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Control design and Lyapunov Functions via Bernstein Approximations: Exact Results ⋆⋆

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Abstract: We study the control problem for polynomial continuous-time dynamical systems. We consider polynomial Lyapunov functions and controllers, both parameterised in Bernstein form. Specifically, we present necessary and sufficient conditions for existence of polynomial controllers and Lyapunov functions of some maximum degree, providing at the same time explicit upper bounds on the degree of the involved Bernstein polynomials. The formulated conditions are a set of algebraic inequalities, in the space of Bernstein coefficients.

Keywords: Bernstein polynomials, control design, certificates of positivity, Lyapunov stability, Control Lyapunov Functions, polynomial systems

1. INTRODUCTION

The problem of verifying stability of closed-loop systems by means of a Lyapunov function is an important step in the formal verification of control systems Tabuada (2009). Many works have been focused on the subject of stability analysis for polynomial systems Ackerman (1993). Bernstein polynomials (BP) have been used in control, among other approaches such as Handelman basis functions, sum-of-squares polynomials for finding Lyapunov stability certificates and stabilizing controllers.

For systems with inputs and in specific affine control systems, the main goal is to provide a practical method for computing control Lyapunov functions such that the closed-loop system is asymptotically stable. Since a Lyapunov function is positive definite and the Lie derivative by Lyapunov stability condition of control systems must be negative semidefinite in the neighborhood of the equilibrium, the problem can be translated to the one of searching for *certificates of positivity (CP)*. The sum-of-squares approach is popular in control society as a CP, and has been explored by many authors, e.g., Jarvis-Wloszek (2003); Tan and Packard (2006) to design controllers, approximate domains of attraction for nonlinear systems etc. This method provides special classes of positive functions, and while it can not be used to certify CP for all positive functions, existing methods provide tight approximation results by considering degree elevation, see e.g., Putinar's Positivstellensatz Putinar (1993).

An appealing alternative for finding CP in control design involves the Bernstein basis approximations, see, e.g., Hamadneh and Wisniewski (2018a,b); Sloth and Wisniewski (2014); Ben Sassi et al. (2015). In Sloth and Wisniewski (2014), the simplicial Bernstein coefficients are utilized for controller design problem and formulated as a linear programming problem over simplices. Bernstein expansions have an important property, making them useful for providing CP, primarily for polynomial functions: their value is lower and upper bounded by the Bernstein expansion coefficients, which can be computed explicitly. In Leth et al. (2017), an algorithm with its convergence for the synthesis of Lyapunov polynomials for polynomial vector fields in the Bernstein bases was given. In Lane and Riesenfeld (1981), an algorithm for isolating the enclosure bound and real roots of a polynomial was presented. Moreover, CP for polynomial functions defined over simplices are addressed in Hamadneh et al. (2019, 2020a); Boudaoud et al. (2008); Leroy (2008). The extension to the tensorial rational case is given in Hamadneh et al. (2019, 2020b). Our approach follows the same path, namely, we study the enforcement of Lyapunov monotone conditions via certificates of positivity induced by polynomials in the Bernstein basis. First, we provide a set of results summarizing and extending known facts of operations on BPs over boxes, such as addition, multiplication, and differentiation. Next, by describing the system dynamics in the tensorial Bernstein basis, we express the Lyapunov conditions as inequalities involving only coefficients of BPs over boxes. Instead of over-approximating the Bernstein basis functions using linear or polynomial relaxations, we work directly on the space of the BP *coefficients*. We derive, to the best of our knowledge for the first time necessary and sufficient conditions for existence of polynomial Lyapunov functions and polynomial controllers of a given maximum degree in the monomial form, using BPs defined over boxes. These conditions are accompanied by a set of algebraic inequalities, whose solvability implies the existence of the Lyapunov function and controller of some maximum degree. In general, the conditions are nonlinear, however, when

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considering the controller or the Lyapunov function fixed, they become linear in the Bernstein coefficient space, recovering results in the literature for the autonomous case Leth et al. (2017). Summarizing, our contributions are as follows.

- Whilst most previous studies have focused on the simplicial Bernstein basis, Hamadneh and Wisniewski (2018a,b); Sloth and Wisniewski (2014) Leth et al. (2017), we study the stability of control systems over boxes in the tensorial Bernstein basis,
- We hold the analytic equivalent Bernstein forms for addition, differentiation, multiplication and degree elevation onto control systems in the tensorial Bernstein basis.
- We provide necessary and sufficient conditions for existence of polynomial controllers and Lyapunov functions of some maximum degree, through Bernstein basis approximations. Moreover, we show equivalence of these conditions with a set of algebraic inequalities. We must note that our strategy of working on the space of Bernstein coefficients is similar to that given in Ribard et al. (2016). In comparison, the work there does not provide the nonlinear system of coefficients appearing on the vertices of a simplex. Therefore, the problem of finding upper bounds on the degree of the Bernstein form of the Lyapunov and control function is not sought after there.

