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A Necessary Condition on Chain Reachable Robustness of Dynamical Systems

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Abstract: It is "folklore" that the solution to a set reachability problem for a dynamical system is only noncomputable because of non-robustness reasons. A robustness condition that can be imposed on a dynamical system is the requirement of the chain reachable set to equal the closure of the reachable set. We claim that this condition necessarily imposes strong conditions on the dynamical system. For instance, if the space is connected and compact and we are computing a chain reachable robust single valued function f then f cannot have an unstable fixed point or unstable periodic cycle.

Keywords: Reachability analysis, Robustness, Computability, Discrete-time dynamics, Stability analysis, Minimal sets

1. INTRODUCTION

Many problems in control theory can be solved immediately if one has access to the reachable set of a dynamical system. Unfortunately, it is often difficult to exactly compute the reachable set. However, there are many algorithms that give approximations to the reachable set, for example see Gan et al. (2017); Rungger and Zamani (2018); Fan et al. (2016); Gao and Zufferey (2016); Lal and Prabhakar (2019).

As exact computation of the reachable set is difficult, researchers have investigated this problem through the lens of computability theory, see Collins (2005, 2007); Bournez et al. (2010); Chen et al. (2015); Fijalkow et al. (2019); Kong et al. (2015). In fact the reachable set of a general dynamical system (both in discrete time and continuous time) is noncomputable. This means that we need to find conditions on a dynamical system in order for the reachable set to be computable.

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It is generally believed that the reachable set is not computable due to the dynamical system being "non-physical" or "artificial"; the dynamical system is some mathematical oddity that would never arise in a practical situation. Informally, we may say the dynamical system is somewhat robust if its reachable set is computable. In this work we will examine the implications of a discrete-time dynamical system being chain reachable robust (intuitively, the dynamics are insensitive to infinitesimal perturbations) first examined in Collins (2005); where the authors show that if a dynamical system is chain reachable robust then the reachable set is computable. In fact the authors showed, in their framework of computability, that chain reachable robustness was also a necessary condition on computability of the reachable set. This robustness condition was also used in Bournez et al. (2010) to prove certain continuous-time dynamical systems have computable reachable sets.

Although the computability result in Collins (2005) is sharp, this work (and other work in the literature to the best of our knowledge) provided no practically verifiable sufficient conditions (or any for that matter) for a dynamical system to be chain reachable robust. Our original intention for this paper was to provide at least one non-trivial practically verifiable sufficient condition for a dynamical system to be chain reachable robust. We have failed in this regard. Instead, we provide a necessary condition on chain reachable robustness and assert that this necessary condition is likely too strong of a condition for practical purposes. More specifically, we claim that chain reachable robustness imposes strong conditions on the long-term behavior of the dynamics. Our main result, Theorem 11, states that the long-term behavior of a chain reachable robust system (in a connected compact metric space) is always stable and that the number of "long-term behaviors [1]" is either one or infinity. In the case where $\mathrm{f} : X \to X$ is a continuous function, X is connected compact set, the dynamics are $x_n = \mathrm{f}(x_{n-1})$, and the system is chain reachable robust, then all of the fixed points and periodic cycles of f are stable. In the case where there is a unique fixed point (or periodic cycle), it is globally asymptotically stable.

In Section 2 we briefly introduce necessary background information concerning chain reachable robustness, multifunctions and minimal sets. In Section 3 we develop several technical results about the reachable set (largely under the assumption the system is chain reachable robust) to prove Theorem 11. Due to space limit, the proofs of the preliminary results are omitted and can be find in Fitzsimmons and Liu (2020).

2. PRELIMINARIES

For simplicity we will work in metric spaces, rather than topological spaces like in Collins (2005, 2007). We will consider discrete time dynamical systems with control and without. Let (X, d) be a metric space, U be a set, and $\mathrm{f} : X \times U \to X$ be a function. Let the dynamics be

$$x_{n+1} = \mathrm{f}(x_n, u_n) \tag{1}$$

for some $\{u_n\}_{n\in\mathbb{N}} \subseteq U$. Another way to write the above is to define a multifunction $\mathrm{F} : X \rightrightarrows X$ by $\mathrm{F}[x] = \mathrm{f}(x, U)$ and the dynamics are $x_{n+1} \in \mathrm{F}[x_n]$. If we wish to not use control, then will simply write $x_{n+1} = \mathrm{f}(x_n)$.

