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Output-Feedback PDE Control of Traffic Flow on Cascaded Freeway Segments

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Abstract: We develop in this paper a boundary output feedback control law for an underactuated network of traffic flow on two connected roads; one incoming and one outgoing road connected by a junction. The macroscopic traffic dynamics on each road segment are governed by Aw-Rascle-Zhang (ARZ) model, consisting of second-order nonlinear partial differential equations (PDEs) of traffic density and velocity. The control objective is to stabilize the traffic network system on both roads around a chosen reference system. Using a ramp metering located at the outlet of the outgoing road, we actuate the traffic flux leaving this considered domain. Boundary measurements of traffic flux and velocity are taken at the junction connecting the two road segments. A delay-robust full state feedback control law and a boundary observer are designed for this under-actuated network of two systems interconnected through their boundaries. Each system consists of two hetero-directional linear first-order hyperbolic PDEs. The exponential convergence to the reference system is achieved.

Keywords: Traffic network; ARZ traffic model; PDE control; Backstepping.

1. INTRODUCTION

Traffic flow networks based on macroscopic modeling have been intensively investigated over the past decades. The macroscopic modeling of traffic dynamics is used to describe the evolution of aggregated traffic state values including traffic density, velocity and flow rate on road. The traffic flow network based on the first-order Lightill-Whitham-Richards (LWR) model is theoretically proposed by Coclite et al. (2005) and studied for numerical simulation and applications in Treiber and Kesting (2013). The LWR model corresponds to a conservation law of the traffic density. It can predict the formation and propagation of traffic shockwaves on freeway, but fails to describe the stop-and-go phenomenon which can be properly addressed by the second-order Aw-Rascle-Zhang (ARZ) model Aw and Rascle (2000) Zhang (2002). This second-order ARZ traffic model consists of a set of nonlinear hyperbolic PDEs describing the evolution of the traffic density and velocity. The macroscopic modeling of road networks based on the ARZ model has been developed in Garavello and Piccoli (2006), Herty and Rascle (2006).

In this paper, we use the state-of-art ARZ model traffic flow network to describe the traffic dynamics of two interconnected freeway roads. The traffic network modeling proposed in Herty and Rascle (2006) is adopted. The modeling of the junction of two connected roads conserves the mass and the other traffic property as detailed later in the paper. This property is not smooth across the junction in Garavello and Piccoli (2006). In comparison, the solution in Herty and Rascle (2006) is a weak solution (in the sense of the conservative variables of the ARZ model) that guarantees the well-posedness of the closed-loop system for our control design.

Traffic network control strategies are developed and implemented for the traffic management infrastructures such as ramp metering and varying speed limits. Previous contributions, including Gomes and Horowitz (2006), Jin and Amin (2018), Goatin et al. (2016), Zhang and Ioannou (2016), mostly focus on the spatially discretized approximation of LWR model, namely cell transmission model and its derivation. The ramp metering optimal control problem is studied to improve various performance indexes including total travel time, ramp metering queue, throughput maximization, avoidance of capacity drop. Boundary feedback control algorithms are developed for traffic PDE modeling of a single freeway segment in Bastin and Coron (2016), Karafyllis and Papageorgiou (2019), Yu and Krstic (2018), Yu and Krstic (2019), Zhang et al. (2019). These control laws are restricted to control problem of one segment and traffic network PDE control problems have not been studied to authors' best knowledge.

In authors' previous work Yu and Krstic (2018)-Yu and Krstic (2019), backstepping boundary control laws are designed for ramp metering and varying speed limits to suppress the stop-and-go traffic oscillations on one freeway segment either upstream or downstream of the ramp. The problem of controlling the downstream and the upstream traffic simultaneously with a single boundary actuation remained open. More recently, Yu et al. (2019) considers a boundary state feedback control problem for a traffic flow network system in its most fundamental form: one incoming and one outgoing road connected by a junction. The macroscopic traffic dynamics on each road segment are governed by Aw-Rascle-Zhang (ARZ) model. This results into a network of two interconnected PDE systems coupled through their boundaries. Each subsystem corresponds to a 2×2 coupled hyperbolic system. The authors have considered a ramp metering located at the connecting junction actuating the traffic flow rate entering from the on-ramp to the mainline junction. The objective is to simultaneously stabilize the upstream and downstream traffic to steady states. In this paper, we consider the same traffic PDE model as the one given in Yu et al. (2019) but the actuation is now located at the outlet

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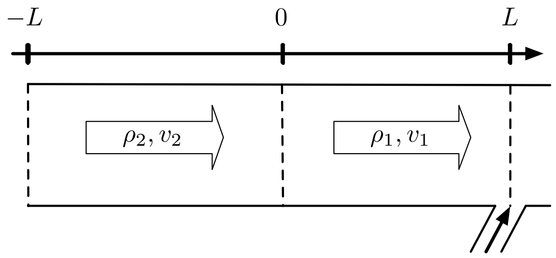

Fig. 1. Traffic flow on an incoming road and an outgoing road connected with a junction, actuation is implemented at the outlet with ramp metering.

of the outgoing road segment. The main contribution of this thread of work is to provide an explicit PDE control design that simultaneously stabilizes the traffic flow on the two connected roads.

The main contribution and novelty of this work, as a counterpart to Yu et al. (2019), lie in the following three aspects: different controlled boundary requires theoretically a new control design; different actuation locations allow the flexibility of implementation of the either of the control algorithms in practice; two control strategies can be evaluated under various performance objectives of the freeway traffic such as fuel consumption, comfort, and total travel time and therefore it is offered a choice when there are trade-offs between these objectives.

