES.

Problem 5. The objective is to establish N functional observers (13) such that

- **(i)** $\lim_{t\to\infty} \mathbf{E}\,\|u - Lx\| = 0$ if $v = 0$,
- **(ii)** the resulting closed-loop system given by (11) and (13) is MSES and satisfies the $\mathcal{H}_\infty$ performance $J_{zv} < 0$ for a given $\gamma > 0$

where L is defined in definition 4 and J_{zv} is given by

$$J_{zv} = \mathbf{E}\left\{\int_0^{+\infty} \left(z^T z - \gamma^2 v^T v\right)\mathrm{d}\, t\right\} < 0, \tag{14}$$

$\forall\, v \in \widehat{L}_2$, $v \neq 0$, $x(0) = 0$ and $\eta(0) = 0$.

4. DESIGN OF THE FUNCTIONAL REDUCED ORDER $\mathcal{H}_\infty$ DECENTRALIZED OBSERVER BASED CONTROL LAW

The synthesis of the functional reduced order $\mathcal{H}_\infty$ decentralized observer based control law (13) is splited down into two steps: first we determine the "state-feedback gains" L_i and second the functional observer matrices M_i, J_i, G_i and E_i are computed.

4.1 Synthesis of the "state-feedback gains" L_i

In this subsection, we assume that the state x is measured in (11), i.e. that $C = I_n$ in (11b). So the closed loop system composed by (11a), (11c) and $u = Lx$ is given by

$$\mathrm{d}\, x = ((A_t + BL)x + B_v v + h(t,x))\,\mathrm{d}\, t + A_w x\,\mathrm{d}\, w \tag{15a}$$
$$z = C_z x + Dv \tag{15b}$$

The design of the gain L is given by the following theorem.

Theorem 6. The closed-loop SDE (15) is MSES and satisfies the $\mathcal{H}_\infty$ criterion (14) if there exist two reals $\gamma > 0$, $\mu_1 > 0$ and, for $i = 1,\dots,N$, matrices $\overline{P}_i = \overline{P}_i^T > 0$, $\overline{P}_i \in \mathbb{R}^{n_i\times n_i}$ and $Y_{L_i} \in \mathbb{R}^{n_i \times m_i}$ such that the following LMI

$$\begin{bmatrix} (a) & \overline{P}H^T & \overline{P}C_z^T & \overline{P}A_w^T & (b) \\ H\overline{P} & -\mu_1^{-1}\Phi & 0 & 0 & 0 \\ C_z\overline{P} & 0 & -I_k & 0 & 0 \\ A_w\overline{P} & 0 & 0 & -\overline{P} & 0 \\ (b)^T & 0 & 0 & 0 & -\gamma^2 I_q + D^T D \end{bmatrix} < 0 \tag{16}$$

is satisfied where

$$(a) = \overline{P}A_t^T + Y_L^T B^T + A_t\overline{P} + BY_L + \mu_1^{-1} I_n,$$
$$(b) = B_v + \overline{P}C_z^T D,$$

and $\overline{P}_1 = \mathrm{bdiag}(\overline{P}_1,\dots,\overline{P}_N)$, $Y_L = \mathrm{bdiag}(Y_{L_1},\dots,Y_{L_N})$.

The gain matrices are given by $L_i = Y_{L_i}\overline{P}_i^{-1}$ with $i = 1,\dots,N$.

Proof. The application of Itô formula (4) on the Lyapunov function $V(x) = x^T P x$, with $P = P^T = \mathrm{bdiag}(P_1,\dots,P_N) > 0$ and $P_i \in \mathbb{R}^{n_i\times n_i}$, for SDE (15) gives

$$\mathrm{d}\, V(x) = \mathfrak{L}V(x)\,\mathrm{d}\, t + 2x^T P A_w X\,\mathrm{d}\, w \tag{17}$$

with

$$\mathfrak{L}V(x) = 2x^T P\left((A + BL)x + B_v v\right) + 2x^T P h(t,x) + \frac{1}{2}\mathrm{tr}\left((A_w x)^T 2P(A_w x)\right). \tag{18}$$

