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Chance-constrained LQG production planning problem under partially observed forward-backward inventory systems.

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Abstract: The target system of our research is a reverse logistics system with imperfect information of inventory variables. This system is affected by two independents and uncorrelated random variables that represent demand and return fluctuations. A Discrete-time, chance-constrained, Linear Quadratic Gaussian Problem under imperfect information of inventory systems (DCLQG) is formulated in order to develop an aggregate manufacturing and remanufacturing plan. Technically, an optimal closed-loop solution for this stochastic problem is possible, but it is not easy to get it, particularly for large size problems. Thus, an open-loop updating approach that provides a quasi-optimal solution is investigated here. This approach considers an equivalent deterministic problem to the DCLQG problem. It is based on the conditional mean value and on variances of inventory variables, which are estimated from a Kalman filter procedure. Such an approach allows managers to build an aggregated production plan, periodically revised, that helps them to make decisions. An open-loop updating approach is compared to a no-updating approach, which depends only on the initial condition of states of the system. An example shows the importance of information gathering to provide sub-optimal solutions for stochastic problems with imperfect information of states. It is also shown that sub-optimal production policies can improve the company's profitability.

1. INTRODUCTION

Nowadays, the rational use of raw-material extracted from the environment is a vital issue that has been debated in academia and industry. The scarcity of raw material and pollution caused by the process of extraction and transformation justifies such a concern (Bouras et al., 2016). As a result, supply chains have included actions for recovering or recycling used products. Thus, planning the closed-loop supply chain has become an essential issue for companies reaching sustainability. Indeed, companies are more and more concerned about preserving the environment without, however, losing sight of their profitability; for such, they are reducing wastage and, simultaneously, recovering or recycling used products (Zhang et al., 2018).

Conceptually, a reverse logistics system shifts products from their end destination to one where they can capture value or be appropriately disposed of. Some products get values by means of the remanufacturing process; they are our interest in this paper. An integrated reverse logistics system can be mathematically described through two different inventory balance equations: (1) a first equation to represent the forward channel, where products are made by original equipment and with available raw-material. They are stored in a serviceable storage unit before being moved to the marketplace; and (2) a second equation to describe the backward channel, where used products are collected from the marketplace and stocked in the recovery storage unit, before being remanufactured or discarded. It is worth mentioning that typical managerial activities applied to forward channels - such as production planning and control, scheduling, inventory control, etc. - can be easily replicated to the backward channel.

Since the 90s of the last century, authors have proposed typologies to classify problems of reverse logistics. A small review is: Fleischmann et al. (1997) proposed a quantitative typology with three kinds of models: the first focus on the collection and transportation of used products, the second is related to the planning and scheduling of remanufacturing process, and the third consists in the planning of reusing items, parts, and products without remanufacturing; Chan et al. (2017) categorized problems of remanufacturing in different classes related to production planning, inventory control management, and manufacturing network design; and Abbey and Guide Jr (2018) introduced a typology based on industrial practices for strategical and operational activities of remanufacturing. Regard to these typologies is possible to classify this paper within the operational area related to the production planning and inventory control of remanufactured products.

Many problems found in the literature have a stochastic nature, see, for instance, Govindan et al. (2015) and Zang et al. (2018). For part of them, the fluctuations of demand and return variables follow a normal distribution of probability (Modak and Kelle, 2019). Since linear balance equations usually describe inventory systems, these systems are Gaussian processes. Moreover, the inventory levels measured in the output of these systems can only be partially observed. According to Bensoussan et al. (2007), such an imperfect measure of inventory variables can be caused for several reasons, such as theft, depreciation, typos, wrong placement in the warehouse, etc. Note that the characteristics above described regard to inventory systems increase the difficulties of getting an optimal closed-loop solution for stochastic production planning problems enormously.

Some authors, as Bertesekas (2000), Bensoussan et al. (2007), Polotski and Kenne (2017), and Cheraghalikhani et al. (2019) have discussed approaches for solving stochastic inventory control problems with imperfect information of states. However, as mentioned above, providing an optimal aggregate production planning policy is not an easy task because of the complex nature of the stochastic problems. According to Bertesekas (2000), the certainty-equivalence principle and sufficient statistics concept can reduce the complexity of stochastic problems, and, as a result, a variety of sub-optimal approaches can be proposed. Such approaches provide sub-optimal solutions that can be a good approximation for closed-loop solutions obtained from original problems.

In this paper, a sequential linear quadratic inventory-production optimization problem for a class of discrete-time stochastic linear inventory systems affected by independent random perturbations is formulated. The inventory levels (states) of these systems are not perfectly observed, and they are subject to probabilistic constraints. The random perturbations are represented by seasonal models: one refers to the demand rate for new products, and the other refers to the return rate of used products that are discarded after their life cycle. For practical application, these random variables can be approximated by a Gaussian stochastic process, with mean and variance known over periods of the planning horizon. Gaussian approximation allows reducing all information required about the system in their first and second statistics moments (Graves, 1999). Thus, both inventory systems in the forward and reverse channels are described by Gaussian stochastic processes. Moreover, the measurement errors in the output of inventory storage unities of the forward and backward channels are also approximated by Gaussian white noises. Since all random variables follow Gaussian distributions of probability, the resulting sequential stochastic problem belongs to the class of discrete-time, chance-constrained Linear Quadratic Gaussian problems with imperfect information of states (DCLQG, as an acronym).