Despite the fact that numerous theoretical results in the literature are focused on the boundary control of this class of hyperbolic system based on the backstepping approach Anfinsen and Aamo (2019) Auriol and Meglio (2016) Coron et al. (2013) Deutscher (2017), the control of the network of PDEs remains a challenging research topic. This is due to the fact that in most cases, these systems are underactuated (only the PDE located at one extremity of the network can be actuated). To tackle this problem, multiple approaches have been proposed: PI boundary controllers by Bastin and Coron (2013); Bastin et al. (2015), flatness based design of feedforward control laws by Schmuck et al. (2011), boundary feedback control using weighted Lyapunov function by Herty and Yong (2016) and more recently backstepping-based control laws by Auriol et al. (2019). In this paper, we will mainly adopt the theoretical result of full-state feedback design of Auriol et al. (2019). In addition, we propose boundary observer design to construct the output feedback controller which is a new theoretical contribution that has not been developed before.

The paper is organized as follows. In Section II, we introduce the freeway traffic flow system on two connected roads under consideration. In particular, we introduce the PDE model and boundary conditions describing the dynamics of the traffic density and velocity. These equations are then linearized around a given steady-state. A stabilizing state-feedback control law is obtained in Section III for this underactuated system using backstepping approach. In Section IV we design the boundary observer. Finally, some concluding remarks are given in Section V.

2. PROBLEM STATEMENT

We consider a model for two connected road segments with unidirectional traffic flow. The outgoing road segment corresponds to $x \in [0,L]$ and the incoming road segment to $x \in [-L,0]$. They are connected at the junction through the boundary $x = 0$. The traffic dynamics of each road segment are described with two systems of PDE models and the junction is given as a set of boundary conditions for the PDE model, which allows weak solutions for the traffic network problem Herty and Rascle (2006).

2.1 ARZ PDE model

The evolution of traffic density $\rho_1(x,t)$ and velocity $v_1(x,t)$ for $(x,t) \in [0,L] \times [0,\infty)$ and traffic density $\rho_2(x,t)$ and velocity $v_2(x,t)$ for $(x,t) \in [-L,0] \times [0,\infty)$ is modeled by the following ARZ model (as shown in Fig. 1):

$$\partial_t \rho_i + \partial_x(\rho_i v_i) = 0, \tag{1}$$

$$\partial_t(\rho_i w_i) + \partial_x(\rho_i v_i w_i) = -\frac{\rho_i(v_i - V_i(\rho_i))}{\tau_i}, \tag{2}$$

where the variable w_i is interpreted as "friction" or drivers' aggressiveness of each vehicle that transports in the traffic flow. The traffic flow velocity v_i is related to w_i by

$$v_i = w_i - p_i(\rho_i). \tag{3}$$

where the traffic pressure is defined as an increasing function of the density, $p_i(\rho_i) = (v_m/\rho_{m,i}^{\gamma_i})\rho_i^{\gamma_i}$. The traffic pressure $p_i(\rho_i)$ can be interpreted as the effect that forces drivers to slow down when there is a denser traffic density. The maximum velocity v_m is assumed to be the same for the two road segments while the maximum density $\rho_{m,i}$ and coefficient $\gamma_i \in \mathbb{R}^+$ representing drivers' property are allowed to vary in the different segments, due to different compositions of drivers and vehicles or road attributes.

The equilibrium density-velocity relation $V_i(\rho_i)$ on each road is given in the form of Greenshield's model

$$V_i(\rho_i) = v_m \left(1 - \left(\frac{\rho_i}{\rho_{m,i}}\right)^{\gamma_i}\right). \tag{4}$$

The time scale of $\tau_i \in \mathbb{R}^+$ is assumed to be constant. If we consider an empty road so that $\rho_i = 0$, then $V(\rho_i) = v_m$ and $w_i = v_i$. Therefore, the state variable w_i represents the heterogeneity of traffic flow, namely, the property of each vehicle, with respect to aggregated equilibrium density-velocity relation $V(\rho)$. We denote the traffic flow rate on each road as

$$q_i = \rho_i v_i. \tag{5}$$

The equilibrium flow rate and density relation, also known as the fundamental diagram, is then given by

$$Q_i(\rho_i) = \rho_i V(\rho_i) = \rho_i v_m \left(1 - \left(\frac{\rho_i}{\rho_{m,i}}\right)^{\gamma_i}\right). \tag{6}$$

For the fundamental diagram in (6), the critical density $\rho_{c,i}$ is given by $\rho_{c,i} = \frac{\rho_{m,i}}{(1+\gamma_i)^{1/\gamma_i}}$ such that $Q_i'(\rho_i)|_{\rho_i=\rho_{c,i}} = 0$. The critical density segregates the free regime and congested regime of equilibrium traffic states. The traffic flow is said to be in free regime when the density satisfies $\rho_i < \rho_{c,i}$. The traffic flow is said to be congested when the density satisfies $\rho_i > \rho_{c,i}$. The traffic flux reaches its maximum value at the critical density $q_{c,i} = Q(\rho_{c,i})$ which is also referred as the road capacity. In this work, we focus on the congested traffic for both segments. Notice that we allow different road capacities $q_{c,i}$ for the two segments. We adopt in this paper the traffic PDE network model given in Herty and Rascle (2006). This model allows weak solutions of the network system. Regarding the boundary conditions at the junctions, the Rankine-Hugoniot condition is satisfied. This implies the conservation of the mass and drivers' property. We assume the continuity of the flux and drivers' property across the boundary condition at $x = 0$,

$$\rho_1(0,t)v_1(0,t) = \rho_2(0,t)v_2(0,t), \tag{7}$$

$$w_2(0,t) = w_1(0,t). \tag{8}$$

We assume for the open-loop system, a constant incoming flow rate $q^\star$ for the inlet boundary $x = -L$ and the same constant outgoing flow rate $q^\star$ at the outlet boundary for $x = L$ in the open-loop system. The outgoing flow rate is actuated by a control input $U(t)$ in the closed-loop system,

$$q_2(-L,t) = q^\star, \tag{9}$$
$$q_1(L,t) = q^\star + U(t). \tag{10}$$

The control objective is to stabilize the traffic flow in both the incoming and outgoing road segments with the single control of the flow rate at the outlet, as shown in Fig. 2.

