Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

# Controllability of linear delay systems and of their sampled versions

**Hugues Mounier** * **Silviu-Iulian Niculescu** *,**

* *University Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), 3, rue Joliot Curie, F-91192, Gif-sur-Yvette, France (e-mail:{Hugues.Mounier,Silviu.Niculescu}@l2s.centralesupelec.fr)*
** *Inria Saclay DISCO Team*

**Abstract:** This paper focuses on the controllability preservation through sampling of linear time-delay systems. We make use of a module theoretic framework acting as a unifying one for most of the existing delay system controllability notions. The controllability properties are envisioned through ring theoretic properties. Some illustrative examples complete the presentation.

*Keywords:* Delay systems, Controllability, Sampling, Module theoretic setting, $\pi$-freeness.

## 1. INTRODUCTION

Delay systems, e.g. systems modeled by differential-difference equations represent the simplest class of infinite-dimensional processes. They naturally arise in technological systems, where the delay is quite often situated in the input or the output. This infinite dimensional character is merely lost through discretization.

Contrarily to the case of finite-dimensional systems, where the controllability notion is unique, there has been numerous, and seemingly unrelated controllability generalizations for delay systems. More specifically, notions have been elaborated either from a functional analytic approach (*see, e.g.,* Bartosiewicz (1984); Manitius and Triggiani (1978); Yamamoto (1989)) or from a more algebraic viewpoint (*see, e.g.,* with a similar formalism as used here (Bourlès and Marinescu (2011); Fabiańska and Quadrat (2007); Mounier and Rudolph (2003); Rudolph et al. (2003)), with systems over rings (Loiseau (2000); Sontag (1976)), with distributional rings Bourlès and Oberst (2015), Vettori and Zampieri (2002)), with dual trajectory modules over an operator ring (see, for instance, Gluesing-Luerssen (2002); Pillai and Shankar (1999)). There has thus also been many bridging notions between continuous delay systems and sampled data systems (Fridman (2014a,b); Louisell (2001)).

In Fliess and Mounier (1998), an algebraic setting was introduced, based on module theoretic properties, which yielded a unifying framework within which most of theses notions could be compared. Having the previous facts in mind, it is natural to ask *which controllability property can be preserved under sampling.*

The main features of our algebraic framework can be summarized as follows:

- The *intrinsic character* of the various notions and definitions.
- Varying the coefficients ring and the module properties yield a *great flexibility* in defining structural notions.
- A framework enabling the *unification of most of* the existing *structural notions* of the delay system's literature (*see* Fliess and Mounier (1998)).
- The *constructive character* of the structural notions, as demonstrated by the numerous computer algebra implementations (*see, e.g.,* Chyzak et al. (2005); Quadrat and Robertz (2014)).
- The *independence* of the obtained structural results *from* often tricky *existence and uniqueness* prerequisites, as is customary for infinite dimensional systems through functional analytic techniques.
- The *same framework* can be used both *for finite- and infinite-dimensional* linear *systems.*

The sampling we shall use are classical ones, and we will not explicitly study the effect of sampled data control on a continuous system.

The aim of this contribution is to investigate whether controllability notions can be preserved through sampling.

The remaining of the paper is organized as follows: In Section 2, general algebraic definitions are given, and the case of finite dimensional and delay systems are given in Section 3. Discretization and interplay between controllability and sampling are studied in Sections 4 and 5. Finally, some concluding remarks end the paper.

## 2. PREREQUISITES AND $R$–SYSTEMS

Most of the commutative algebra we use may be found in standard textbooks, such as Eisenbud (1995); Lang (71); Rotman (1979).

In particular, the basic definitions of module, ideal, free module, . . . , can be found for example in pages 11 to 17 of Eisenbud (1995). The notion of torsion freeness is in (Lang, 71, p. 388) or (Rotman, 1979, p. 224). For projectivity, *see,* *e.g.,* (Eisenbud, 1995, A3.2 p. 615) or (Rotman, 1979, Th. 3.11 and 3.14, p. 62–63).

### *2.1 Preliminary definitions*

All the rings and algebras we shall consider are commutative, with unity, and without zero divisors.