In literature, inventory control and production planning problems based on non-constrained stochastic optimization models with perfect information of states are often found. However, a general class of discrete-time chance-constrained stochastic models under imperfect information on states is not so often. The reason is that many papers address only one or another characteristic associated with this class of stochastic problems; for instance: Soudjani and Majumdar (2018) discuss aspects related to chance-constrained on decision variables. On the other hand, Polotski and Kenne (2017) consider the problem with imperfect information on inventory and demand variables. In this paper, the DCLQG problem is considered. It encompasses many characteristics of the general class of stochastic problems. The assumption of the inventory systems is a Gaussian stochastic process allow reducing the DCLQG problem to an equivalent, but a deterministic problem. This equivalent problem is built by the conditional first and second statistical moments that are provided by the Kalman filter estimator. This strategy is possible due to the linearity and Gaussian nature of the system and the certainty-equivalence principle. The equivalent problem is easier to be solved than the stochastic one, and their original properties,

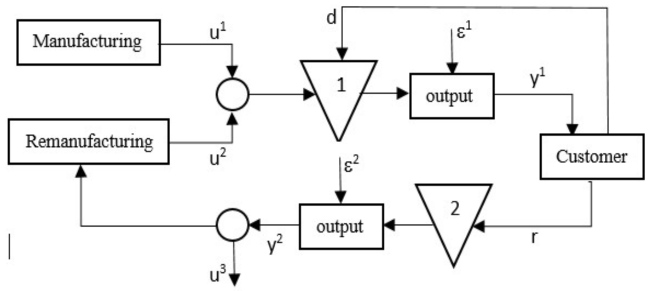

Fig. 1. Diagram of the logistics system with two channels

like linearity and convexity, are preserved. Different mathematical programming techniques can be applied to solve this deterministic problem. The main difference among them is the capacity to use the available information over the periods in order to update the optimal solution, see Bertesekas (2000)

One purpose of this paper is to show that a feasible suboptimal solution for the DCLQG problem is possible by solving the equivalent deterministic problem in an open-loop updating approach. Such an approach consists of subdivided the planning horizon of the problem into a certain number of stages. At each stage, since the state of the stochastic system is measured, a sequence of control inputs are provided from the solution of the deterministic equivalent problem; however, only the first component this sequence is applied to the stochastic system, and a new state is observed. This procedure keeps on until the last stage of the planning horizon be reached. In literature, this approach is commonly named the "shrinking horizon," a kind of rolling horizon scheme, for more details, see Farahani et al. (2017). An example shows that the use of an approach with a simple feedback structure can improve the solution of the original stochastic problem significantly when compared to no-updating approaches.

The next sections of the paper are distributed as follows: section 2 introduces a dynamic time-discrete model to represents forward-backward logistics systems. In sequence, a Chance-constrained, Linear Quadratic Gaussian Problem (DCLQG) is formulated. In section 3, the Kalman filter approach is considered to estimate serviceable and recovery levels of inventory by means of conditional first and second statistical moments. Based on such estimates, an equivalent deterministic problem is proposed. In section 4, an open-loop updating approach is considered to provide an aggregated production plan. At last, section 5 presents a simple example to illustrate the use of an equivalent problem with an open-loop updating sub-optimal approach and to compare this kind of "closed-loop" solution with a "static" (i.e., no-updating) one. Section 6 concludes and introduces future works.

2. A LOGISTICS SYSTEM

Fig. 1 illustrates the forward and backward channels of a make-to-stock company. Note that there are two storage units in this system: the first represents a serviceable inventory (unit 1) containing manufactured and remanufactured products that should promptly meet the demand. The second describes a returnable inventory (unit 2), where the collected products are stored. After quality inspection, some of these used products can be moved to remanufacture processing, and others can be disposed of. Note that there are two main reasons to disposal: (1) a low-quality of the used products, which makes them inappropriate for remanufacturing, and (2) the cost of maintenance of returnable inventory that reduces the profitability of the company.

Some features of this logistics system are:

- The demand and return rates are seasonal and follow independent, no correlated Gaussian distribution of probabilities.
- Both manufacturing and remanufacturing unities have finite processing capacity.
- There are minimum storage limits (i.e., safety-stocks) for serviceable and returnable inventory systems.
- Serviceable and returnable Inventory levels are partially observed from output systems.
- Output measurement errors associated with inventory units are Gaussian white noises.
- Used products that do not pass the quality process are automatically sent to industrial waste, thus avoiding additional costs for the company.

2.1. Inventory-production system

The logistics system shown in Fig. 1 can mathematically be modeled by two interrelated discrete-time stochastic balance equations, which are constituted by two state variables that describe inventory levels in storage units 1 and 2; and three decision variables that are related to manufacturing, remanufacturing, and discard rates. There are also two outputs variables, from which the states of the system are observed. These set of equations are given by:

$$x_{k+1}^1 = x_k^1 + u_k^1 + u_k^2 - d_k \tag{1}$$

$$x_{k+1}^2 = x_k^2 - u_k^2 - u_k^3 + r_k \tag{2}$$

$$y_k^1 = x_k^1 + \varepsilon_k^1 \tag{3}$$

$$y_k^2 = x_k^2 + \varepsilon_k^2 \tag{4}$$

where, for each period k, the notation is given as follows:

x_k^1 = serviceable inventory level;
x_k^2 = inventory level of used products;
u_k^1 = production rate of manufacturing machine;
u_k^2 = production rate for remanufacturing machine;
u_k^3 = disposal rate;
d_k = demand rate for serviceable products;
r_k = return rate of used products;
y_k^1 = partial observation of serviceable inventory level;
y_k^2 = partial observation of inventory level of used products;
ε_k^1 and ε_k^2 = white noises of output systems 1 and 2.
$\Im_k$ = denote the vector of available information of system 1 and 2 at each period k, more details on section 3.1.