2.2 Steady states $(\rho_1^\star, v_1^\star, \rho_2^\star, v_2^\star)$

In this paper, we want to stabilize the traffic flow in the two segments around the corresponding steady states. These arbitrary steady states $(\rho_1^\star, v_1^\star, \rho_2^\star, v_2^\star)$ are chosen such that the boundary conditions (7) and (8) are satisfied, i.e.

$$\rho_1^\star v_1^\star = \rho_2^\star v_2^\star = q^\star, \tag{11}$$
$$w_1^\star = w_2^\star = v_m, \tag{12}$$

where the steady state velocities satisfy the equilibrium density-velocity relation $v_i^\star = V_i(\rho_i)$. The constant flux $q^\star$ in (11) satisfies ($q^\star = Q_1(\rho_1^\star) = Q_2(\rho_2^\star)$) according to (6). The steady state densities $\rho_1^\star$ and $\rho_2^\star$ are chosen such that this relation is satisfied. The constant driver's property in (12) implies that we have the same maximum velocity v_m for the two segments (which corresponds to our initial assumption).

2.3 Linearized model in the Riemann coordinates

The linearized model is given in the following Riemann variables defined as

$$\tilde{w}_i = w_i - w_i^\star, \tag{13}$$
$$\tilde{v}_i = v_i - v_i^\star. \tag{14}$$

where $w_i^\star$ and $v_i^\star$ are the steady states value defined above. We then apply a spatial transformation $\bar{w}_i(x,t) = \exp\left(\frac{x}{\tau_i v_i^\star}\right)\tilde{w}_i(x,t)$ to simplify the linearized model $(\tilde{w}_i, \tilde{v}_i)$. One can check that with such a change of variable, the system (1),(2), (7),(8),(9), (10) rewrite (for $i \in \{1,2\}$) as

$$\partial_t \bar{w}_i + v_i^\star \partial_x \bar{w}_i = 0, \tag{15}$$
$$\partial_t \tilde{v}_i - (\gamma_i p_i^\star - v_i^\star)\partial_x \tilde{v}_i = c_i(x)\bar{w}_i, \tag{16}$$
$$\tilde{v}_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\bar{w}_1(L,t) \tag{17}$$
$$+ \frac{v_1^\star(1-r_1)}{q^\star}U(t), \tag{18}$$
$$\bar{w}_1(0,t) = \bar{w}_2(0,t), \tag{19}$$
$$\bar{w}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\tilde{v}_2(-L,t), \tag{20}$$
$$\tilde{v}_2(0,t) = \delta\frac{r_2}{r_1}\tilde{v}_1(0,t) + r_2(1-\delta)\bar{w}_2(0,t), \tag{21}$$

where the spatially varying coefficients $c_i(x)$ are defined as $c_i(x) = -\frac{1}{\tau_i}\exp\left(-\frac{x}{\tau_i v_i^\star}\right)$. The constant coefficient δ is defined by $\delta = \frac{\gamma_2 p_2^\star}{\gamma_1 p_1^\star} > 0$. It represents the ratio between the drivers' aggressiveness and the traffic pressure of the two segments. The constant coefficients r_i are defined as $r_i = -\frac{v_i^\star}{\gamma_i p_i^\star - v_i^\star}$ and

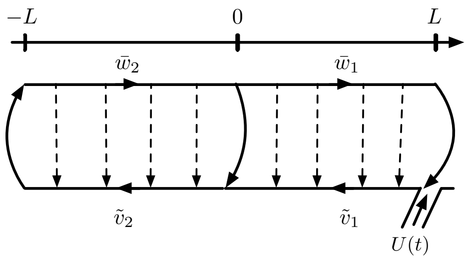

Fig. 2. Control diagram for the closed-loop system.

$p_i^\star = p_i(\rho_i^\star)$. For the congested regime we have $\rho_i^\star > \frac{\rho_{m,i}}{(1+\gamma_i)^{1/\gamma_i}}$ so that the characteristic speed $\gamma_i p_i^\star - v_i^\star > 0$. Thus the following inequalities are satisfied,

$$-1 < r_i < 0. \tag{22}$$

The more traffic is congested, the smaller the absolute value of the ratio constant r_i. Detailed calculations regarding the linearization and spatial transformation can be obtained following Yu and Krstic (2019). The control diagram for the closed-loop system (15)-(21) is given in Fig. 2.