Definition 1. Let (X, d) be a metric space, $C \subseteq X$, and $\mathrm{F} : X \rightrightarrows X$ be a multifunction. Define the reachable set

$$\begin{aligned}\mathrm{R}[\mathrm{F}, C] = \{x \in X : \exists\{x_n\}_{n=0}^N, N \geq 0, \text{ s.t.}\\ x_i \in \mathrm{F}[x_{i-1}], 1 \leq i \leq N, x_0 \in C,\\ \text{and } x = x_N\}.\end{aligned}$$

If the multifunction is understood, we may instead write $\mathrm{R}[C]$ to be the reachable set.

We can see that $\mathrm{R}[\mathrm{F}, C] = \bigcup_{n=0}^\infty \mathrm{F}^{\circ n}[C]$, where $\mathrm{F}[C] = \bigcup_{c\in C} \mathrm{F}[c]$, $\mathrm{F}^{\circ 0}[x] = \{x\}$, and $\mathrm{F}^{\circ n}[x] = \mathrm{F}\left[\mathrm{F}^{\circ(n-1)}[x]\right]$.

For $\epsilon > 0$ and a set $A \subseteq X$, we use the notation $A_\epsilon = \mathcal{B}_\epsilon(A) = \bigcup_{a\in A} \mathcal{B}_\epsilon(a)$, where $\mathcal{B}_\epsilon(x) = \{y \in X : \mathrm{d}(x, y) < \epsilon\}$.

Definition 2. Let (X, d) be a metric space, $C \subseteq X$, and $\mathrm{F} : X \rightrightarrows X$ be a multifunction. Let $\epsilon > 0$, we define an ϵ-chain of $[\mathrm{F}, C]$ to be $\{y_n\}_{n=0}^N$, $N \geq 0$, with $y_i \in \mathrm{F}_\epsilon[y_{i-1}] := \mathcal{B}_\epsilon(\mathrm{F}[y_{i-1}])$, $1 \leq i \leq N$, and $y_0 \in C$.

Define the chain reachable set

$$\begin{aligned}\mathrm{CR}[\mathrm{F}, C] = \{x \in X : \forall\epsilon > 0, \exists\{y_n\}_{n=0}^N, \text{ an}\\ \epsilon\text{-chain of } [\mathrm{F}, C], \text{ s.t } x = y_N\}.\end{aligned}$$

If the multifunction is understood, we may instead write $\mathrm{CR}[C]$ to be the chain reachable set. The reachable set $\mathrm{R}[\mathrm{F}, C]$ is said to be chain reachable robust or simply robust if $\overline{\mathrm{R}[\mathrm{F}, C]} = \mathrm{CR}[\mathrm{F}, C]$.

The chain reachable set is closed assuming that f is continuous in both its variables and U is a compact set. In view of (1), an ϵ-chain of $[F, C]$ is also of the form: $\{y_n\}_{n=0}^N$, $N \geq 0$, and

$$\mathrm{d}(y_i, \mathrm{f}(y_{i-1}, u_{i-1})) < \epsilon,$$

for $1 \leq i \leq N$, where $\{u_n\}_{n=0}^{N-1} \subseteq U$, and $y_0 \in C$. Additionally, if we define $\mathrm{F}_\epsilon[x] = \mathcal{B}_\epsilon(\mathrm{F}[x])$, then $\mathrm{CR}[\mathrm{F}, C] = \bigcap_{\epsilon>0} \bigcup_{n=0}^\infty \mathrm{F}_\epsilon^{\circ n}[C]$. In Collins (2007) the authors showed that the chain reachable set is an optimal over-approximation of the reachable set.

The idea of using ϵ-chains or perturbed dynamics to study the true dynamics is widely used in verification and control of dynamical systems, for example see Kong et al. (2015); Liu (2017); Li and Liu (2018b,a).