The equation (1) describes the serviceable inventory system that represents the forward flow channel, where products must be available to meet demand promptly. On the other hand, equation (2) describes the returnable inventory system, which is related to used products waiting to be remanufactured or disposed of. Fluctuations of demand and return variables follow the Gaussian distribution of probabilities. This means that demand variables d_k can be approximated by its mean value $\hat{d}_k$ and finite variance $\sigma_d^2 \gg 0$. Similarly, the return variable r_k can also be approximated by its first and second statistical moments, that is, $\hat{r}_k$ and finite $\sigma_r^2 \gg 0$. Besides, according to Graves (1999), such an approximation makes it easier to deal with stochastic variables, without loss of generality. Note that the stochastic nature of systems (1) - (2) is necessarily Gaussian due to the linearity of the systems. Moreover, considering that the measurement errors of the outputs of these systems are white Gaussian noises, it is possible to estimate the states using the Kalman filter.

2.2. The overall functional criterion

For each period k of planning horizon N, a quadratic function is considered to describe the inventory/production costs for operating (1)-(2). The function, denoted by $\mathcal{F}_k$, is given by:

$$\mathcal{F}_k(.) = E\{h_1(x_k^1)^2 + h_2(x_k^2)^2 | \Im_k\} + c_1(u_k^1)^2 + c_2(u_k^2)^2 + c_3(u_k^3)^2 \tag{5}$$

where h_1 and h_2 are holding costs associated with serviceable and returnable inventory systems, respectively; c_1, c_2, c_3 denote, respectively, the production costs associated with manufacturing, remanufacturing, and discarding. The symbol $E\{.\}$ is the expected value operator. The information vector $\Im_k$ contains all *sufficient statistics* measured from systems (1)-(2), at each period k (Bertesekas, 2000). At the end period k=T, the inventory cost is $\mathcal{F}_T = E\{h_1(x_T^1)^2 + h_2(x_T^2)^2 | \Im_T\}$.

It is worth mentioning that other cost models can be used to represent $\mathcal{F}_k(.)$. However, quadratic costs have some advantages when compared to other costs, for instance: it is a convex and smooth function; It is easy to specify the break-even point where the cost function should be minimized; and it gives to decision variables high penalties when they deviate too much from their source of origin and small penalties when minor deviations occur. In literature, production operations quadratic approximations are often used (Krajewski, 2017).

2.3. Probabilistic physical constraints

Serviceable and recoverable inventories – represented by storage units 1 and 2 in Fig. 1 – are physically limited spaces that receive manufactured-remanufactured products and used products collected from the marketplace, respectively. Usually, these physical storage spaces are designed to have a minimum and a maximum number of products, which satisfy the demand. In general, this minimum number of products is associated with a safety stock that reduces the risk of scarcity.

The number of products to be repaired and returned is information extracted by two measuring devices, placed at the output of systems 1 and 2 (see Fig. 1). These devices record the value of output variables y_k^1 and y_k^2 that contain the states of systems x_k^1 and x_k^2. Note that devices (3) and (4) have errors of observation that are described by normal white noises ε_k^1 and ε_k^2. Thus, systems (1)-(2) are stochastic processes under imperfect information of inventory state variables x_k^1 and x_k^2.

The stochastic nature of systems (1)-(2) and output devices (3)-(4) make that lower bounds (i.e., safety stock) of the observed levels of serviceable and returnable inventories can be violated at any period of planning horizon (Silva Filho and Andres, 2017). According to Jiang et al. (2019), this characteristic of stochastic systems has an important impact on the customer satisfaction level. More details on probabilistic constraints are added in the following.

Based on the above, the lower bound of products in the serviceable storage unit (i.e., safety stock level $\underline{x}_1$) is used in order to guarantee that the demand will be satisfied anytime, even if unexpected events occur, such as a huge growth of demand. Usually, there is no upper bound since it is assumed that there is enough storage capacity in the serviceable storage unit. Similarly, in storage 2, where only returnable products are stored, a lower bound $\underline{x}_2$ is also possible to be considered. Usually, $\underline{x}_2$ is equal to zero, but for some companies, it may be interesting to maintain a level of safety stock in order to complement their production for remanufacturing products. However, there is a problem to be considered that is the stochastic nature of input-output systems (1)-(4). Such a nature implies in a real possibility of both observed inventory levels, that is, y_k^1 (serviceable) and y_k^2 (returnable), violated their lower bounds $\underline{x}_1$ and $\underline{x}_2$, respectively. Thus, to guaranty no violation of these physical constraints and also to preserve them explicitly in the formulation of the stochastic problem, they will be considered in probability as follow: $\text{Prob}\,(y_k^1 \geq \underline{x}_1 \,|\, \mathfrak{I}_k) \geq 1-\alpha_1$ and $\text{Prob}\,(y_k^2 \geq \underline{x}_2 \,|\, \mathfrak{I}_k) \geq 1-\alpha_2$. Note that $\text{Prob}[\cdot \,|\, \mathfrak{I}_k]$ is the conditional probability operator, the indexes α_1 and α_2 are probability indexes provided by the manager in the range [0,1).

2.4. Formulating the stochastic problem

A discrete-time chance-constrained linear quadratic Gaussian problem under imperfect information of states is here considered to provide an optimal sequential aggregate production plan {$u_1(k)$, $u_2(k)$, and $u_3(k)$; k = 0,1,2, …, T-1} for a reverse logistics problem of Fig.1. This problem, denoted as DCLQG, is formulated as follows:

$$\text{Min}\, \mathcal{F}(.) = \sum_{k=0}^{N} E\left\{ \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix}^T \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} \,\middle|\, \mathfrak{I}_k \right\} + \sum_{k=0}^{N-1} \left\{ \begin{pmatrix} u_k^1 \\ u_k^2 \\ u_k^3 \end{pmatrix}^T \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{pmatrix} u_k^1 \\ u_k^2 \\ u_k^3 \end{pmatrix} \right\}$$

such that, (6)

$$\begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{pmatrix} = \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{pmatrix} u_k^1 \\ u_k^2 \\ u_k^3 \end{pmatrix} + \begin{pmatrix} -d_k \\ r_k \end{pmatrix}$$