The corresponding initial condition are denoted by $(\tilde{v}_0)_i = \tilde{v}_i(\cdot,0)$ and $(\bar{w}_0)_i = \bar{w}_i(\cdot,0)$. The objective of this paper is to design a control law U that stabilizes the system (15)-(21) around its equilibrium in the sense of the L^2-norm. Such an interconnected system has already been considered for full-state feedback design in Auriol et al. (2019) for a a general framework in the case of an actuator located at one of the extremity of the network. It has been proved in Logemann et al. (1996) that a system can be delay-robustly stabilized only if its open-loop transfer function has a finite number of zeros on the complex right half plane. For the considered class of linear hyperbolic system, it has been proved in Auriol and Meglio (2019) that such a condition is equivalent to requiring (15)-(21) with zero in-domain couplings (i.e. $c_1 \equiv c_2 \equiv 0$) to be exponentially stable in the open-loop. This requirement can be expressed in terms of the following assumption (see Auriol and Meglio (2019), Yu et al. (2019) for details)

Assumption 1. The boundary couplings of the system (15)-(21) are such that

$$|1-\delta|\exp\left(\frac{-L}{\tau_2 v_2^\star}\right) + \delta\exp\left(\frac{-L}{\tau_1 v_1^\star}\right)\exp\left(\frac{-L}{\tau_2 v_2^\star}\right) < 1. \tag{23}$$

Indeed, if $c_i \equiv 0$, it is straightforward (using the method of characteristics) to express $\bar{w}_2(t,0)$ as a solution of a neutral system whose stability is only guaranteed if Assumption 1 holds. Due to the transport structure of (15)-(21), the convergence to zero of $\bar{w}_2(t,0)$ implies the stabilization of the system. Then, (15)-(21) with zero in-domain couplings is exponentially stable in open-loop and the system can be robustly stabilized. Here, we consider that $\delta < 1$ which means the drivers are more aggressive and traffic pressure is greater in segment 2 than that of segment 1. Thus, Assumption 1 is satisfied. If we consider the traffic conditions in the two segments to be the same, then $\delta = 1$ and (23) becomes $\exp\left(\frac{-L}{\tau_2 v_2^\star} + \frac{-L}{\tau_1 v_1^\star}\right) < 1$. The Assumption 1 easily holds and thus the system can be delay-robustly stabilized.

3. STATE FEEDBACK CONTROL DESIGN

In this section, we consider a full state feedback law that stabilizes the system (15)-(21) in the sense of the L^2-norm. Our approach is directly adjusted from Auriol et al. (2019), which explains why we will only present the main ideas of the design. Inspired by Coron et al. (2013), we first consider the two invertible backstepping transformations

$$\bar{\alpha}_1(t,x) = \bar{w}_1(t,x), \tag{24}$$

$$\bar{\beta}_1(t,x) = \tilde{v}_1(t,x) - \int_0^x \bar{K}_1^{vw}(x,\xi)\bar{w}_1(t,\xi)d\xi - \int_0^x \bar{K}_1^{vv}(x,\xi)\tilde{v}_1(t,\xi)d\xi, \tag{25}$$

$$\bar{\alpha}_2(t,x) = \bar{w}_2(t,x), \tag{26}$$

$$\bar{\beta}_2(t,x) = \tilde{v}_2(t,x) - \int_{-L}^x \bar{K}_2^{vw}(x,\xi)\bar{w}_2(t,\xi)d\xi - \int_{-L}^x \bar{K}_2^{vv}(x,\xi)\tilde{v}_2(t,\xi)d\xi, \tag{27}$$

where the kernels $\bar{K}_1^{vw}$ and $\bar{K}_1^{vv}$ are defined on the set $\mathscr{T}_1 = \{(x,\xi) \in [0,L]^2,\ \xi \le x\}$, the kernels $\bar{K}_2^{vw}$ and $\bar{K}_2^{vv}$ are defined on the set $\mathscr{T}_2 = \{(x,\xi) \in [-L,0]^2,\ \xi \le x\}$. They satisfy the following set of PDEs on their corresponding domains of definition

$$(\gamma_i p_i^\star - v_i^\star)\partial_x \bar{K}_i^{vw} - v_i^\star \partial_\xi \bar{K}_i^{vw} = c_i(\xi)\bar{K}_i^{vv}, \tag{28}$$

$$\partial_x \bar{K}_i^{vv} + \partial_\xi \bar{K}_i^{vv} = 0, \tag{29}$$

along with the boundary conditions

$$\bar{K}_i^{vw}(x,x) = -\frac{c_i(x)}{\gamma_i p_i^\star},\ \bar{K}_1^{vv}(x,0) = 0, \tag{30}$$

$$\bar{K}_2^{vv}(x,-L) = \frac{v_2^\star}{\gamma_2 p_2^\star - v_2^\star}\exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\bar{K}_2^{vw}(x,-L). \tag{31}$$

We have the following lemma.

Lemma 1. Coron et al. (2013) For system (28)-(31), there exists a unique solution $\bar{K}_1^{vw}$, $\bar{K}_1^{vv}$ in $L^\infty(\mathscr{T}_1)$ $\bar{K}_2^{vw}$, $\bar{K}_2^{vv}$ in $L^\infty(\mathscr{T}_2)$. Moreover, the transformations (27)-(27) are invertible, that is there exists $\bar{L}_1^{\beta\alpha}$, $\bar{L}_1^{\beta\beta}$ in $L^\infty(\mathscr{T}_1)$ and $\bar{L}_2^{\beta\alpha}$ and $\bar{L}_2^{\beta\beta}$ in $L^\infty(\mathscr{T}_2)$ such that

$$\tilde{v}_1(t,x) = \bar{\beta}_1(t,x) - \int_0^x \bar{L}_1^{\beta\alpha}(x,\xi)\bar{\alpha}_1(t,\xi)d\xi - \int_0^x \bar{L}_1^{\beta\beta}(x,\xi)\bar{\beta}_1(t,\xi)d\xi \tag{32}$$

$$\tilde{v}_2(t,x) = \bar{\beta}_2(t,x) - \int_{-L}^x \bar{L}_2^{\beta\alpha}(x,\xi)\bar{\alpha}_2(t,\xi)d\xi - \int_{-L}^x \bar{L}_2^{\beta\beta}(x,\xi)\bar{\beta}_2(t,\xi)d\xi \tag{33}$$