[1] By this we are referring to minimal sets, see Subsection 2.2.

2.1 Multifunctions

A multifunction from X to Y is a function from X to $2^Y \setminus \emptyset$. If F is a multifunction from X to Y, we write $\mathrm{F} : X \rightrightarrows Y$ and, for all $S \subseteq X$, we define $\mathrm{F}[S] = \bigcup_{s \in S} \mathrm{F}[s]$.

Definition 3. Let X, Y be sets and $\mathrm{F} : X \rightrightarrows Y$. Define, for all $B \subseteq Y$, the upper pre-image of F as

$$\mathrm{F}^+[B] = \{x \in X : \mathrm{F}[x] \subseteq B\}.$$

and the lower pre-image of F as

$$\mathrm{F}^-[B] = \{x \in X : \mathrm{F}[x] \cap B \neq \emptyset\}.$$

Often, the lower pre-image is called the inverse multifunction of F; note that F^- is a multifunction in its own right, while F^+ is not.

Definition 4. Let $(X, \tau), (Y, \rho)$ be topological spaces, and $\mathrm{F} : X \rightrightarrows Y$. We say that F is lower semicontinuous (l.s.c.) if, for all V open in Y, $\mathrm{F}^-[V]$ is open in X. We say that F is upper semicontinuous (u.s.c.) if, for all V open in Y $\mathrm{F}^+[V]$ is open in X.

If F is both lower and upper semicontinuous, then we call F continuous.

We would like to note that, in Collins (2005, 2007), they assume that the multifunctions being computed are closed-valued continuous multifunctions.

Proposition 1. Let $(X, \mathrm{d}), (Y, \rho)$ be metric spaces and $\mathrm{F} : X \rightrightarrows Y$. Then

(1) F is l.s.c. if and only if, for all $S \subseteq X$, we have $\mathrm{F}[\overline{S}] \subseteq \overline{\mathrm{F}[S]}$.
(2) Assume that F is compact-valued. Then F is u.s.c. if and only if, for every compact set $C \subseteq X$ and every $\epsilon > 0$, there is a $\delta > 0$ such that $\mathrm{F}[C_\delta] \subseteq \mathrm{F}_\epsilon[C]$.
(3) Assume that F is compact-valued. Then F is u.s.c. if and only if, for every point $x \in X$ and every $\epsilon > 0$, there is a $\delta > 0$ such that $\mathrm{F}[\mathcal{B}_\delta(x)] \subseteq \mathrm{F}_\epsilon[x]$.
(4) F is u.s.c. if and only if, for every closed set $C \subseteq Y$ we have that $\mathrm{F}^-[C]$ is closed.
(5) If F is l.s.c., then $\mathrm{R}[F, x]$ and $\overline{\mathrm{R}[F, x]}$ are l.s.c. multifunctions of x.

If we have two multifunctions $\mathrm{F} : X \rightrightarrows Y$ and $\mathrm{G} : Y \rightrightarrows Z$, define the composition multifunction $\mathrm{G} \circ \mathrm{F} : X \rightrightarrows Z$ by $\mathrm{G} \circ \mathrm{F}[x] = \mathrm{G}[\mathrm{F}[x]]$. The composition of l.s.c. (u.s.c.) multifunctions is again l.s.c. (u.s.c.). Suppose that P is a property sets can have (i.e. closed, open, convex, finite etc.). We say F is P-valued if, for all $x \in X$, $\mathrm{F}[x]$ has the property P. Instead of saying F is singleton-valued we will say F is single/point-valued. We define $\overline{\mathrm{F}} = \mathrm{clF} : X \rightrightarrows Y$ to be $\overline{\mathrm{F}}[x] = \mathrm{clF}[x] = \overline{\mathrm{F}[x]}$ for all $x \in X$.

Note both the chain reachable sets and reachable sets are multifunctions for a fixed $\mathrm{F} : X \rightrightarrows X$. In this case $\mathrm{R}, \mathrm{CR} : X \rightrightarrows X$, $\mathrm{R}[x] = \bigcup_{n=0}^{\infty} \mathrm{F}^{\circ n}[x]$, and $\mathrm{CR}[x] = \bigcap_{\epsilon > 0} \mathrm{R}[\mathrm{F}_\epsilon, x]$.