Using the theorem of Fubini for a mesurable stochastic process x Chen [1985], we have

$$\mathbf{E}\left\{\int_0^T x\,\mathrm{d}\,t\right\} = \int_0^T \mathbf{E}\left\{x\,\mathrm{d}\,t\right\}$$

and the performance index J_{zv} in (14) can be written as follows

$$\begin{aligned}J_{zv} = \int_0^{+\infty} &\mathbf{E}\{(z^T z - \gamma^2 v^T v)\,\mathrm{d}\,t + \mathrm{d}\,V(x)\}\\ &-\mathbf{E}\{V(x)\}_{t=+\infty} + \mathbf{E}\{V(x)\}_{t=0},\end{aligned}$$

Taking the expectancy on the both sides of the equation (17) and using $\mathbf{E}\{\mathrm{d}\,w\} = 0$, we obtain

$$\mathbf{E}\{\mathrm{d}\,V(x)\} = \mathbf{E}\{\mathfrak{L}V(x)\}.$$

Since $\mathbf{E}\{V(x)\}_{t=0} = 0$ because $x(0) = 0$ and $\mathbf{E}\{V(x)\}_{t=+\infty} \geqslant 0$, we have

$$J_{zv} \leqslant \int_0^{+\infty} \mathbf{E}\{(z^T z - \gamma^2 v^T v)\,\mathrm{d}\,t + \mathfrak{L}V(x)\,\mathrm{d}\,t\}. \tag{19}$$

Using inequality (12) and Lemma 3, the term $2x^T(t)Ph(t,x)$ can be bounded as follows

$$\begin{aligned}2x^T Ph(t,x) &\leqslant \mu_1 h^T(t,x)h(t,x) + \mu_1^{-1}x^T PPx\\ &\leqslant \mu_1 x^T H^T \Phi^{-1} Hx + \mu_1^{-1}x^T PPx\end{aligned} \tag{20}$$

where $\mu_1 > 0$ is a given real. Then using (18) yields

$$\begin{aligned}\mathfrak{L}V(x) \leqslant\ & x^T(P(A_t + BL) + (A_t + BL)^T P + \mu_1 H^T \Phi^{-1} H\\ &+ \mu_1^{-1}PP + A_w^T PA_w)x + 2X^T PB_v v.\end{aligned} \tag{21}$$

There exists a real $c_3 > 0$ such that the condition $\mathfrak{L}V(x) \leqslant c_3 V(x)$ in Lemma 2 is satisfied if $\mathfrak{L}V(x) < 0$, i.e. if there exist a matrix $P = P^T = \mathrm{bdiag}(P_1, \ldots, P_N) > 0$ and a gain L such that

$$\begin{aligned}z^T z - \gamma^2 v^T v + x^T\big(&(A_t + BL)^T P + P(A_t + BL) + A_w^T PA_w\\ &+\mu_1 H^T \Phi^{-1} H + \mu_1^{-1}PP\big)x + 2x^T PB_v v < 0.\end{aligned}$$

Using (15b), the previous inequality can be rewritten as

$$\begin{aligned}&(C_z x + Dv)^T(C_z x + Dv) - \gamma^2 v^T v\\ &\quad + x^T\big((A_t + BL)^T P + P(A_t + BL) + A_w^T PA_w\\ &\quad +\mu_1 H^T \Phi^{-1} H + \mu_1^{-1}PP\big)x + 2x^T PB_v v < 0\end{aligned}$$

and is equivalent to

$$\begin{aligned}x^T\big(&(A_t + BL)^T P + P(A_t + BL) + A_w^T PA_w + \mu_1 H^T \Phi^{-1} H\\ &+\mu_1^{-1}PP + C_z^T C_z\big)x - \gamma^2 v^T v\\ &+ 2x^T PB_v v + 2x^T C_z^T Dv + v^T D^T Dv < 0.\end{aligned} \tag{22}$$

So inequality (19) holds if condition (22) is satisfied. Applying the Schur lemma Boyd et al. [1994] on inequality (22) gives the following inequality

$$\Upsilon = \begin{bmatrix} \Upsilon_1 & H^T & C_z^T & A_w^T P & P & \Upsilon_2\\ H & -\mu_1^{-1}\Phi & 0 & 0 & 0 & 0\\ C_z & 0 & -I_k & 0 & 0 & 0\\ PA_w & 0 & 0 & -P & 0 & 0\\ P & 0 & 0 & 0 & -\mu_1 I_n & 0\\ \Upsilon_2^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q + D^T D\end{bmatrix} < 0 \tag{23}$$

where $\Upsilon_1 = (A_t + BL)^T P + P(A_t + BL)$ and
$\Upsilon_2 = PB_v + C_z^T D$.