*Definition 2.1.* Let $R$ be a ring. An *R–linear system*, or an *R–system*, or a (*linear*) *system over* $R$, $\Sigma$ is a finitely generated $R$–module. □

In other words, an $R$–system is an $R$–module with finite free presentation (*see, e.g.,* (Eisenbud, 1995, p.17), (Rotman, 1979, p. 90)):

$$E \xrightarrow{\phi} W \longrightarrow \Sigma \longrightarrow 0.$$

A *presentation matrix* $P_\Sigma$ of $\Sigma$ is one associated with $\phi$.

Notation. We denote by $[\xi]$ the $R$–submodule spanned by a subset $\xi$ of $\Sigma$.

*Definition 2.2.* An *R–linear dynamics*, or an *R–dynamics*, or a (*linear*) *dynamics over* $R$, is an $R$–system $\Sigma$ equipped with an *input* $\boldsymbol{u} = (u_1, \ldots, u_m)$, *i.e.*, a finite subset of $\Sigma$, such that the quotient module $\Sigma/[\boldsymbol{u}]$ is torsion. The input $\boldsymbol{u}$ is said to be *independent* if $[\boldsymbol{u}]$ is a free $R$–module of rank $m$. □

*Definition 2.3.* An *output* $\boldsymbol{y} = (y_1, \ldots, y_p)$ is a finite subset of $\Sigma$. An $R$–system equipped with an input and an output is called an *input–output R–system.* □

### *2.2 General controllability notions*

Varying the rings under consideration through suitable tensor products yields various controllability notions.

*Definition 2.4.* Let $A$ be an $R$–algebra. The $R$–system $\Sigma$ is said to be *A–torsion free controllable* (resp. *A–reflexive controllable, A–projective controllable, A–free controllable*) if the $A$–module $A \otimes_R \Sigma$ is torsion free (resp. projective, free). □

Elementary homological algebra Rotman (1979) yields:

*Proposition 2.1.* The $A$–free controllability implies the $A$–projective controllability, which implies the $A$–reflexive controllability, which implies the $A$–torsion free controllability. □

Localisation (*e.g.* inversion of some elements) preserves module properties in the following sense (*see, e.g.,* (Rotman, 1979, 4.81, p. 198)):

*Proposition 2.2.* If $S \subseteq R$ is multiplicative, then localisation $\Sigma \mapsto S^{-1}\Sigma = S^{-1}R \otimes_R \Sigma$ defines an exact functor $\mathbf{Mod}_R \to \mathbf{Mod}_{S^{-1}R}$. Thus if $\Sigma$ is free (resp. projective), then $S^{-1}\Sigma$ is free (resp. projective). □

The following result is borrowed from (Rowen, 1991, Proposition 2.12.17, p. 233):

*Proposition 2.3.* Let $\Sigma$ be an $R$–system, $A$ an $R$–algebra, and $S$ a multiplicative part of $A$ such that $\Sigma$ is $S^{-1}R$–free controllable. Then, there exists an element $\pi$ in $S$ such that $\Sigma$ is $R[\pi^{-1}]$–free controllable. □

*Definition 2.5.* Let $\Sigma$ be an $R$–system, $A$ an $R$–algebra, and $S$ a multiplicative part of $A$ containing an element $\pi$ such that $\Sigma$ is $R[\pi^{-1}]$–free controllable. Such a system $\Sigma$ will be called *π–free.* □

We have the following direct consequence:

*Corollary 2.1.* Let $\Sigma$ be an $R$–torsion free controllable $R$–system and $S$ a multiplicative part of $R$ such that $S^{-1}R$ is a principal ideal ring. Then, there exists $\pi \in S$ such that $\Sigma$ is $R[\pi^{-1}]$–free controllable and $\Sigma$ is $\pi$–free. □

*Remark 2.1.* Note that the previous definitions allow for an extreme flexibility in the choice of the structural notions to be considered. First, three distinct algebraic notions are underlined (torsion freeness, projectivity and freeness) and last, but not least, the base change, through extension of scalars, yields a vast number of notions. □

Let us end this subsection by another notion, the *spectral controllability*, very useful in the infinite-dimensional case. The following definition (given in Woittennek and Mounier (2010)) extends previous ones (*see, e.g.,* Kirillova and Churakova (1967), Asmykovich and Marchenko (1976), Bartosiewicz (1984), Rocha and Willems (1994)) in our context.