$$\begin{pmatrix} y_{k+1}^1 \\ y_{k+1}^2 \end{pmatrix} = \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} + \begin{pmatrix} \varepsilon_k^1 \\ \varepsilon_k^2 \end{pmatrix}$$

$$\text{Prob}\,(y_k^1 \geq \underline{x}_1 \,|\, \mathfrak{I}_k) \geq 1-\alpha_1$$

$$\text{Prob}\,(y_k^2 \geq \underline{x}_2 \,|\, \mathfrak{I}_k) \geq 1-\alpha_2$$

$$0 \leq u_1^k \leq \bar{u}_1;\; u_2^k \leq \bar{u}_2;\; u_3^k \geq 0$$

where Prob(·) is the probabilistic operator, the indexes α_1 and α_2 are percentages of probability provided by the manager in the range [0,1). Lower bounds on inventories variables (i.e., safety-stocks) are represented by $\underline{x}_1$ and $\underline{x}_2$, and upper bounds of processing capacities of manufacturing and remanufacturing variables are given by $\bar{u}_1$ and $\bar{u}_2$. The finite planning horizon is described by N>0 periods. Note that the variables x_k^1 and x_k^2 are partially observed from output systems (3)-(4), and, consequently, the inventory costs give in (7) must be estimated for each period k with the base on the vector $\mathfrak{I}_k$.

2.4. A suboptimal approach for (7)

An optimal closed-loop solution for (6) is not a trivial task. The classical dynamic programming algorithm, proposed by Bellman, cannot be applied in reason of its large time-consuming. Thus, Bellman's approach is "quasi-prohibitive" for the majority of practical applications. On the other hand, suboptimal approaches can be a good alternative to solve (6). Such approaches use statistical moments information to formulate equivalent problems to the original stochastic problem, that are easier to be solved (Bertesekas, 2000). Considering that stochastic problem (6) assumes that all independent and dependent random variables follow Gaussian distributions, then all required information to convert this problem into a deterministic equivalent is the first and second statistical moments of them (Silva Fo. and Andres, 2017).

3. Discrete-time Constrained Linear Quadratic Problem

The first essential step to transform a stochastic problem into a deterministic equivalent one is to get all available statistical information extracted from the stochastic systems. Since the stochastic problem (6) is under imperfect state information, a previous necessary step is to estimate the statistics about these variables. These estimates are derived from mechanisms that use the information vector $\mathfrak{I}_k$. This vector contains all information available about observed states and decision variables. As mentioned before, all variables of inventory systems (1)-(2) and output systems (3)-(4) are approximated by Gaussian random variables. As a result, these systems are completely driven by their first and second statistics moments (i.e., means and variances of random variables). As another important consequence, the estimations of inventory variables can be provided by using the Kalman estimator.

Before formulating the deterministic problem, let us start estimating the states of systems (1)-(2).

3.1. Estimating the Inventory Level by Kalman Filter

The information vector $\mathfrak{I}_k$ contains all current and past data related to output variables (i.e., y_k^1 and y_k^2) and to decision variables (i.e., u_k^1, u_k^2, u_k^3). It is defined as follows:

$$\mathfrak{I}_k = \{u_0^1, u_0^2, u_0^3, \ldots, u_k^1, u_k^2, u_k^3, y_0^1, y_0^2 \ldots, y_k^1, y_k^2\} \supset \mathfrak{I}_{k-1} \quad (7)$$

From $\mathfrak{I}_k$, it is possible to compute the all-sufficient statistics required to estimate inventory variables. Based on these estimations, an equivalent deterministic problem to the stochastic problem (6) can be formulated.

Thus, an estimator that uses the information vector $\mathfrak{I}_k$ can produce expected values for inventory variables x_k^1 and x_k^2.

Due to the Gaussian nature of the problem (6), only the conditional first and second statistical moments of the inventory variables need to be calculated. As a result, such estimations can be calculated by the Kalman filter. The mean conditional inventory estimations are:

$$x^1_{k+1|k+1} = x^1_{k|k} + u^1_k + u^2_k - \hat{d}_k + V^{k+1|k}_{x^1} \cdot (V^{k+1|k}_{x^1} + \sigma^2_{\varepsilon^1})^{-1} . [y^1_{k+1} - x^1_{k+1|k} - u^1_k - u^2_k + \hat{d}_k] \quad (8)$$

$$x^2_{k+1|k+1} = x^2_{k|k} - u^2_k - u^3_k + \hat{r}_k + V^{k+1|k}_{x^2} \cdot (V^{k+1|k}_{x^2} + \sigma^2_{\varepsilon^2})^{-1} . [y^2_{k+1} - x^2_{k+1|k} + u^2_k + u^3_k - \hat{r}_k] \quad (9)$$

and, conditional covariances evolutions (10) and (11):

$$\begin{cases} V^{k+1|k+1}_{x^1} = V^{k+1|k}_{x^1} - V^{k+1|k}_{x^1} \cdot (V^{k+1|k}_{x^1} + \sigma^2_{\varepsilon^1})^{-1} \cdot V^{k+1|k}_{x^1} \\ V^{k+1|k}_{x^1} = V^{k|k}_{x^1} + \sigma^2_{d} \end{cases} \quad (10)$$

$$\begin{cases} V^{k+1|k+1}_{x^2} = V^{k+1|k}_{x^2} - V^{k+1|k}_{x^2} \cdot (V^{k+1|k}_{x^2} + \sigma^2_{\varepsilon^2})^{-1} \cdot V^{k+1|k}_{x^2} \\ V^{k+1|k}_{x^2} = V^{k|k}_{x^2} + \sigma^2_{r} \end{cases} \quad (11)$$

where $x^i_{k|k} = E\{x^i_k|\Im_k\}$ and $V^{k|k}_{x^i} = E\{(x^i_k - x^i_{k|k})^2\}$ denote, respectively, the conditional estimates of mean and covariance of the inventory variables (i=1, 2). Note that $\sigma^2_\eta \geq 0$ denotes the variance of white noise η.