Pre- and post-multiplying the above inequality by $\begin{bmatrix} \overline{P} & 0 & 0 & 0 & 0 & 0\\ 0 & I_\ell & 0 & 0 & 0 & 0\\ 0 & 0 & I_k & 0 & 0 & 0\\ 0 & 0 & 0 & \overline{P} & 0 & 0\\ 0 & 0 & 0 & 0 & I_n & 0\\ 0 & 0 & 0 & 0 & 0 & I_q\end{bmatrix}$ gives the inequality

$$\Theta = \begin{bmatrix} \Theta_1 & \overline{P}H^T & \overline{P}C_z^T & \overline{P}A_w^T & I_n & \Theta_2\\ H\overline{P} & -\mu_1^{-1}\Phi & 0 & 0 & 0 & 0\\ C_z\overline{P} & 0 & -I_k & 0 & 0 & 0\\ A_w\overline{P} & 0 & 0 & -\overline{P} & 0 & 0\\ I_n & 0 & 0 & 0 & -\mu_1 I_n & 0\\ \Theta_2^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q + D^T D\end{bmatrix} < 0 \tag{24}$$

where $\overline{P} = P^{-1}$ and

$$\begin{aligned}\Theta_1 &= \overline{P}A_t^T + Y_L^T B^T + A_t\overline{P} + BY_L,\\ \Theta_2 &= B_v + \overline{P}C_z^T D,\\ Y_L &= L\overline{P}\end{aligned}$$

Inequality (24) is equivalent to LMI (16). □

4.2 Synthesis of the functional reduced order matrices

Using item (i) of Definition 4, the filtering error can be defined as

$$e_r = Lx - u = \Psi x - \eta \tag{25}$$

where

$$\Psi = L - EC = \mathrm{bdiag}(L_1 - E_1 C_1, \ldots, L_1 - E_N C_N). \tag{26}$$

The expression of the dynamics of the filtering error is given as follows

$$\begin{aligned}\mathrm{d}\,e_r = (Me_r + (\Psi A_t - M\Psi - JC)x + (\Psi B - G)u+)\,\mathrm{d}\,t\\ \Psi B_v v\,\mathrm{d}\,t + (\Psi B_v v + \Psi h(t,x))\,\mathrm{d}\,t + \Psi A_w x\,\mathrm{d}\,w\end{aligned} \tag{27}$$

In order to ensure that the dynamics error is exponentially stable in mean square and to remove the maximum of dependent terms of the state x in SDE (27), we will determine the value of matrices M, J, G and E imposing that the following Sylvester constraints

$$0 = \Psi A_t - M\Psi - JC, \tag{28}$$

$$0 = \Psi B - G. \tag{29}$$

are verified.

Using approach developed in Souley Ali et al. [2006], we define a matrix $S = \mathrm{bdiag}(S_1, \ldots, S_n)$ given by

$$S = J - ME \tag{30}$$

where S_i has the same dimension as J_i for $i = 1, \ldots, N$. Since Ψ is a block diagonal matrix, the Sylvester equation (28) can be rewritten as

$$L_i A_{t_i} = \mathcal{M}_i \mathcal{C}_i, \qquad i = 1, \ldots, N \tag{31}$$

where

$$\mathcal{M}_i = [M_i\ S_i\ E_i],$$

$$\mathcal{C}_i = \begin{bmatrix} L_i\\ C_i\\ C_i A_{t_i}\end{bmatrix}.$$

From Rao and Mitra [1971], equation (31) has a solution $\mathcal{M}_i$ if and only if

$$\text{rang}\left(\begin{bmatrix} L_iA_t \\ L_i \\ C_i \\ C_iA_{t_i} \end{bmatrix}\right) = \text{rang}\left(\begin{bmatrix} L_i \\ C_i \\ C_iA_{t_i} \end{bmatrix}\right), \tag{32}$$