*Definition 2.6.* Let $R$ be any ring that is isomorphic to a subring of the ring $\mathscr{O}$ of entire functions with pointwise defined multiplication. Denote the embedding $R \to \mathscr{O}$ by $\mathscr{L}$. A finitely presented $R$-system with presentation matrix $P$ is said to be *spectrally controllable* if the $\mathscr{O}$-matrix $\hat{P} = \mathscr{L}(P)$ satisfies

$$\exists k \in \mathbf{N} : \forall \sigma \in \mathbf{C} : \mathrm{rk}_{\mathbf{C}} \hat{P}(\sigma) = k$$

The following proposition establishes a first algebraic characterisation for spectral controllability

*Proposition 2.4.* (See Woittennek and Mounier (2010)). Let $R$ be any ring that is isomorphic to a subring of the ring $\mathscr{O}$ of entire functions with pointwise defined multiplication. A finitely presented $R$-system is *spectrally controllable* if, and only if, the module $\Sigma_{\mathscr{O}} = \mathscr{O} \otimes_R \Sigma$ is torsion free.

The next proposition strongly links spectral controllability and $R$-freeness, which yields a fairly constructive character.

*Proposition 2.5.* (See Woittennek and Mounier (2010)). Let $R$ be any Bézout domain that is isomorphic to a subring of $\mathscr{O}$ with the embedding $R \to \mathscr{O}$ denoted by $\mathscr{L}$. Then the notions of spectral controllability and $R$-free controllability are equivalent if and only if $\mathscr{L}$ maps non-units in $R$ to non-units in $\mathscr{O}$.

### *2.3 Decomposition*

The classical decomposition of a module over a PID into its torsion and torsion free parts generalizes to Bézout domains (see, e.g. (Hazewinkel, 1995, p. 381)).

*Proposition 2.6.* Let $R$ be a Bézout domain, and $\Sigma$ an $R$-module. One has the following decomposition:

$$\Sigma = t\Sigma \oplus \Sigma/t\Sigma$$

where $t\Sigma$ is the torsion submodule of $\Sigma$. □

## 3. FINITE DIMENSIONAL AND DELAY SYSTEMS

### *3.1 Finite dimensional systems*

Two common instances of $R$-systems are given when $R$ is the ring of differential operators $\mathbf{R}[\frac{d}{dt}]$, which yields continuous time linear finite-dimensional systems Fliess (1990), and when $R$ is $\mathbf{R}[\delta_h]$, with $\delta_h$ the shift operator of delay $h$, $h$ being the sampling interval, with $\delta_0$ being the identity operator. This yields linear discrete time finite-dimensional systems Fliess (1992).

On such rings (being common examples of principal ideal rings), all the notions of torsion freeness, projectivity and freeness coincide, resulting in the well known Kalman controllability notion. See Fliess (1990); Fliess and Bourlès (1997) for several fruitful consequences of this framework.

### 3.2 Various rings and delay systems

We shall consider the following rings in the sequel:

- $\mathcal{R}_p = \mathbf{R}[\frac{d}{dt}, \delta_L]$, the polynomial ring; a ring on which delay systems are defined. In the incommensurate delay case, it becomes $\mathcal{R}_{\mathbb{p}} = \mathbf{R}[\frac{d}{dt}, \boldsymbol{\delta_L}]$, where $\boldsymbol{\delta_L} = (\delta_{L_1}, \delta_{L_2}, \ldots, \delta_{L_r})$ and the dimension of the $\mathbf{Q}$ vector space $[L_1, \ldots, L_r]_{\mathbf{Q}}$ is $r$.
- $\mathcal{R}_e = \mathbf{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$, the ring of exponential polynomials, where an analytic relationship between the ring indeterminates is emphasized. This ring is a subring of $\mathbf{R}[\![\frac{d}{dt}]\!]$, the power series ring in $\frac{d}{dt}$. It is commutative, since $\frac{d}{dt}e^{-h\frac{d}{dt}} = e^{-h\frac{d}{dt}}\frac{d}{dt}$. In the incommensurate delay case, it becomes $\mathcal{R}_{\mathbb{e}} = \mathbf{R}[\frac{d}{dt}, e^{-\boldsymbol{L}\frac{d}{dt}}]$.
- $\mathcal{R}_a = \mathbf{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}]$, the ring of exponential delays and advances. In the incommensurate delay case, it becomes $\mathcal{R}_{\mathbb{a}} = \mathbf{R}[\frac{d}{dt}, e^{-\boldsymbol{L}\frac{d}{dt}}, e^{\boldsymbol{L}\frac{d}{dt}}]$.
- $\mathcal{R}_d = \mathbf{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathscr{O}$, the distributed delay ring, where $\mathscr{O}$ denotes the ring of entire functions. This ring includes distributed delays. In the incommensurate delay case, it becomes $\mathcal{R}_{\mathbb{d}} = \mathbf{R}(\frac{d}{dt})[e^{-\boldsymbol{L}\frac{d}{dt}}, e^{\boldsymbol{L}\frac{d}{dt}}] \cap \mathscr{O}$.
- $\mathcal{R}_s = \mathbf{R}[\delta_h]$, the discrete delay, a natural ring when considering purely discrete systems. This ring is isomorphic to $\mathcal{R}_s^e = \mathbf{R}[e^{-h\frac{d}{dt}}]$. In the incommensurate delay case, it becomes $\mathcal{R}_{\mathbb{s}} = \mathbf{R}[\boldsymbol{\delta_h}]$. The latter ring will not be used here, since in computer implementations, only commensurate delays are used.

### 3.3 Delay systems

The definition we shall adopt is the following.

*Definition 3.1.* A *pointwise linear delay system* is a finitely generated $\mathcal{R}_{\mathbb{p}}$-module. A *pointwise linear commensurate delay system* is a finitely generated $\mathcal{R}_p$-module. □

*Definition 3.2.* Let $\Lambda$ be a pointwise linear delay system. We shall call the *exponential version* of $\Lambda$, denoted by $\Lambda_e$ the $\mathcal{R}_{\mathbb{e}}$-system obtained by the functor isomorphism $\psi_e^*$ from $\mathbf{Mod}_{\mathcal{R}_{\mathbb{p}}}$ to $\mathbf{Mod}_{\mathcal{R}_{\mathbb{e}}}$ (where $\mathbf{Mod}_R$ is the category of $R$-modules, for a ring $R$), with $\psi_e$ the ring isomorphism:

$$\begin{aligned}\psi_e : \mathcal{R}_{\mathbb{p}} &\longrightarrow \mathcal{R}_{\mathbb{e}}\\ \boldsymbol{\delta_L} &\longmapsto e^{-\boldsymbol{L}\frac{d}{dt}}\end{aligned}$$

□

*Definition 3.3.* Let $\Lambda$ be a pointwise linear delay system. The extension of scalars $\mathcal{R}_{\mathbb{a}} \otimes_{\mathcal{R}_{\mathbb{e}}} \Lambda_e$ is called the *advanced extension*, or $\mathcal{R}_{\mathbb{a}}$*-extension* of $\Lambda$, and is denoted as $\Lambda_a$. The extension of scalars $\mathcal{R}_{\mathbb{d}} \otimes_{\mathcal{R}_{\mathbb{e}}} \Lambda_e$ is called the *distributed extension*, or $\mathcal{R}_{\mathbb{d}}$*-extension* of $\Lambda$, and is denoted as $\Lambda_d$. □

## 4. DISCRETIZATION

We shall define the discretization in the following way.