The initial conditions for (8)-(11) are given by:

$$\begin{cases} x^i_{0|0} = \hat{x}^i_0 + V^{0|0}_{x^i_0} \cdot (\sigma^2_{\varepsilon^i_0})^{-1} \cdot (y^i_0 - \hat{x}^i_0) \\ V^{0|0}_{x^i_0} = V^0_{x^i_0} - V^0_{x^i_0} \cdot (V^0_{x^i_0} - \sigma^2_{\varepsilon^i_0})^{-1} \cdot V^0_{x^i_0} \end{cases} \quad (12)$$

where $\hat{x}^i_0 = E\{x^i_0\}$ and $V^0_{x^i_0} = E\left\{\left(x^i_0 - \hat{x}^i_0\right)^2\right\}$ denote the mean value and variance of the initial state x^i_0, with i = 1 and 2.

The inventory variable estimators, given by equations (8)-(12), are used to formulate the equivalent deterministic problem. Below are introduced each step of the transformation process.

3.2. Constrained Linear Quadratic (CLQ) problem

Based on conditional mean value and variance (i.e., equations (8)-(12)), the following transformations can be performed:

3.2.a) Linear Quadratic (LQ) criterion:

$$E\left\{\sum_{k=0}^{N} (h_i \cdot (x^i_k)^2 | \Im_k)\right\} = h_i \cdot \sum_{k=0}^{N} E\left\{(x^i_k)^2 | \Im_k\right\} = $$
$$= h_i \cdot \sum_{k=0}^{N} \int_{-\infty}^{+\infty} f\left((x^i_k)^2 + \tau | \Im_k\right) \cdot \rho_{\delta_k}(\tau) \cdot \partial\tau =$$
$$= h_i \cdot \sum_{k=0}^{N} (\hat{x}^i_k)^2 + \sum_{k=1}^{N} K^i_k \quad (13)$$

where, for i=1,2 follows that: $\hat{x}^i_k = E\{x^i_{k|k}\} = E\{E[x^i_k|\Im_k]\}$ is the mean estimated value of inventory variables x^1_k and x^2_k; $\rho_{\delta_k}(\tau) = \frac{\partial}{\partial\tau}\Phi_{\delta_k}(.)$ is the probability density function of the random variable $\delta_k = x^i_{k|k} - \hat{x}^i_k$; and the term K^i_k denotes a partial integration constant of ith inventory cost at period k that depends on the temporal evolution of the conditional variance $V^{k|k}_{x^i}$. Thus, the general constant is $K = \sum_{k=1}^{N} K^i_k$.

3.2.b) Equivalent inventory systems

Applying the principle of certainty equivalence, the random variables of systems (1) - (2), without any loss of generality, can be replaced by their respective first moments. Thus, considering $d_k = \hat{d}_k$ and $r_k = \hat{r}_k$, the equivalent deterministic systems are given by:

$$\hat{x}^1_{k+1} = \hat{x}^1_k + \hat{u}^1_k + \hat{u}^2_k - \hat{d}_k; \quad \hat{x}^1_0 = E(x^1_0|\Im_k) \quad (14)$$

$$\hat{x}^2_{k+1} = \hat{x}^2_k - \hat{u}^2_k - \hat{u}^3_k + \hat{r}_k; \quad \hat{x}^2_0 = E(x^2_0|\Im_k) \quad (15)$$

It is assumed that $\{\hat{u}^i_k = u^i_k, i = 1,2,3\}$ with $V^k_{u^1} = V^k_{u^2} = V^k_{u^3} = 0$. This means that decision variables of systems (14)-(15) are completely deterministic variables. Besides, output measurement systems (3)-(4) are given by $\hat{y}^1_k = \hat{x}^1_k$ and $\hat{y}^2_k = \hat{x}^2_k$, because ε^1_k and ε^2_k are white noises, implying that $E\{\varepsilon^1_k\} = E\{\varepsilon^2_k\} = 0$.

3.2.c) Equivalent deterministic constraints

The chance-constraints $\text{Prob.}(y^i_k \geq \underline{x}_i|\Im_k) \geq 1 - \alpha_i$ can be rewritten as $\text{Prob.}(x^i_k \geq \underline{x}_i - \varepsilon^i_k|\Im_k) \geq 1 - \alpha_i$, for i=1,2. Then, using conditional estimations of mean and variance (8)-(12), the probabilistic operator can be mathematically handled in order to obtain the following deterministic inequalities:

$$\hat{x}^1_k \geq \underline{x}_1 + \sqrt{V^{k|k}_{x^1}} \cdot \Phi^{-1}_{\varepsilon}(\alpha_1) = \underline{x}^1_{\alpha_1}(k) \quad (16)$$

$$\hat{x}^2_k \geq \underline{x}_2 + \sqrt{V^{k|k}_{x^2}} \cdot \Phi^{-1}_{\varepsilon}(\alpha_2) = \underline{x}^i_{\alpha_2}(k) \quad (17)$$

where $V^{k|k}_{x^1} \cong k.\sigma^2_d$ and $V^{k|k}_{x^2} \cong k.\sigma^2_r$ are variances of $\hat{x}^1_k$ and $\hat{x}^2_k$, respectively; and $\{\underline{x}^i_{\alpha_i}(k),\ i=1,2\}$ denote safety-stock levels of serviceable and returnable storage units 1 and 2 (see Fig. 1). Note that magnitudes of $\underline{x}^1_{\alpha_1}(k)$ and $\underline{x}^2_{\alpha_2}(k)$ tend to increase proportionally with the evolution of k over the planning horizon N. The reason is that systems (1)-(2) run in an open loop that is without any updating during each new period k. The probabilistic indexes $\alpha_1 \in [0,1)$, and $\alpha_2 \in [0,1)$ are chosen by the user and represent confidence levels of safety-stock for serviceable and returnable storage units.