The transformation (25)-(27) maps the original system (15)-(21) to the decoupled target system

$$\partial_t \bar{\alpha}_1 + v_1^\star \partial_x \bar{\alpha}_1 = 0, \tag{34}$$

$$\partial_t \bar{\beta}_1 - (\gamma_1 p_1^\star - v_1^\star)\partial_x \bar{\beta}_1 = -v_1^\star \bar{K}_1^{vw}(x,0)\alpha_2(t,0), \tag{35}$$

$$\partial_t \bar{\alpha}_2 + v_2^\star \partial_x \bar{\alpha}_2 = 0, \tag{36}$$

$$\partial_t \bar{\beta}_2 - (\gamma_2 p_2^\star - v_2^\star)\partial_x \bar{\beta}_2 = 0, \tag{37}$$

with boundary conditions

$$\bar{\beta}_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\bar{\alpha}_1(L,t) + \frac{v_1^\star(1-r_1)}{q^\star}\bar{U}(t) \tag{38}$$

$$\bar{\alpha}_1(0,t) = \bar{\alpha}_2(0,t), \tag{39}$$

$$\bar{\alpha}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\bar{\beta}_2(-L,t), \tag{40}$$

$$\bar{\beta}_2(0,t) = \delta\frac{r_2}{r_1}\bar{\beta}_1(0,t) + r_2(1-\delta)\bar{\alpha}_2(0,t) + \int_{-L}^0 (\bar{L}_2^{\beta\alpha}(0,\xi)\bar{\alpha}_2(t,\xi) + \bar{L}_2^{\beta\beta}(0,\xi)\bar{\beta}_2(t,\xi))d\xi. \tag{41}$$

In what follows, $U(t)$ is given related to $\bar{U}(t)$ in (38) by

$$U(t) = \bar{U}(t) + \frac{q^\star}{v_1^\star(1-r_1)}\int_0^L \bar{L}_1^{\beta\alpha}(L,\xi)\bar{\alpha}_1(t,\xi) + \bar{L}_1^{\beta\beta}(L,\xi)\bar{\beta}_1(t,\xi)d\xi. \tag{42}$$

Let us now consider the following affine transformation

$$\bar{\beta}_1(t,x) = \eta_1(t,x) - \int_0^x G(x,\xi)\eta_1(t,\xi)d\xi - \int_{-L}^0 M^\alpha(x,\xi)\bar{\alpha}_2(t,\xi) + M^\beta(x,\xi)\bar{\beta}_2(t,\xi)d\xi \tag{43}$$

$$\bar{\beta}_2(t,x) = \eta_2(t,x), \tag{44}$$

where the kernels $M^\alpha(x,\xi)$ and $M^\beta(x,\xi)$ are defined on the domain $\mathscr{T} = \{(x,\xi) \in [0,L]\times[-L,0]\}$ while $G(x,\xi)$ is defined on $\mathscr{T}_1$. They satisfy the following set of PDEs

$$(\gamma_1 p_1^\star - v_1^\star)\partial_x M^\alpha - v_2^\star \partial_\xi M^\alpha = 0, \tag{45}$$

$$(\gamma_1 p_1^\star - v_1^\star)\partial_x M^\beta + (\gamma_2 p_2^\star - v_2^\star)\partial_\xi M^\beta = 0, \tag{46}$$

$$\partial_x G + \partial_\xi G = 0, \tag{47}$$

with the boundary conditions

$$M^\alpha(0,\xi) = \frac{r_1}{\delta r_2}\bar{L}_2^{\beta\alpha}(0,\xi),\ M^\beta(0,\xi) = \frac{r_1}{\delta r_2}\bar{L}_2^{\beta\beta}(0,\xi), \tag{48}$$

$$M^\alpha(x,0) = (1-\delta)M^\beta(x,0) - \frac{v_1^\star}{v_2^\star}\bar{K}_1^{vw}(x,0)) \tag{49}$$

$$M^\beta(x,-L) = -\exp\left(\frac{-L}{\tau_2 v_2^\star}\right)M^\alpha(x,-L) \tag{50}$$

$$G(x,0) = \delta\frac{r_2}{r_1}\frac{\gamma_2 p_2^\star - v_2^\star}{\gamma_1 p_1^\star - v_1^\star}M^\beta(x,0). \tag{51}$$

We have the following lemma.

Lemma 2. Auriol et al. (2019) Consider system (45)-(51), there exists a unique solution M^α, M^β in $L^\infty(\mathscr{T})$, G in $L^\infty(\mathscr{T}_1)$.

The transformation (43) maps (34)-(41) to the system

$$\partial_t \bar{\alpha}_i + v_i^\star \partial_x \bar{\alpha}_i = 0, \tag{52}$$

$$\partial_t \bar{\eta}_i - (\gamma_i p_i^\star - v_i^\star)\partial_x \bar{\eta}_i = 0, \tag{53}$$

with the boundary condition

$$\eta_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\bar{\alpha}_1(L,t) + \frac{v_1^\star(1-r_1)}{q^\star}\bar{U}(t) + \int_{-L}^0 M^\alpha(L,\xi)\bar{\alpha}_2(t,\xi) + M^\beta(L,\xi)\eta_2(t,\xi)d\xi + \int_0^L G(L,\xi)\eta_1(t,\xi)d\xi, \tag{54}$$

$$\bar{\alpha}_1(0,t) = \bar{\alpha}_2(0,t), \tag{55}$$

$$\bar{\alpha}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\eta_2(-L,t), \tag{56}$$

$$\eta_2(0,t) = \delta\frac{r_2}{r_1}\eta_1(0,t) + r_2(1-\delta)\bar{\alpha}_2(0,t). \tag{57}$$