*Definition 4.1.* Let $\Lambda$ be a delay system, let $h$ be a real number dividing $L$ (*i.e.* such that $L = Nh$ with $N \in \mathbf{N}$). Consider then the ring morphism $\phi_{\mathbb{D}_{p,q,h}}$, where $p, q \in \mathcal{R}_s$

$$\begin{aligned}\phi_{\mathbb{D}_{p,q,h}} : \mathcal{R}_p &\to (q)^{-1}\mathcal{R}_s\\ \frac{d}{dt} &\mapsto \frac{p(\delta_h)}{q(\delta_h)}\end{aligned}$$

this morphism corresponding to an identity approximation, *i.e.*, $p$ and $q$ being such that

$$\lim_{h \mapsto 0} \frac{p(e^{-h\frac{d}{dt}})}{q(e^{-h\frac{d}{dt}})} = \frac{d}{dt}$$

A *discretizer of* $\Lambda$ *with sampling period* $h$ is a functor $\mathbb{D}_{p,q,h}$: $\mathbf{Mod}_{\mathcal{R}_p} \to \mathbf{Mod}_{(q)^{-1}\mathcal{R}_s}$ built in the following way:

$$\widetilde{\Lambda} = \mathbf{R}[\tfrac{d}{dt}, \delta_h, q(\delta_h)^{-1}] \otimes_{\mathcal{R}_p} \Lambda \quad \text{(extn)}$$
$$\mathbb{D}_{p,q,h}\Lambda = \phi^*_{\mathbb{D}_{p,q,h}} \widetilde{\Lambda} \quad \text{(rest)}$$

where (extn) is an extension of scalars, (rest) a restriction of scalars, and $\phi^*_{\mathbb{D}_{p,q,h}}$ is the restriction of scalars functor associated to $\phi_{\mathbb{D}_{p,q,h}}$. □

*Remark 4.1.* Let us remark that

$$\mathbf{R}[\delta_L] = \mathbf{R}[\delta_h^N] \subset \mathbf{R}[\delta_h] = \mathcal{R}_s$$

Let us give three examples of this general definition.

*Examples 4.1.*

- Backward difference discretization. In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is
$$\phi_{\mathbb{D}_{p,q,h}}(\tfrac{d}{dt}) = \frac{1 - \delta_h}{h}$$
*i.e.* $p = (1 - \delta_h)/h$, and $q = 1$. We shall denote $\mathbb{D}_{B,h}$ this discretization.
- Forward difference discretization. In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is
$$\phi_{\mathbb{D}_{p,q,h}}(\tfrac{d}{dt}) = \frac{\delta_h^{-1} - 1}{h} = \frac{1 - \delta_h}{h\delta_h}$$
*i.e.* $p = (1 - \delta_h)$, and $q = h\delta_h$. We shall denote $\mathbb{D}_{F,h}$ this discretization.
- Tustin discretization. In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is
$$\phi_{\mathbb{D}_{p,q,h}}(\tfrac{d}{dt}) = \frac{2(1 - \delta_h)}{h(1 + \delta_h)}$$
*i.e.* $p = 2(1 - \delta_h)$, and $q = h(1 + \delta_h)$. We shall denote $\mathbb{D}_{T,h}$ this discretization. □

## 5. INTERPLAY BETWEEN CONTROLLABILITY AND SAMPLING

### 5.1 Ring and dimension level analyis

On a gross level, one has the following proposition, whose proof follows directly from the results in Section 2.

*Proposition 5.1.* Let $\Lambda$ be a delay system. We have the following:

(1) For $\Lambda$: $\mathcal{R}_{\mathbb{p}}$-free $\Leftrightarrow$ $\mathcal{R}_{\mathbb{p}}$-projective $\rightleftarrows\!\!\!\!/$ $\mathcal{R}_{\mathbb{p}}$-torsion free

(2) For $\Lambda_e$: $\mathcal{R}_e$-free $\Leftrightarrow$ $\mathcal{R}_e$-projective $\not\Leftarrow\!\!\Rightarrow$ $\mathcal{R}_e$-torsion free
(3) For $\Lambda_{\mathcal{R}_d}$: $\mathcal{R}_d$-free $\Leftrightarrow$ $\mathcal{R}_d$-projective $\Leftrightarrow$ $\mathcal{R}_d$-torsion free
(4) For $\mathbb{D}_*\Lambda$: $\mathcal{R}_s$-free $\Leftrightarrow$ $\mathcal{R}_s$-projective $\Leftrightarrow$ $\mathcal{R}_s$-torsion free, where $\mathbb{D}_* = \mathbb{D}_{p,q,h}$, with $p, q \in \mathcal{R}_s$.