3.2.d) Equivalent deterministic problem – CLQ

The purpose is to find an optimal sequence $\{\hat{u}^{1*}_k, \hat{u}^{2*}_k$ and $\hat{u}^{3*}_k$, $k=0,\cdots,N-1\}$ that solves the following equivalent deterministic (CLQ) problem:

$$\text{Min } \hat{F} = \sum_{k=0}^{N} \left\{ \begin{pmatrix} \hat{x}^1_k \\ \hat{x}^2_k \end{pmatrix}^T \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{pmatrix} \hat{x}^1_k \\ \hat{x}^2_k \end{pmatrix} \right\} + \sum_{k=0}^{N-1} \left\{ \begin{pmatrix} u^1_k \\ u^2_k \\ u^3_k \end{pmatrix}^T \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{pmatrix} \hat{u}^1_k \\ \hat{u}^2_k \\ \hat{u}^3_k \end{pmatrix} \right\} + K$$

Such that (18)

$$\begin{pmatrix} \hat{x}^1_{k+1} \\ \hat{x}^2_{k+1} \end{pmatrix} = \begin{pmatrix} \hat{x}^1_k \\ \hat{x}^2_k \end{pmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{pmatrix} \hat{u}^1_k \\ \hat{u}^2_k \\ \hat{u}^3_k \end{pmatrix} + \begin{pmatrix} -\hat{d}_k \\ \hat{r}_k \end{pmatrix}$$

$$\hat{x}^1_k \geq \underline{x}^1_{\alpha_1}(k); \ \hat{x}^2_k \geq \underline{x}^i_{\alpha_2}(k)$$

$$0 \leq \hat{u}^1_k \leq \bar{u}_1; \ \hat{u}^2_k \leq \bar{u}_2; \ \hat{u}^3_k \geq 0$$

At last, it is worth mentioning that problem (18) preserves the linearity and convexity characteristics of the original problem (6), and its solution is an open-loop solution for problem (6).

4. A SUBOPTIMAL APPROACH

Open-loop solutions for the problem (6) can be provided by solving the problem (18) through different sub-optimal approaches, see Bertesekas (2000) and Silva Fo & Andres (2017). Two of these approaches are briefly discussed below.

Approach 1 – open-loop no-updating – considers that states of inventory systems (1)-(2) are estimated from outputs (3)-(4) only once at period k=0, that is, $\hat{x}_0^1 = x_{0|0}^1$ and $\hat{x}_0^2 = x_{0|0}^2$. Based on this initial information, problem (18) is solved without considering any additional information on the inventory levels of systems (1) - (2) over the remaining periods k of planning horizon N. This type of solution is known as "static" solution. It is an open-loop solution to the problem (6).

Approach 2 – open-loop updating – considers that for each period k≥0, the current states of inventories are observed from (3)-(4) and their values estimated through Kalman filter, that is, $x_{k|k}^1$ and $x_{k|k}^2$. Follows then that $\hat{x}_k^1 = x_{k|k}^1$ and $\hat{x}_k^2 = x_{k|k}^2$ are used as an initial condition, to provide an optimal sequential open-loop policy $\{(\hat{u}_j^{1*}, \hat{u}_j^{2*}, \hat{u}_j^{3*}),\ j \in [k, N\text{-}1]\}$ that minimizes the CLQ problem (18). It is important to realize that only values of the optimal policy of period k, that is, $(\hat{u}_k^{1*}, \hat{u}_k^{2*}, \hat{u}_k^{3*})$ are effectively used as input to systems (1)-(2). This means that the remaining sequence (i.e., $(\hat{u}_j^{1*}, \hat{u}_j^{2*}, \hat{u}_j^{3*})$ $\forall$ j ∈ [k+1, N+1]) is completely ignored.

As soon as new measures are taken from the outputs (3)-(4) of systems (1)-(2), the above procedure is repeated. As a result, the equivalent deterministic CLQ problem must be solved N times. It means that such an approach follows a "kind" of rolling horizon scheme, where the optimal open-loop production plan is provided by solving (18) in the following ranges: [0, N], [1, N], ⋯, [N-1, N].

Fig. 2 illustrates the open-loop updating approach during a given period k of planning horizon N.

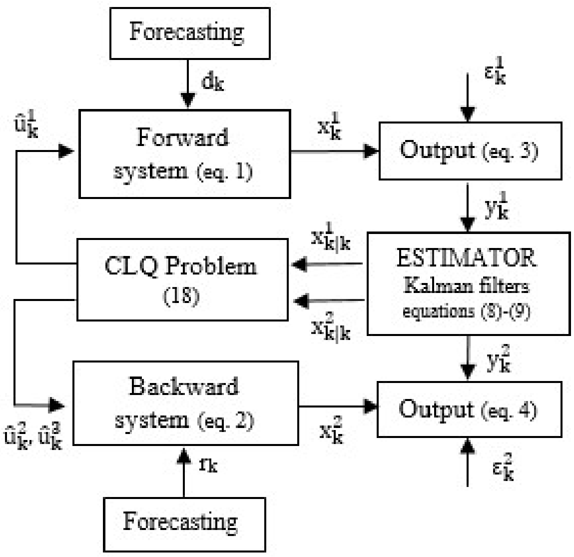

Fig. 2: The open-loop updating approach

5. USE CASE

Consider a make-to-stock company whose products belong to the same family. After their life cycle, part of these products is collected from users and can be repaired or remanufactured. An example of a product with such a characteristic is a reusable container. Recently, the company implemented an efficient policy of reverse logistics, where the collected products are inspected before sending them to remanufacture or disposal. Besides, the unitary cost of remanufacturing is assumed to be slightly cheaper than the cost of manufacturing. Thus, it is expected that remanufacturing becomes an important alternative to reduce production costs and improve the company's profitability. Essentially, the company looks for an aggregate production policy that optimizes its profitability by means of reducing operational costs.