Let us now define $\bar{U}(t)$ in (54) as

$$\bar{U}(t) = \frac{q^\star}{v_1^\star(1-r_1)}\left(\int_0^L G(L,\xi)\eta_1(\xi,t)d\xi + \int_{-L}^0 M^\alpha(L,\xi)\bar{\alpha}_2(\xi,t) + M^\beta(L,\xi)\eta_2(\xi,t)d\xi\right), \tag{58}$$

Therefore the full state feedback control law $U(t)$ is obtained by substituting (58) into (42). With this control law the target system (52)-(57) is exponentially stable due to Assumption 1 (see Auriol and Meglio (2019) for detials). Due to the invertibility of the backstepping transformations (25)-(27) and (43), it can easily be proved that there exist L^∞ functions R_2^w, R_2^v, R_1^w and R_1^v such that the resulting stabilizing control law $U(t)$ can be rewritten

$$U(t) = \int_{-L}^{0} R_2^w(\xi)\bar{w}_2(\xi,t) + R_2^v(\xi)\tilde{v}_2(\xi,t)d\xi + \int_0^L R_1^w(\xi)\bar{w}_1(\xi,t) + R_1^v(\xi)\tilde{v}_1(\xi,t)d\xi. \quad (59)$$

As this control law is only composed of integral terms, it is strictly proper. It can be shown following the ideas of Auriol et al. (2019) that it is robust to delays in the actuation.

4. OBSERVER DESIGN

In this section we design an observer that relies on the measurement of $\tilde{q}_i$ and $\tilde{v}_i$ at the left side of the junction. Since we have $\bar{w}_2(0,t) = \frac{\gamma_2 p_2^\star}{q^\star}\tilde{q}_2(0,t) - \frac{1}{r_2}\tilde{v}_2(0,t)$, the measurement $Y_0(t) = \bar{w}_2(0,t)$ is obtained.

4.1 Observer equations

Inspired by R. Vazquez and Coron (2011), the observer equations read as follows

$$\partial_t \hat{w}_i + v_i^\star \partial_x \hat{w}_i = \mu_i(x)(\bar{w}_2(0,t) - \hat{w}_i(0,t)), \quad (60)$$

$$\partial_t \hat{v}_i - (\gamma_i p_i^\star - v_i^\star)\partial_x \hat{v}_i = c_i(x)\hat{w}_i + \nu_i(x)(\bar{w}_2(0) - \hat{w}_i(0)), \quad (61)$$

with boundary conditions

$$\hat{v}_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\hat{w}_1(L,t) + \frac{v_1^\star(1-r_1)}{q^\star}U(t), \quad (62)$$

$$\hat{w}_1(0,t) = \hat{w}_2(0,t), \quad (63)$$

$$\hat{w}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\hat{v}_2(-L,t), \quad (64)$$

$$\hat{v}_2(0,t) = \delta\frac{r_2}{r_1}\hat{v}_1(0,t) + (1-\delta)r_2\hat{w}_2(0,t), \quad (65)$$

where $\hat{w}_i(x,t), \hat{v}_i(x,t)$ are the estimates of the state variables $\bar{w}_i(x,t)$ and $\tilde{v}_i(x,t)$. The terms μ_i and ν_i are output injection gains that have to be designed. They are L^∞ functions respectively defined on $([0,L])^2$ and $([-L,0])^2$. The corresponding initial conditions are L^2 functions. Defining the error estimates $\check{w}_i = \hat{w}_i - \bar{w}_i$ and $\check{v}_i = \hat{v}_i - \tilde{v}_i$ and using the fact that $\hat{w}_1(0,t) = \hat{w}_2(0,t)$, the error system is obtained by subtracting the observer equations in (60)-(65) from (15)-(21),

$$\partial_t \check{w}_i + v_i^\star \partial_x \check{w}_i = \mu_i(x)\check{w}_i(0,t), \quad (66)$$

$$\partial_t \check{v}_i - (\gamma_i p_i^\star - v_i^\star)\partial_x \check{v}_i = c_i(x)\check{w}_i + \nu_i(x)\check{w}_i(0,t), \quad (67)$$

$$\check{v}_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\check{w}_1(L,t), \quad (68)$$

$$\check{w}_1(0,t) = \check{w}_2(0,t), \quad (69)$$

$$\check{w}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\check{v}_2(-L,t), \quad (70)$$

$$\check{v}_2(0,t) = \delta\frac{r_2}{r_1}\check{v}_1(0,t) + (1-\delta)r_2\check{w}_2(0,t), \quad (71)$$

4.2 Backstepping transformations

Let us consider the following backstepping transformations

$$\check{w}_1(t,x) = \check{\alpha}_1(t,x) - \int_0^x N_1^{ww}(x,\xi)\check{\alpha}_1(t,\xi)d\xi, \quad (72)$$

$$\check{v}_1(t,x) = \check{\beta}_1(t,x) - \int_0^x N_1^{vw}(x,\xi)\check{\alpha}_1(t,\xi)d\xi, \quad (73)$$

$$\check{w}_2(t,x) = \check{\alpha}_2(t,x) - \int_x^0 N_2^{ww}(x,\xi)\check{\alpha}_2(t,\xi)d\xi, \quad (74)$$

$$\check{v}_2(t,x) = \check{\beta}_2(t,x) - \int_x^0 N_2^{vw}(x,\xi)\check{\alpha}_2(t,\xi)d\xi, \quad (75)$$