and all the solutions to this equation are given by

$$\mathcal{M}_i = L_iA_{t_i}\mathcal{C}_i^{\dagger} + \mathcal{Z}_i(I_{m_i+2p_i} - \mathcal{C}_i\mathcal{C}_i^{\dagger}) \tag{33}$$

where $\mathcal{C}_i^{\dagger}$ is any generalized inverse of matrix $\mathcal{C}_i$ [1] and $\mathcal{Z}_i \in \mathbb{R}^{m_i\times(m_i+2p_i)}$ is an arbitrary matrix. There exists a permutation matrix W such that

$$[M \;\; S \;\; E] = \mathcal{F}W + \mathcal{Z}\mathcal{T}W \tag{34}$$

where

$$\begin{aligned}
\mathcal{Z} &= \text{bdiag}(\mathcal{Z}_1, \dots, \mathcal{Z}_N),\\
\mathcal{F} &= \text{bdiag}(L_1A_{t_1}\mathcal{C}_1^{\dagger}, \dots, L_NA_{t_N}\mathcal{C}_N^{\dagger}),\\
\mathcal{T} &= \text{bdiag}((I_{m_1+2p_1} - \mathcal{C}_1\mathcal{C}_1^{\dagger}), \dots, I_{m_N+2p_N} - \mathcal{C}_N\mathcal{C}_N^{\dagger})).
\end{aligned}$$

Using (34), the matrices M, S and E are given by

$$\begin{aligned}
M &= \mathcal{F}WU_M + \mathcal{Z}\mathcal{T}WU_M\\
&= M_a + \mathcal{Z}M_b, && (35\text{a})\\
S &= \mathcal{F}WU_S + \mathcal{Z}\mathcal{T}WU_S\\
&= S_a + \mathcal{Z}S_b, && (35\text{b})\\
E &= \mathcal{F}WU_E + \mathcal{Z}\mathcal{T}WU_E\\
&= E_a + \mathcal{Z}E_b, && (35\text{c})
\end{aligned}$$

with

$$U_M = \begin{bmatrix} I_m \\ 0_{p\times m} \\ 0_{p\times m} \end{bmatrix}, \;\; U_S = \begin{bmatrix} 0_{m\times p} \\ I_p \\ 0_{p\times p} \end{bmatrix} \;\text{ and }\; U_E = \begin{bmatrix} 0_{m\times p} \\ 0_{p\times p} \\ I_p \end{bmatrix}.$$

By inserting (34) and (35) in equation (26) and SDE (27), we obtain the following SDE

$$\begin{aligned}
\mathrm{d}\, e_r &= (Me_r + \Psi h(t,x) + \Psi B_v v)\,\mathrm{d}\, t + \Psi A_w x\,\mathrm{d}\, w\\
&= ((\mathcal{M}_a + \mathcal{Z}\mathcal{M}_b)e_r + (\mathcal{N}_a + \mathcal{Z}\mathcal{N}_b)h(t,x))\,\mathrm{d}\, t\\
&\quad + (\mathcal{N}_a + \mathcal{Z}\mathcal{N}_b)v\,\mathrm{d}\, t + (\mathcal{N}_a + \mathcal{Z}\mathcal{N}_b)x\,\mathrm{d}\, w
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\mathcal{M}_a &= \mathcal{F}WU_M,\\
\mathcal{M}_b &= \mathcal{T}WU_M,\\
\mathcal{N}_a &= L - \mathcal{F}WU_EC,\\
\mathcal{N}_b &= \mathcal{T}WU_EC.
\end{aligned}$$

Using the above developments, the closed-loop SDE composed by (11) and (13) can be written in the following compact form

$$\mathrm{d}\, X_r = (\mathbf{A}_{t_r}X_r + \mathbf{H}_r h_r(t,X_r) + \mathbf{B}_{v_r}v)\,\mathrm{d}\, t + \mathbf{A}_{w_r}X_r\,\mathrm{d}\, w \tag{37a}$$