It is assumed that the company has the same production structure as detached in Fig. 1. The main drawback of this structure is that serviceable and returnable inventory variables are not perfectly measured over the periods of the planning horizon. However, under the supposition of the Gaussian process for systems (1)-(4), a Kalman filter estimator (8)-(12) can be used to determine the current state of these variables. By means of such estimations and from the data given in Table 1, the CLQ-problem (18) is formulated. As a result, optimal open-loop solutions can be provided as an alternative to the closed-loop optimal solution of the original problem (6).

5.1. Numerical data

In this use case, seasonal forecasting models are considered to describe fluctuations of demand and return variables. These models are given by $d_k=\hat{d}_k+10(\sin(\pi.k/3)+5.\nu_k)$ and $r_k = \hat{r}_k-5(\sin(\pi.k/3)+2.\eta_k)$, respectively. The ν_k and η_k are error measures of forecasting described as white Gaussian noises. The demand and return rates, to be applied to systems (1)-(2), were generated from these forecasting models. Note that they are frozen for this application, as exhibited in Table 1.

Table 1. Problem's data

Months (k)	1	2	3	4	5	6	7	8	9	10	11	12
Demand (d)	537	509	508	507	500	513	543	531	511	499	518	504
Return (r)	473	477	476	497	495	480	465	484	473	488	483	487

$\hat{x}_0^1 = \hat{x}_0^2 = 100;\ \underline{x}_k^1 = 75;\ \underline{x}_k^2 = 17;\ \bar{u}_1 = 300;\ \bar{u}_2 = 300$

$N = 12$ months $\sigma_d = 15$ and $\sigma_r = 10$

Confidence indexes: $\alpha_1 = 95\%$ and $\alpha_2 = 95\%$

Mean values of demand and return rates: $\hat{d}_k = \hat{r}_k = 500;\ \forall k$

Inventory cost: $\mathcal{F}_x = 2.\left(\hat{x}_k^1\right)^2 + 3.\left(\hat{x}_k^2\right)^2$

Production cost: $\mathcal{F}_u = 0.60\left(\hat{u}_k^1\right)^2 + 0.55\left(\hat{u}_k^2\right)^2 + 0.35\left(\hat{u}_k^3\right)^2$

It is important to note that the choices of the holding inventory, production and disposal costs were all arbitrarily defined, but with some practical reasons, for instance: i) the holding cost of serviceable inventory (h_1=2$) be less than the holding cost of returnable inventory (h_2=3$), reflects the fact of this last one contains costs with used product collection and inspection processes; and ii) the fact of the cost of manufacturing products (c_1=0,60$) is slightly higher than the cost of remanufacturing used products (c_2=0,55$) expresses the company's interest in getting “value” from remanufactured products, which has direct implications for its sustainability and, perhaps, "green" image in relation to concurrence.

5.2. Scenarios 1: open-loop updating approach

Trajectories exhibited in Fig. 3 and Fig. 4 show the behavior of estimated serviceable and returnable inventory variables. Note that estimates inventory variables $x^1_{k|k}$ and $x^2_{k|k}$ follow their mean values $\hat{x}^1_k$ and $\hat{x}^2_k$, which are provided as a solution of problem (18) at each new period k of planning horizon N.

On the other hand, the trajectories exhibited in Fig. 5 illustrate the behavior of decision variables, that is, manufacturing (u^1_k), remanufacturing (u^2_k), and disposing of (u^3_k). It is possible to observe the remanufacturing decision variable u^2_k dominates the production process, with almost 100% of its total monthly capacity is being used ($i.e., u^2_k \cong 300$), while the manufacturing decision variable u^1_k uses approximately 85% of total capacity (i.e., $u^1_k \cong 250$). The main justification for the intense use of the remanufacturing process is its slightly lower cost compared to the cost of manufacturing. Also, it is worth observing that 39% of returnable products are being discarded. Improper quality of these used-products or/and costs reduction policy can explain such a discard. In the next section, these results are compared to the static solution of the open-loop no-updating approach.

5.2. Scenarios 2: the open-loop no-updating approach

This type of open-loop approach provides a "static" solution. The reason is that during the optimization process of the problem (18), only the estimated initial states of the systems (1)-(2) are available. Fig. 6 shows the state trajectories, that is, the serviceable and returnable inventory levels, and Fig. 7 presents their respective decisions rates.

5.3. Comparing suboptimal solutions

Now solutions based on updating and no-updating solutions are compared with their trajectories of states and decision variables and also with their total costs. From comparing Fig. 5 and Fig. 7, it is possible to realize that the updated approach provides variations of decision variables smoothly, while the no-updating approach provides strong variations on these variables. The main reason is that the no-updating approach needs to maintain high levels of safety stock to reduce future uncertainties about demand and return fluctuation levels. Indeed, from Fig. 6, it is possible to see that serviceable and returnable inventory trajectories grow over periods of the planning horizon created large safety-stocks. Such a feature can be justified considering that safety constraints (16) and (17) are strongly affected by the evolution of their respective second statistic moments. As dis $\hat{x}^2_k$ d in subsection 3.2.c), these variances increase when the systems (1)-(2) is running in open-loop operation, i.e., $V^{k+1}_{x^1} \geq V^k_{x^1}$ and $V^{k+1}_{x^2} \geq V^k_{x^2}$. Consequently, the levels of safety-stock of the inventory variables also tend to increase proportionally, that is, $\underline{x}^1_{\alpha_1}(k+1) \geq \underline{x}^1_{\alpha_1}(k)$ and $\underline{x}^2_{\alpha_2}(k+1) \geq \underline{x}^2_{\alpha_2}(k)$. It is worth understanding that the purpose of the growth of inventory lower-boundaries is to try ensuring a feasible open-loop solution for the problem (18). Thus, it is possible to conclude that such growth has an impact on decision variables, causing instability in the decision-making process. This explains the strong variations observed in Fig. 7.