2.2 Minimal Sets

Suppose that X is a metric space and $\mathrm{F} : X \rightrightarrows X$ is a multifunction. Then a set $A \subseteq X$ is said to be a minimal set of F, or simply a minimal set, if it is a minimal closed, nonempty, invariant set of F. That is, A is closed, nonempty and satisfies $\mathrm{F}[A] \subseteq A$. Further, for all $B \subseteq A$ that is closed, nonempty and satisfies $\mathrm{F}[B] \subseteq B$, we must have that $B = A$. In a compact space with $\mathrm{F} = \{\mathrm{f}\}$ being single-valued, a minimal set is where all the long-term behavior of the sequence $\{\mathrm{f}^{\circ n}\}_{n \in \mathbb{N}}$ "happens". In this section we state a number of results about minimal sets of a l.s.c. multifunction.

Proposition 2. Let (X, d) be a metric space, $A \subseteq X$ be a set, and $\mathrm{F} : X \rightrightarrows X$ be a l.s.c. multifunction. Then the following are equivalent:

(1) A is a minimal set of F.
(2) $A \neq \emptyset$ and for all $a \in A$ we have $\overline{\mathrm{R}[\mathrm{F}, a]} = A$.

Furthermore, if $\overline{\mathrm{R}[\mathrm{F}, x]}$ is compact for some $x \in X$, then there is a compact minimal set $A \subseteq \overline{\mathrm{R}[\mathrm{F}, x]}$.

In the case that $\mathrm{F} = \{\mathrm{f}\}$ is single-valued, minimal sets are typically fixed points of f (even in the multi-valued case, we have $\overline{\mathrm{F}[A]} = A$ if A is minimal) or limit cycles of f, one of which must be the case if the minimal set is finite.

Example 1. Let X be the unit circle in the complex plane with the usual metric. Every point in X can be uniquely represented in the form $e^{2\pi i x}$, where $x \in [0, 1)$ and $i^2 = -1$. Define the map

$$\mathrm{f}\left(e^{2\pi i x}\right) = e^{2\pi i (x + \theta)}$$

for $x, \theta \in [0, 1)$ and $\mathrm{f} : X \to X$. If $\theta = \frac{p}{q}$ for $p, q \in \mathbb{Z}, q \neq 0$ and p, q are relatively prime, then the minimal sets of f are all of the form $\{z, \mathrm{f}(z), \ldots, \mathrm{f}^{\circ q}(z)\}$, where z could be any point in X. In fact, every point in X belongs to a minimal set. If θ is irrational, then the unique minimal set of f is X (this follows from the relatively well-known fact that the sequence $\{(x + n\theta) \bmod 1\}_{n\in\mathbb{N}}$ is dense on $[0, 1]$ when θ is irrational). This is an example of a minimal set that is not a fixed point or periodic cycle.

Definition 5. Let (X, τ) be a topological space, $A \subseteq X$ be a set, and $\mathrm{F} : X \rightrightarrows X$ be a multifunction. Then A is said to be Lyapunov stable if, for every open set $V \supseteq A$, there is a open set $W \supseteq A$ with $\mathrm{R}[W] \subseteq V$.

A Lyapunov stable minimal set is the place where the long-term behavior of the dynamics from Equation (1) happens, assuming that the dynamics reach the minimal set in the long-term.

Proposition 3. Let (X, d) be a metric space space, U be a set, and $\mathrm{f} : X \times U \to X$ be a function such that, for all $u \in U$, we have that $\mathrm{f}(\cdot, u) = \mathrm{f}_u(\cdot)$ is continuous. Furthermore, let A be a Lyapunov stable compact set and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by Equation (1) with $\overline{\{x_n\}_{n\in\mathbb{N}}}$ compact. Then we have

$$\overline{\{x_n\}_{n\in\mathbb{N}}} \cap A \neq \emptyset \implies \bigcap_{N\in\mathbb{N}} \overline{\{x_n\}_{n=N}^{\infty}} \subseteq A$$

and, in the case where U is singleton (no control) and A is a minimal set of $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ where f is continuous, we have $\overline{\{x_n\}_{n\in\mathbb{N}}} \cap A \neq \emptyset \implies \bigcap_{N\in\mathbb{N}} \overline{\{x_n\}_{n=N}^{\infty}} = A$.