$$z = \mathbf{C}_rX_r + Dv \tag{37b}$$

where

$$\begin{aligned}
\mathbf{A}_{t_r} &= \begin{bmatrix} A_t + BL & BL \\ 0 & \mathcal{M}_a + \mathcal{Z}\mathcal{M}_b \end{bmatrix},\\
\mathbf{A}_{w_r} &= \begin{bmatrix} A_w & 0 \\ (\mathcal{N}_a + \mathcal{Z}\mathcal{N}_b)A_w & 0 \end{bmatrix},\\
\mathbf{B}_{v_r} &= \begin{bmatrix} B_v \\ (\mathcal{N}_a + \mathcal{Z}\mathcal{N}_b)B_v \end{bmatrix},\\
\mathbf{C}_r &= [C_z \;\; 0],\\
\mathbf{H}_r &= \begin{bmatrix} I_n \\ \mathcal{N}_a + \mathcal{Z}\mathcal{N}_b \end{bmatrix},\\
h_r(t,X_r) &= h(t,x),\\
X_r &= \begin{bmatrix} x \\ e_r \end{bmatrix}.
\end{aligned}$$

We can state the main theorem.

Theorem 7. Assume that

- **(i)** the rank condition (32) holds for $i = 1, \dots, N$,
- **(ii)** LMI (16) has been satisfied and the gain $L = \text{bdiag}(L_1, \dots, L_N)$, given in Theorem 6, has been calculated.

Problem 5 is solved if there exist two reals $\gamma > 0$, $\mu_2 > 0$ and, for $i = 1, \dots, N$, matrices $Q_{x_i} = Q_{x_i}^T > 0$, $Q_{e_i} = Q_{e_i}^T > 0$, $Q_{x_i} \in \mathbb{R}^{n_i\times n_i}$, $Q_{e_i} \in \mathbb{R}^{m_i\times m_i}$ and $Y_{Z_i} \in \mathbb{R}^{(m_i+2p_i)\times m_i}$ such that the following LMI

$$\begin{bmatrix}
(a) + (a)^T & \overline{H}^T & \mathbf{C}_r^T & (b)^T & (c) & (d) \\
\overline{H} & -\mu_2^{-1}\Phi & 0 & 0 & 0 & 0 \\
\mathbf{C}_r & 0 & -I_k & 0 & 0 & 0 \\
(b) & 0 & 0 & -Q & 0 & 0 \\
(c)^T & 0 & 0 & 0 & -\mu_2 I_n & 0 \\
(d)^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q + D^TD
\end{bmatrix} < 0 \tag{38}$$

is satisfied where

$$\begin{aligned}
(a) &= \begin{bmatrix} Q_x(A_t + BL) & Q_xBL \\ 0 & Q_e\mathcal{M}_a + Y_Z\mathcal{M}_b \end{bmatrix},\\
(b) &= \begin{bmatrix} Q_xA_w & 0 \\ (Q_e\mathcal{N}_a + Y_Z\mathcal{N}_b)A_w & 0 \end{bmatrix},\\
(c) &= \begin{bmatrix} Q_x \\ Q_e\mathcal{N}_a + Y_Z\mathcal{N}_b \end{bmatrix},\\
(d) &= \begin{bmatrix} Q_xB_v + C_z^TD \\ (Q_e\mathcal{N}_a + Y_Z\mathcal{N}_b)B_v \end{bmatrix},\\
\overline{H} &= [H \;\; 0],
\end{aligned}$$

and $Q_x = \text{bdiag}(Q_{x_1}, \dots, Q_{x_n})$, $Q_e = \text{bdiag}(Q_{e_1}, \dots, Q_{e_n})$, $Y_Z = \text{bdiag}(Y_{Z_1}, \dots, Y_{Z_N})$.

The matrices M_i, J_i, J_2 and E_i of the N decentralized functional reduced order observer (13) are given in equations (29), (30) and (35) by using $\mathcal{Z}_i = Q_{e_i}^{-1}Y_{Z_i}$ with $i = 1, \dots, N$.

Proof.

First, we assume that LMI (16) has been satisfied and that the gain $L = \text{bdiag}(L_1, \dots, L_N)$ has been calculated (see Theorem 6).

Let $\mathcal{V}(X_r) = X_r^TQX_r$ be a Lyapunov function candidate where $Q = Q^T = \text{bdiag}(Q_x, Q_e) > 0$, $Q_x =$

[1] A generalized inverse $\mathcal{C}_i^{\dagger}$ is any matrix satisfying $\mathcal{C}_i = \mathcal{C}_i\mathcal{C}_i^{\dagger}\mathcal{C}_i$.