where the kernels N_1^{ww} and N_1^{wv} are L^∞ functions defined on the set $\mathscr{T}_1$, while the kernels N_2^{ww} and N_2^{wv} are L^∞ functions defined on the set $\mathscr{T}_3 = \{(x,\xi), \in [-L,0]^2,\ \xi \geq x\}$. On their corresponding domains of definition, they satisfy the following set of PDEs:

$$\partial_x N_i^{ww} + \partial_\xi N_i^{ww} = 0, \quad (76)$$

$$(\gamma_i p_i^\star - v_i^\star)\partial_x N_i^{vw} - v_i^\star \partial_\xi N_1^{vw} = -c_i(x)N_i^{ww}, \quad (77)$$

along with the boundary conditions

$$N_1^{vw}(x,x) = \frac{c_1(x)}{\gamma_1 p_1^\star}, \quad N_2^{vw}(x,x) = \frac{-c_2(x)}{\gamma_2 p_2^\star}, \quad (78)$$

$$N_1^{ww}(L,x) = \frac{1}{r_1}\exp\left(\frac{L}{\tau_1 v_1^\star}\right)N_1^{vw}(L,x), \quad (79)$$

$$N_2^{ww}(-L,x) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}N_2^{vw}(-L,x). \quad (80)$$

The well-posedness of this kernel PDE-system is guaranteed by the following lemma.

Lemma 3. R. Vazquez and Coron (2011) Consider system (76)-(80). There exists a unique solution N_1^{vw}, N_1^{vv} in $L^\infty(\mathscr{T}_1)$ and N_2^{vw}, N_2^{vv} in $L^\infty(\mathscr{T}_3)$.

Let us now define the output injection gains μ_i and ν_i as

$$\nu_1(x) = -v_1^\star N_1^{vw}(x,0), \quad \mu_1(x) = -v_1^\star N_1^{ww}(x,0), \quad (81)$$

$$\nu_2(x) = v_2^\star N_2^{vw}(x,0), \quad \mu_2(x) = v_2^\star N_2^{ww}(x,0). \quad (82)$$

With this choice of injection gains, differentiating the transformations (72)-(73) and (74)-(75) with respect to time and space, the error system (66)-(71) is mapped to the following system

$$\partial_t \check{\alpha}_i + v_i^\star \partial_x \check{\alpha}_i = 0, \quad (83)$$

$$\partial_t \check{\beta}_i - (\gamma_i p_i^\star - v_i^\star)\partial_x \check{\beta}_i = 0, \quad (84)$$

$$\check{\beta}_1(L,t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^\star}\right)\check{\alpha}_1(L,t), \quad (85)$$

$$\check{\alpha}_1(0,t) = \check{\alpha}_2(0,t), \quad (86)$$

$$\check{\alpha}_2(-L,t) = \exp\left(\frac{-L}{\tau_2 v_2^\star}\right)\frac{1}{r_2}\check{\beta}_2(-L,t), \quad (87)$$

$$\check{\beta}_2(0,t) = \delta\frac{r_2}{r_1}\check{\beta}_1(0,t) + (1-\delta)r_2\check{\alpha}_2(0,t). \quad (88)$$

This system is exponentially stable due to Assumption 1. Due to the invertibility of the Volterra transformations (72)-(73) and (74)-(75), we have the following theorem

Theorem 4. Consider the PDE system (60)-(65) with the output injections gains defined in (81)-(82). Then for any initial condition $(\hat{w}_i(\cdot,0), \hat{v}_i(\cdot,0))$, the states $(\hat{w}_i, \hat{v}_i)$ exponentially converge to the states $(\bar{w}_i, \tilde{v}_i)$.

4.3 Output feedback law

We are now able to give the main theorem of this paper.

Theorem 5. Consider the system (15)-(21) with the control law

$$U(t) = \int_{-L}^{0} R_2^w(\xi)\hat{w}_2(\xi,t) + R_2^v(\xi)\hat{v}_2(\xi,t)d\xi + \int_0^L R_1^w(\xi)\hat{w}_1(\xi,t) + R_1^v(\xi)\hat{v}_1(\xi,t)d\xi. \quad (89)$$

where the states $\hat{w}_i$ and $\hat{v}_i$ satisfy the observer equations (60)-(65). Then for any L^2 initial condition, the states $\bar{w}$ and $\tilde{v}$ exponentially converge to zero. This implies the local convergence of the initial states of ρ_i and v_i to the steady states $\rho_i^\star$ and $v_i^\star$.

Proof. The proof uses the same ideas as the ones given in (Lamare et al., 2018, Theorem 5). As we have $\tilde{v} = -\check{v}_i + \hat{v}_i$ and $\bar{w} = -\check{w}_i + \hat{w}_i$, the control law (89) can be rewritten as

$$U(t) = \int_{-L}^{0} R_2^w(\xi)\bar{w}_2(\xi,t) + R_2^v(\xi)\tilde{v}_2(\xi,t)d\xi + \int_{0}^{L} R_1^w(\xi)\bar{w}_1(\xi,t) + R_1^v(\xi)\tilde{v}_1(\xi,t)d\xi + \mathscr{D}(t),$$

where $\mathscr{D}$ is given by U in (89) in which the terms $\hat{v}_i$ and $\hat{w}_i$ have been replaced by $\check{v}_i$ and $\check{w}_i$ respectively. Since $\check{v}_i$ and $\check{w}_i$ converge to zero due to Theorem 4, the term $\mathscr{D}$ can be seen as a bounded disturbance that converges to zero. The rest of the proof is based on Assumption 1 and on the variation of constants formula (Theorem 7.6, page 32 in Hale and Lunel (1993)). It is omitted here due to space restrictions.