Effectively we know that if the dynamics "touch" a Lyapunov stable set we know the long-term behavior (the limit points of the dynamics) must also be in this Lyapunov stable set. At this point it is natural to ask when a set is Lyapunov stable.

Proposition 4. Let (X, d) be a metric space, $\mathrm{F} : X \rightrightarrows X$ be a multifunction, and $A \subseteq X$ be a compact invariant set. If R or $\overline{\mathrm{R}}$ is u.s.c., then A is Lyapunov stable. In particular, every compact minimal set is Lyapunov stable.

Later, we will use some results about the set

$$\mathcal{W}(A) = \Big\{x \in X : A \subseteq \overline{\mathrm{R}[\mathrm{F}, x]}\Big\}. \qquad (2)$$

Theorem 5. Let (X, d) be a metric space, $\mathrm{F} : X \rightrightarrows X$ be a l.s.c. multifunction, and A be a minimal set of of F. Then the following holds for any local basis $B(a)$, $a \in A$ (a local basis of a point x is a collection of sets with the following property: for any open $V \ni x$, there is a $U \in B(x)$ with $x \in \mathrm{int}(U) \subseteq U \subseteq V$):

(1) $\mathcal{W}(A) = \mathrm{clR}^{-}[A]$.
(2) For any $a \in A$, $\mathcal{W}(A) = \bigcap_{V\in B(a)} \mathrm{R}^{-}[V]$.
(3) $\mathrm{clR}^{-}[\mathcal{W}(A)] = \mathcal{W}(A)$.
(4) $\mathcal{W}(A)$ is open if and only if $\mathcal{W}(A)$ is a neighborhood of some $a \in A$.
(5) If $\overline{\mathrm{R}} = \mathrm{clR}$ is u.s.c. then $\mathcal{W}(A)$ is closed.

3. NECESSARY CONDITIONS ON ROBUSTNESS IN COMPACT SPACES

For the purposes of this section, we will typically be working in a connected compact metric space X and considering a robust multifunction F; we will call F robust if for all $x \in X$ the set $\mathrm{R}[\mathrm{F}, x]$ is robust; that is, $\overline{\mathrm{R}[\mathrm{F}, x]} = \mathrm{CR}[\mathrm{F}, x]$. From a mathematical point of view, this ends up being a strong condition.

Lemma 6. Let (X, d) be a compact metric space and $\mathrm{F} : X \rightrightarrows X$ be a multifunction. Then for all $x \in X$:

$\mathrm{CR}[x] = \overline{\mathrm{R}[x]}$ if and only if, for every $\epsilon > 0$, there is a $\delta > 0$ for which

$$\mathrm{R}[\mathrm{F}_\delta, x] = \bigcup_{n=0}^{\infty} \mathrm{F}_\delta^{\circ n}[x] \subseteq \mathrm{R}_\epsilon[x].$$

It's unclear to us how to interpret of the ϵ-δ condition in the above lemma. Certainly, the condition has implications on safety problems. We say $[\mathrm{F}, x]$ is safe if $\mathrm{R}[\mathrm{F}, x] \subseteq S$ where $S \subseteq X$ is interpreted as a "safe" set. If the ϵ-δ condition is satisfied for $[\mathrm{F}, x]$ and, for some $\epsilon > 0$, we have that $\mathrm{R}_\epsilon[x] \subseteq S$ (i.e., $[\mathrm{F}, x]$ is ϵ-safe), then the δ-perturbed system $[\mathrm{F}_\delta, x]$ is safe, since $\mathrm{R}[\mathrm{F}_\delta, x] \subseteq \mathrm{R}_\epsilon[x] \subseteq S$. That is, ϵ-safety of $[\mathrm{F}, x]$ implies safety of $[\mathrm{F}_\delta, x]$ for some $\delta > 0$. This contrasts to a common use of these δ-perturbed systems: since $\mathrm{R}[\mathrm{F}, x] \subseteq \mathrm{R}[\mathrm{F}_\delta, x]$ for any $\delta > 0$, if $[\mathrm{F}_\delta, x]$ is safe, then $\mathrm{R}[\mathrm{F}, x]$ is safe. In other words, δ-perturbed systems can be used to determine the safety of the real system. In contrast, for chain reachable robust systems, the safety of the δ-perturbed system is guaranteed by the ϵ-safety of the real system.