$\mathrm{bdiag}(Q_{x_1},\ldots,Q_{x_n})$, $Q_e = \mathrm{bdiag}(Q_{e_1},\ldots,Q_{e_n})$, $Q_{x_i} \in \mathbb{R}^{n_i\times n_i}$ and $Q_{e_i} \in \mathbb{R}^{m_i\times m_i}$.

Since the rank condition (32) holds for $i = 1,\ldots,N$, the SDE (37) corresponds to the closed-loop composed by (11) and (13).

Notice that item (i) of Problem 5 is satisfied if the SDE is MSES due to the definition of e_r in (25).

Using the similarity of the structure of SDE (15) and (37), we can go back to the proof of Theorem 6 up to equation (23). So, the Problem 5 is solved by the functional reduced order observer (13) if the following inequality

$$\Omega = \begin{bmatrix} \Omega_1 & \overline{H}^T & \mathbf{C}_r^T & \mathbf{A}_{w_r}^T Q & Q\mathbf{H}_r & \Omega_2 \\ \overline{H} & -\mu_2^{-1}\Phi & 0 & 0 & 0 & 0 \\ \mathbf{C}_r & 0 & -I_k & 0 & 0 & 0 \\ Q\mathbf{A}_{w_r} & 0 & 0 & -Q & 0 & 0 \\ \mathbf{H}_r^T Q & 0 & 0 & 0 & -\mu_2 I_n & 0 \\ \Omega_2^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q + D^T D \end{bmatrix} < 0 \tag{39}$$

holds, where $\Omega_1 = \mathbf{A}_{t_r}^T Q + Q\mathbf{A}_{t_r}$ and $\Omega_2 = Q\mathbf{B}_{v_r} + \mathbf{C}_r^T D$. Inequality (39) corresponds to (23) where the following replacements were made

$$\begin{aligned} P &\longrightarrow Q, \\ A_t + BL &\longrightarrow \mathbf{A}_{t_r}, \\ A_w &\longrightarrow \mathbf{A}_{w_r}, \\ B_v &\longrightarrow \mathbf{B}_{v_r}, \\ C_z &\longrightarrow \mathbf{C}_r, \\ H &\longrightarrow \overline{H} = [H \ \ 0], \\ \mu_1 &\longrightarrow \mu_2, \end{aligned}$$

and inequality (20) has been replaced by

$$\begin{aligned} 2X_r^T Q\mathbf{H}_r h_r(t,X_r) &\leqslant \mu_2 h_r^T(t,X_r) h_r(t,X_r) \\ &\quad + \mu_2^{-1} X_r^T Q\mathbf{H}_r \mathbf{H}_r^T Q X_r \\ &\leqslant \mu_1 X_r^T \overline{H}^T \Phi^{-1} \overline{H} X_r \\ &\quad + \mu_2^{-1} X_r^T Q\mathbf{H}_r \mathbf{H}_r^T Q X_r \end{aligned} \tag{40}$$

where $\mu_2 > 0$ is a given real. The theorem is proved since inequality (39) is equivalent to LMI (38). □

Remark 8. If we put matrix $E_i = 0$ in the decentralized functional reduced order observer (13), matrix Ψ in (26) becomes $\Psi = L$ and equation (30) is not useful since $S = J$. In this case, we have $\mathcal{M}_i = [M_i \ \ J_i]$ and $\mathcal{C}_i^T = \begin{bmatrix} L_i^T & C_i^T \end{bmatrix}$. In this case, the rank condition (32) becomes

$$\mathrm{rang}\left(\begin{bmatrix} L_i A_t \\ L_i \\ C_i \end{bmatrix}\right) = \mathrm{rang}\left(\begin{bmatrix} L_i \\ C_i \end{bmatrix}\right). \tag{41}$$

It is easy to see that (41) $\Rightarrow$ (32), but (32) $\not\Rightarrow$ (41). The matrix E_i therefore plays an important role in the existence of observer (13). □