5. CONCLUSION

In this paper, we develop a boundary state feedback control law for an interconnected two-segment traffic PDE system and then construct the output feedback with a collocated observer design. The control input acts on the traffic flow rate from the downstream outlet and stabilizes the traffic states to segment-specific steady states using the measurement of density and velocity at the inlet. Comparison between this result with Yu et al. (2019) (different actuation locations) over performance objectives is of authors' interest. The control design of this paper can be extended to the traffic PDE model of multiple connected road segments.

REFERENCES

Anfinsen, H. and Aamo, O.M. (2019). *Adaptive Control of Hyperbolic PDEs*. Springer.

Auriol, J. and Meglio, F.D. (2016). Minimum time control of heterodirectional linear coupled hyperbolic pdes. *Automatica*, 71, 300–307.

Auriol, J. and Meglio, F.D. (2019). An explicit mapping from linear first order hyperbolic PDEs to difference systems. *Systems & Control Letter*, 123, 144–150.

Auriol, J., Meglio, F.D., and Bribiesca-Argomedo, F. (2019). Delay robust stabilization of an underactuated network of two interconnected PDEs systems. *American Control Conference*.

Aw, A. and Rascle, M. (2000). Resurrection of "second order" models of traffic flow. *SIAM journal on applied mathematics*, 60, 916–938.

Bastin, G. and Coron, J.M. (2013). Exponential stability of networks of density-flow conservation laws under PI boundary control. *IFAC Proceedings Volumes*, 46, 221–226.

Bastin, G. and Coron, J.M. (2016). *Stability and boundary stabilization of 1-d hyperbolic systems*, volume 88. Birkhäuser, Switzerland.

Bastin, G., Coron, J.M., and Tamasoiu, S.O. (2015). Stability of linear density-flow hyperbolic systems under PIboundary control. *Automatica*, 53, 37–42.

Coclite, G., Garavello, M., and Piccoli, B. (2005). Traffic flow on a road network. *SSIAM journal on mathematical analysis*, 36(6), 1862–1886.

Coron, J.M., R.Vazquez, M.Krstic, and G.Bastin (2013). Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization*, 51(3), 2005–2035.

Deutscher, J. (2017). Finite-time output regulation for linear 2×2 hyperbolic systems using backstepping. *Automatica*, 75, 54–62.

Garavello, M. and Piccoli, B. (2006). Traffic flow on a road network using the aw–rascle model. *Communications in Partial Differential Equations*, 31(2), 243–275.

Goatin, P., Göttlich, S., and Kolb, O. (2016). Speed limit and ramp meter control for traffic flow networks. *Engineering Optimization*, 48(7), 1121–1144.

Gomes, G. and Horowitz, R. (2006). Optimal freeway ramp metering using the asymmetric cell transmission model. *Transportation Research Part C: Emerging Technologies*, 14(4), 244–262.

Hale, J. and Lunel, S.V. (1993). *Introduction to functional differential equations*. Springer-Verlag.

Herty, M. and Rascle, M. (2006). Coupling conditions for a class of second-order models for traffic flow. *SIAM Journal on mathematical analysis*, 38(2), 595–616.

Herty, M. and Yong, W. (2016). Feedback boundary control of linear hyperbolic systems with relaxation. *Automatica*, 69, 12–17.

Jin, L. and Amin, S. (2018). Analysis of a stochastic switching model of freeway traffic incidents. *IEEE Transactions on Automatic Control*, 64(3), pp.1093–1108.

Karafyllis, I. and Papageorgiou, M. (2019). Feedback control of scalar conservation laws with application to density control in freeways by means of variable speed limits. *Automatica*, 105, pp.228–236.

Lamare, P.O., Auriol, J., Meglio, F.D., and Aarsnes, U. (2018). Robust output regulation of 2 x 2 hyperbolic systems: Control law and input-to-state stability. In *American and Control Conference*.

Logemann, H., Rebarber, R., and Weiss, G. (1996). Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM Journal on Control and Optimization*, 34(2), 572–600.

R. Vazquez, M.K. and Coron, J.M. (2011). Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system. *50th IEEE Conference on pp. 4937-4942*.

Schmuck, C., Woittennek, F., Gensior, A., and Rudolph, J. (2011). Flatness-based feed-forward control of an HVDC power transmission network. *Telecommunications Energy Conference (INTELEC), 2011 IEEE 33rd International*, 1–6.

Treiber, M. and Kesting, A. (2013). *Traffic Flow Dynamics—Data, Models and Simulation*. Springer, Heidelberg.

Yu, H., Auriol, J., and Krstic, M. (2019). Simultaneous stabilization of traffic flow on two connected roads. *arXiv preprint*, arXiv:1910.13885.

Yu, H. and Krstic, M. (2018). Varying speed limit control of aw-rascle-zhang traffic model. *In 2018 21st IEEE International Conference on Intelligent Transportation Systems(ITSC)*, 1846–1851.

Yu, H. and Krstic, M. (2019). Traffic congestion control for aw-rascle-zhang model. *Automatica*, 100, 38–51.

Zhang, H.M. (2002). A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, 36(3), 275–290.

Zhang, L., Prieur, C., and Qiao, J. (2019). P I boundary control of linear hyperbolic balance laws with stabilization of arz traffic flow models. *Systems & Control Letters*, 123, 85–91.

Zhang, Y. and Ioannou, P.A. (2016). Combined variable speed limit and lane change control for highway traffic. *IEEE Transactions on Intelligent Transportation Systems*, 18(7), 1812–1823.