Lemma 7. Let (X, d) be a metric space and $\mathrm{F} : X \rightrightarrows X$ be a u.s.c. multifunction. Then for all $\epsilon > 0$ and all compact sets $C \subseteq X$, there is $\delta > 0$ for all $n \in \mathbb{N}$ such that

$$\mathrm{F}_\delta^{\circ n}[\mathcal{B}_\delta(C)] \subseteq \mathrm{F}_\epsilon^{\circ n}[C].$$

Note that $F_\epsilon^{\circ n}$ is the n-fold composition of F_ϵ and not the epsilon enlargement of $F^{\circ n}$.

The lemma above allows us to show that the ϵ-chains in the definition of the chain reachable set are allowed to have initial points within ϵ distance of a point in the initial set.

Proposition 8. Let (X, d) be a metric space, $\mathrm{F} : X \to X$ be a u.s.c. multifunction and C a compact set of X. Then

$$\mathrm{CR}[\mathrm{F}, C] = \bigcap_{\epsilon>0} \bigcup_{n=0}^{\infty} \mathrm{F}_\epsilon^{\circ n}[C] = \bigcap_{\epsilon>0} \bigcup_{n=0}^{\infty} \mathrm{F}_\epsilon^{\circ n}[\mathcal{B}_\epsilon(C)].$$

We now can show some necessary conditions on chain reachable robustness.

Theorem 9. Let (X, d) be a compact metric space and $\mathrm{F} : X \rightrightarrows X$ be a robust u.s.c. multifunction. Then the following hold:

(1) $\overline{R}$ is u.s.c.
(2) $\overline{R}$ (and R) is l.s.c. whenever F is l.s.c.

The multifunction $\overline{\mathrm{R}}$ being continuous (in every sense we discuss here) ends up being a rather strong condition. In particular, it implies some strange things about the minimal sets of F. In Subsection 2.2, Propositions 2 and 4 showed that there are minimal sets of F all of which are Lyapunov stable if $\overline{\mathrm{R}}$ is u.s.c. and X is compact. When we consider the simpler case of $\mathrm{F} = \{\mathrm{f}\}$ being single-valued with f continuous and assume all minimal sets are fixed points of f, the fact that *all* the fixed points are Lyapunov stable is already a strong condition. Already, we can tell $\mathrm{f}(x) = x^2$ on $[0, 1]$ is not robust since $\bar{x} = 1$ is not Lyapunov stable. This necessity of Lyapunov stability actually gets stranger. Theorem 5 gives conditions for the set $\mathcal{W}(A)$ to be both open and closed. Already we can tell that $\mathcal{W}(A)$ is closed since $\overline{\mathrm{R}}$ is u.s.c. But under the assumption that the minimal sets are bounded away from each other we can also show that $\mathcal{W}(A)$ is open.

Lemma 10. Let (X, d) be a compact metric space and $\mathrm{F} : X \rightrightarrows X$ be a l.s.c. multifunction with clR being u.s.c. Suppose that A is a minimal set of F for which there is an open set $V \supseteq A$ such that V contains no other minimal sets except A; that is, A is isolated from other minimal sets. Then $\mathcal{W}(A)$ is open.