5. APPLICATION OF THEOREM 7 TO THE CASE OF DECENTRALIZED FULL ORDER OBSERVERS

The decentralized functional reduced order observer (13) is replaced by the following decentralized full order observer

$$\mathrm{d}\,\widehat{x}_i = M_i \widehat{x}_i \,\mathrm{d}t + J_i y_i \,\mathrm{d}t + G_i u_i \,\mathrm{d}t, \tag{42}$$

where $\widehat{x}_i \in \mathbb{R}^{n_i}$ is the estimate of the state x_i, and the control law is given by

$$u_i = L\widehat{x}_i. \tag{43}$$

With observer (42), matrix Ψ in (26) becomes $\Psi = I_n$ since there does not exist a functional $u = Lx$ to be estimated due to the fact that $\widehat{x}_i$ is given by the observer (42) and the control law by (43). This has several consequences:

- The Sylvester equation (28) becomes
$$M = A_t - JC, \qquad i = 1,\ldots,N \tag{44}$$
- The rank condition (32) is always satisfied since matrix $\mathcal{C}_i$ is of full column rank. This rank condition can be removed from Theorem 7.
- $S = J$ in (30) and equation (31) becomes
$$A_{t_i} = \mathcal{M}_i \mathcal{C}_i, \qquad i = 1,\ldots,N \tag{45}$$
with $\mathcal{M}_i = [M_i \ \ J_i]$ and $\mathcal{C}_i^T = \begin{bmatrix} I_{n_i} & C_i^T \end{bmatrix}$. So the rank condition (32) is always satisfied since matrix $\mathcal{C}_i$ is of full column rank and can be removed from Theorem 7.
- By choosing $\mathcal{C}_i^\dagger = [I_{n_i} \ \ 0]$, equation (33) becomes
$$\begin{aligned} \mathcal{M}_i = [M_i \ \ J_i] &= A_{t_i}\,[I_{n_i} \ \ 0] + \mathcal{Z}_i \begin{bmatrix} 0 & 0 \\ -C_i & I_{p_i} \end{bmatrix} \\ &= [(A_{t_i} - J_i C_i) \ \ J_i] \end{aligned} \tag{46}$$
and $\mathcal{Z}_i = [\mathcal{Z}_{i_a} \ \ J_i]$.
- The LMI (38) in Theorem 7 is simplified as follows
$$\begin{bmatrix} (a)+(a)^T & \overline{H}^T & \mathbf{C}_r^T & (b)^T & (c) & (d) \\ \overline{H} & -\mu_2^{-1}\Phi & 0 & 0 & 0 & 0 \\ \mathbf{C}_r & 0 & -I_k & 0 & 0 & 0 \\ (b) & 0 & 0 & -Q & 0 & 0 \\ (c)^T & 0 & 0 & 0 & -\mu_2 I_n & 0 \\ (d)^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q + D^T D \end{bmatrix} < 0 \tag{47}$$
where
$$\begin{aligned} (a) &= \begin{bmatrix} Q_x(A_t + BL) & Q_x BL \\ 0 & Q_e A_t + Y_J C \end{bmatrix}, \\ (b) &= \begin{bmatrix} Q_x A_w & 0 \\ Q_e A_w & 0 \end{bmatrix}, \\ (c) &= \begin{bmatrix} Q_x \\ Q_e \end{bmatrix}, \\ (d) &= \begin{bmatrix} Q_x B_v + C_z^T D \\ Q_e B_v \end{bmatrix}, \\ \overline{H} &= [H \ \ 0], \end{aligned}$$
and $Y_J = \mathrm{bdiag}(Y_{J_1},\ldots,Y_{J_N})$. The observer gain is $J_i = Q_{e_i}^{-1} Y_{J_i}$ with $i = 1,\ldots,N$.

6. CONCLUSION

In this contribution, we propose a reduced order decentralized $\mathcal{H}_\infty$ observer based control for a large scale nonlinear interconnected stochastic system. The considered system is affected by multiplicative noises and is composed by N subsystems which are interconnected by nonlinear functions. The design is decoupled into two steps: in a first time, we calculate a state feedback gain and, in a second time, we determine the matrices of the decentralized observer based controller. This decoupling is justified by the fact that the above mentioned state feedback gain is used to solve constraints between observer matrices (see (33)), so it must be calculated firstly.

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