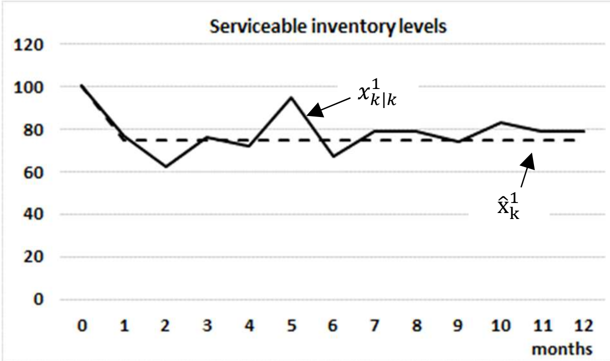

Fig. 3. Serviceable inventory trajectories - updating approach

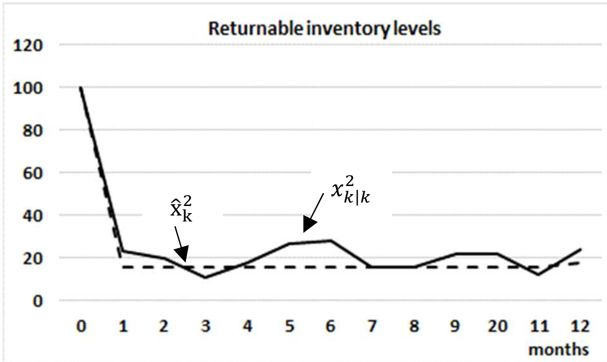

Fig. 4. Returnable inventory trajectories - updating approach

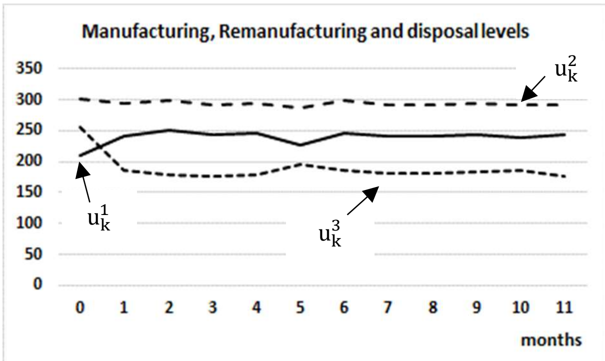

Fig. 5. Trajectories of decision variables – updating approach

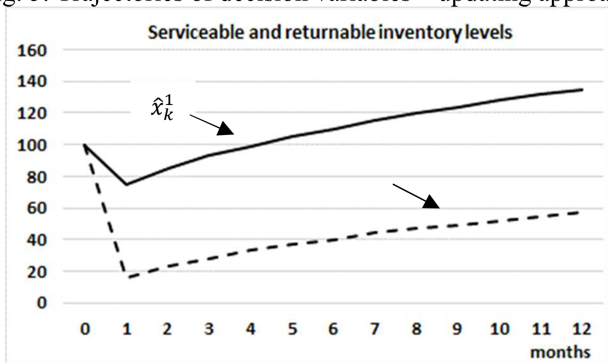

Fig. 6. Inventory levels trajectories – no updating approach

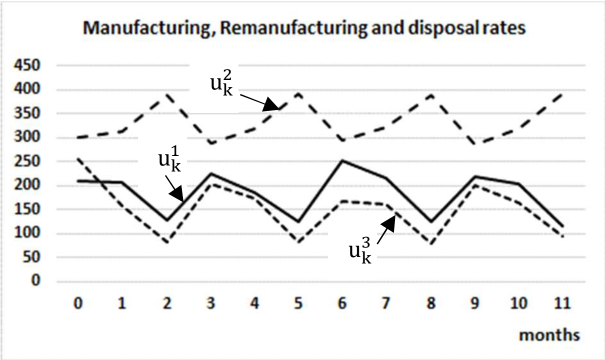

Fig. 7. Decision rates trajectories – no updating approach

At last, the total cost of each approach is compared. It was observed that the cost of open-loop updating was $ 1,281,309, while the cost of the no-updating approach was $ 1,428,569, a difference of $ 147,260. This means that the use of an updating mechanism to get information about the system (1)-(2) can be an interesting managerial practice to help in the process of improving the profitability of the company. Besides, managers can develop different scenarios and obtain important insights into the aggregate production process.

6. CONCLUSION

A discrete-time Linear Quadratic Gaussian problem with chance-constraints has been proposed in order to provide an optimal aggregate production plan. The whole system considers two logistic channels under imperfect information of states (i.e., about their inventory variables). The Gaussian nature of these two channels allows using the Kalman filter to estimate their current inventory levels. The complexity of the stochastic problem leads to the application of the certainty-equivalence principle in order to formulate an equivalent deterministic problem. An open-loop updating approach was considered to provide a suboptimal solution to the original stochastic problem. A simple example compared the open-loop updating approach to the no-updating approach. In conclusion, it was possible to verify that the periodically revised solution allows the manager to obtain an aggregate production plan that can improve the profitability of their companies. Future studies must consider other suboptimal approaches of literature, as discussed in Bertesekas (2000), and more complex real examples.

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