Theorem 11. Let (X, d) be a compact connected metric space and $\mathrm{F} : X \rightrightarrows X$ is a robust continuous multifunction. Then either:

(1) F possesses a unique minimal set that is Lyapunov stable.
(2) F possesses infinitely many minimal sets (every minimal set is Lyapunov stable). Further, for every minimal set A and every open $V \supseteq A$, there is a minimal set B with $B \subseteq V \setminus A$.

In the first case, if in addition $\mathrm{F} = \{\mathrm{f}\}$ is single-valued, then the unique minimal set is globally attractive; for all $x \in X$ we have that $\bigcap_{N \in \mathbb{N}} \overline{\{\mathrm{f}^{\circ n}(x)\}_{n=N}^{\infty}}$ is the unique minimal set.

Proof. Either there are finitely many minimal sets or there are infinitely many. Suppose that there are finitely many, and that A is one of these minimal sets. We will show that $\mathcal{W}(A)$ is a nonempty, closed and open set, then concluding that $\mathcal{W}(A) = X$ by connectedness. By Theorems 9 & 5, $\mathcal{W}(A)$ is closed, it is nonempty since $\emptyset \neq A \subseteq \mathcal{W}(A)$ and since there are finitely many minimal sets there is an open set of A that contains no other minimal sets. Thus, $\mathcal{W}(A)$ is also open by Lemma 10 and so $X = \mathcal{W}(A)$. Now suppose that B is another minimal set of F. Then $b \in B \subseteq \mathcal{W}(A)$ and by definition of $\mathcal{W}(A)$ we have that $A \subseteq \overline{\mathrm{R}}[b] = B$ (the equality follows from B being minimal and item (2) of Proposition 2). But B, A are minimal, so by definition $A = B$ and A is the unique minimal set in X.

In the case where there is at least one isolated minimal set we may apply the above argument. Hence if there are infinitely many minimal sets there can be no isolated minimal sets. Meaning that, for every minimal set A and open set $V \supseteq A$ there is a minimal set B with $B \subseteq V$ $B \neq A$, since both B, A are minimal if $A \cap B \neq \emptyset$ then $A \cap B$ is a closed nonempty invariant set and $A \cap B \subseteq A$. But A is minimal so $A = A \cap B \subseteq B$ and B is minimal as well, hence, $B = A$ which is a contradiction. Therefore, $A \cap B = \emptyset$ and $B \subseteq V \setminus A$.

If F is single-valued, the result follows from Proposition 3. □

The above theorem gives us a dramatic dichotomy about the number and properties of the minimal sets of a continuous robust multifunction. In our opinion, the case where F =

{f} is single-valued with all of its minimal sets being fixed points is the easiest case to imagine. In this case it may not be immediately clear if any such functions can satisfy item (2), given the stability requirements on the fixed points. The obvious and easy to forget example of such a function is the identity map. With further assumptions, this is in fact the only example.

Corollary 12. Let $X = [a, b] \subseteq \mathbb{R}$, with $a < b$, be equipped with the normal metric. Assume that f is an analytic function whose minimal sets are all fixed points of f. If f is robust, then either f is the identity function on X or f has a unique attracting fixed point on X.

Proof. Suppose that f is robust, by Theorem 11 there are only two cases: either f has a unique attracting minimal set or f has infinitely many minimal sets—none of which are isolated. By assumption, all these minimal sets are fixed points. It follows from the identity theorem that an analytic function on a connected and compact set with an infinite number of fixed points is the identity function. □

4. DISCUSSION AND CONCLUSIONS

Since chain reachable robustness in compact spaces implies that all minimal sets (specifically fixed points and periodic cycles) must be stable, chain reachable robustness is an unusable condition on any dynamics suspected of having unstable behavior; which is a realistic assumption to have when we do not allow for control. Even if we allow control we should would expect that point to point controllability would not hold if all minimal sets are stable (unless the unique minimal set is the space).

That being said, some "real" systems may actually be chain reachable robust and any nontrivial sufficient condition for this would be of interest in order to check for computability of the reachable set. A starting point could be that the functions $\mathrm{f}(x, u)$ are non-expansive functions of x for each $u \in U$, which guarantees the necessary condition $\overline{\mathrm{R}}$ is u.s.c. and so all the minimal sets of F would be stable.

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