*Remark 5.1.* Thus, from these general module properties implications and equivalences, one is tempted to conclude that from a controllability viewpoint, to work with $\Lambda_{\mathcal{R}_d}$ in a continuous setting or to work on $\mathbb{D}_*\Lambda$ in a discrete setting is equivalent. We shall see in the sequel that this is not true. □

### *5.2 A Disturbing situation*

On the ring and dimension level analysis, things appear to be easy to handle. But these levels are quite coarse, and when one looks deeper into a finer, Fitting ideal analysis, the situation appears to be more tricky, as the following examples highlight.

*Example 5.1.* Consider the following example

$$\dot{x}(t) = s_0 x(t) + (1 - hs_0)^N u(t) - u(t - L)$$

whose associated module is $\Lambda$. The associated presentation matrix is

$$\left(\tfrac{d}{dt} - s_0 \quad (1 - hs_0)^N - e^{-L\frac{d}{dt}}\right) = \left(m_1(\tfrac{d}{dt}, e^{-L\frac{d}{dt}}) \quad m_2(\tfrac{d}{dt}, e^{-L\frac{d}{dt}})\right)$$

Hence, the system is spectrally controllable if $s_0 \neq 0$ and not spectrally controllable if $s_0 = 0$. Through backward difference discretization, we get

$$\left(\frac{1 - \delta_h}{h} - s_0 \quad (1 - hs_0)^N - \delta_h^N\right) = (1 - hs_0 - \delta_h)\left(\frac{1}{h} \quad r(\delta_h)\right)$$

Thus $\mathbb{D}_{B,h}\Lambda_e$ is non controllable, whatever the value of $s_0$ is. □

*Example 5.2.* Another example is

$$h\dot{x}(t) = x(t) + u(t - L)$$

whose associated presentation matrix is

$$\left(1 - h\frac{d}{dt} \quad e^{-L\frac{d}{dt}}\right)$$

and thus, the system is spectrally controllable. Its backward difference discretization has the presentation matrix

$$\left(1 - h\,\frac{1 - \delta_h}{h} \quad \delta_h^N\right) = \left(\delta_h \quad \delta_h^N\right)$$

Hence, the discretized system is non controllable. □

*Example 5.3.* Let $p$ and $s_0$ such that $p(s_0) = 0$. Consider the system

$$\dot{x}(t) = s_0 x(t) + p(e^{-L\frac{d}{dt}})(u)(t)$$

The associated presentation matrix is

$$(\frac{d}{dt} - s_0 \quad p(e^{-L\frac{d}{dt}}))$$

and the system is not spectrally controllable. The discretization $\mathbb{D}_{p,q,h}$ has the following presentation matrix

$$\left(\frac{p(\delta_h)}{q(\delta_h)} - s_0 \quad p(\delta_h)^N\right)$$

which has no common factor. Hence, the discretized system is controllable. □

## 6. CONCLUSION

We have exhibited some discretizer notion of for delay systems envisioned through an algebraic module theoretic setting. A ring theoretic and dimension analysis has been conducted, under which strong similarities occur between properties of the continuous system on a distributed delay ring and of the discrete system on the difference operator ring. Several simple examples showed that this analysis is not sufficient and one needs to investigate further through finer tools, such as the Fitting ideal. This is the topic of further research, in progress.

## ACKNOWLEDGMENTS

The authors thank an anonymous reviewer for most valuable and helpful comments that helped the authors to improve the overall quality of the paper.

## REFERENCES

Asmykovich, I. and Marchenko, V.M. (1976). Spectrum control in systems with delay. *Autom. Remote Control*, 37(7), 975–984.

Bartosiewicz, Z. (1984). Approximate controllability of neutral systems with delays in control. *J. Diff. Eq.*, 51, 295–325.

Bourlès, H. and Marinescu, B. (2011). *Linear Time-Varying Systems*. Springer Berlin Heidelberg.

Bourlès, H. and Oberst, U. (2015). Generalized convolution behaviors and topological algebra. *Acta Applicandae Mathematicae*, 141(1), 107–148.

Chyzak, F., Quadrat, A., and Robertz, D. (2005). Effective algorithms for parametrizing linear control systems over Ore algebras. *Applicable Algebra in Engineering, Communications and Computing*, 16(5), 319–376.

Eisenbud, D. (1995). *Commutative Algebra with a view toward Algebraic Geometry*. Number 150 in Graduate Texts in Math. Springer-Verlag, New York.

Fabiańska, A. and Quadrat, A. (2007). Applications of the quillen-suslin theorem to multidimensional systems theory. In H. Park and G. Regensburger (eds.), *Gröbner Bases in Control Theory and Signal Processing*, volume 3 of *Radon Series on Computation and Applied Mathematics*, 23–106. de Gruyter.

Fliess, M. (1990). Some basic structural properties of generalized linear systems. *Systems Control Lett.*, 15, 391–396.

Fliess, M. (1992). Reversible linear and nonlinear discrete-time dynamics. *IEEE Trans. Automat. Control*, AC–37, 1144–1153.

Fliess, M. and Bourlès, H. (1997). Finite poles and zeros of linear systems: an intrinsic approach. *Internat. J. Control*, 68, 897–922.

Fliess, M. and Mounier, H. (1998). Controllability and observability of linear delay systems: an algebraic approach. *ESAIM: COCV (Control, Optimisation and Calculus of Variations)*, 3, 301–314.

Fridman, E. (2014a). Introduction to time-delay and sampled-data systems. In *European Control Conference*, 1428–1433. Strasbourg, France.

Fridman, E. (2014b). *Introduction to Time-Delay Systems – Analysis and Control*. Springer, Cham, Switzerland.

Gluesing-Luerssen, H. (2002). *Linear Delay-Differential Systems with Commensurate Delays: An Alebraic Approach*, volume 1770 of *Lect. Notes Math.* Springer6verlag, Berlin.

Hazewinkel, M. (ed.) (1995). *Encyclopaedia of mathematics*, volume 1. Springer, Dordrecht.

Kirillova, F. and Churakova, S. (1967). On the problem of controllability of linear systems with aftereffect. *Differ. Uravn.*, 3, 436–445.

Lang, S. (71). *Algebra.* Addison Wesley, Reading, MA.

Loiseau, J.J. (2000). Algebraic tools for the control and stabilization of time-delay systems. *Annual Reviews in Control*, 24, 135–149.

Louisell, J. (2001). Delay differential systems with time-varying delay: new directions for stability theory. *Kybernetika*, 37, 239–251.

Manitius, A. and Triggiani, R. (1978). Function space controllability of retarded systems: a derivation from abstract operator conditions. *SIAM J. Control Optim.*, 16, 599–645.

Mounier, H. and Rudolph, J. (2003). Time delay systems. *Encyclopaedia of Life and Support Systems*, 6.43.19.4.

Pillai, H.K. and Shankar, S. (1999). A behavioral approach to control of distributed systems. *SIAM J. Control Optim.*, 37, 388–408.

Quadrat, A. and Robertz, D. (2014). A constructive study of the module structure of rings of partial differential operators. *Acta Applicandae Mathematicae*, 133(1), 187–234.

Rocha, P. and Willems, J.C. (1994). Controllability for delay-differential systems. In *Proc. 33rd IEEE Conf. Decision Contr.*, 2894–2897. Lake Buena Vista, Florida.

Rotman, J. (1979). *An Introduction to Homological Algebra.* Academic Press, Orlando.

Rowen, L.H. (1991). *Ring Theory.* Academic Press, Boston.

Rudolph, J., Winkler, J., and Woittennek, F. (2003). *Flatness based control of distributed parameter systems: Examples and computer exercises from various technological domains.* Berichte aus der Steuerungs- und Regelungstechnik. Shaker Verlag, Aachen.

Sontag, E.D. (1976). Linear systems over commutative rings: a survey. *Ricerche Automatica*, 7, 1–34.

Vettori, P. and Zampieri, S. (2002). Module theoretic approach to controllability of convolutional systems. *Linear Algebra Appl.*, 351/352(2/3), 739–759.

Woittennek, F. and Mounier, H. (2010). Controllability of networks of spatially one-dimensional second order PDE – an algebraic approach. *SIAM J. Contr.*, 48, 3882–3902.

Yamamoto, Y. (1989). Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems. *SIAM J. Control Optim.*, 27, 217–234.