2. PRELIMINARIES

In this section, we introduce the expansion of polynomials into Bernstein form in the state space and highlight important properties.

2.1 Tensorial Bernstein Basis

We consider the Bernstein expansion of a polynomial function f over a general n-dimensional box Q in the set of real intervals $\mathbb{I}(\mathbb{R})^n$,

$$Q = [\underline{q}_1, \overline{q}_1] \times .... \times [\underline{q}_n, \overline{q}_n]$$

with

$$\underline{q}_\mu \leq \overline{q}_\mu, \ \mu = 1, ..., n.$$

The *width* of Q is denoted by $w(Q)$,

$$w(Q) := \overline{q} - \underline{q}.$$

Comparisons and arithmetic operations on *multiindices* $i = (i_1, ..., i_n)$ are defined component-wise. For $x \in \mathbb{R}^n$ and a multiindex j, its monomial is $x^j := x_1^{j_1} ... x_n^{j_n}$. Using compact notation, $D = (D_1, ..., D_n)$, we have $\sum_{j=0}^{D} := \sum_{j_1=0}^{D_1} \cdots \sum_{j_n=0}^{D_n}$ and $\binom{D}{i} := \prod_{\mu=1}^{n} \binom{D_\mu}{i_\mu}$. An n-variate polynomial function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is expressed in the *monomial form* as $f(x) = \sum_{j=0}^{d} a_j x^j$, where $d = (d_1, ..., d_n)$, and can be represented in the *Bernstein form* by

$$f(x) = \sum_{i=0}^{D} b_i^{(D)}(f) \ B_i^{(D)}(x), \quad x \in Q. \tag{1}$$

In (1), the i^{th} Bernstein polynomial of degree $D \geq d$ is

$$B_i^{(D)}(x) = \binom{D}{i} (x - \underline{q})^i (\overline{q} - x)^{D-i} w(Q)^{-D}. \tag{2}$$

Moreover, the *Bernstein coefficients* $b_i^{(D)}$ of f of degree D over Q are given analytically by the formula

$$b_i^{(D)}(f) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{D}{j}} \ s_j, \quad 0 \leq i \leq D, \tag{3}$$

where

$$s_j = w(Q)^j \sum_{\tau=j}^{d} \binom{\tau}{j} a_\tau \underline{q}^{\tau-j}, \ a_j = 0 \text{ for } d < j. \tag{4}$$

The Bernstein basis polynomials are by construction non-negative for all $x \in Q$, i.e., $B_i^{(D)}(x) \geq 0, \forall i = 0, ..., D$. We underline that 0 is the multiindex with all components equal to 0. Without loss of generality, we can consider the domain of f to be the unit box $U = [0, 1]^n$, since any compact non-empty box in $\mathbb{R}^n$ can be mapped thereupon by an affine transformation. Hence, the expression of f as (1) can be simplified with

$$B_i^{(D)}(x) = \binom{D}{i} x^i (1 - x)^{D-i}, \ x \in U, \tag{5}$$

and

$$b_i^{(D)}(f) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{D}{j}} a_j, \ 0 \leq i \leq D. \tag{6}$$

We highlight two important properties of Bernstein polynomials, namely, the *endpoint interpolation property*

$$b_{\hat{i}}^{(D)}(f) = f\left(\frac{\hat{i}}{D}\right),$$

for some $\hat{i}$, where $0 \leq \hat{i} \leq D$ satisfy $\hat{i}_j \in \{0, D_j\}, j = 1, ..., n$, and the *enclosing property* (Garloff, 1986)

$$\min_{0 \leq i \leq D} b_i^{(D)}(f) \leq f(x) \leq \max_{0 \leq i \leq D} b_i^{(D)}(f),$$

for all $x \in U$.

2.2 Convergence and tightness of approximation bounds

The *range* of f on Q is defined by $R(Q) := [\min_{x \in Q} f(x), \max_{x \in Q} f(x)] =: [\underline{f}, \overline{f}]$, while the *enclosure bound* of the Bernstein form is defined by $E^{(D)}(f, Q) := [\min_{0 \leq i \leq D} b_i^{(D)}(f), \max_{0 \leq i \leq D} b_i^{(D)}(f)]$. In this subsection, we provide linear and quadratic convergence of the interval range $R(Q)$ of a polynomial f to its enclosure bound. For this purpose, we define the (Hausdorff) distance H between the two intervals $E^{(D)}(f, Q)$ and $R(Q)$ as

$$H(E^{(D)}(f, Q), R(Q)) =$$
$$\max \left\{ \left| \min_{0 \leq i \leq D} b_i^{(D)}(f) - \underline{f} \right|, \left| \max_{0 \leq i \leq D} b_i^{(D)}(f) - \overline{f} \right| \right\}.$$

Theorem 1. (Hamadneh, 2018, Theorem 4.3.1) For $D \geq d$, the following bound holds for the overapproximation of the range $R(Q)$ of f over Q by the Bernstein form:

$$H\left(R(Q), E^{(D)}(f, Q)\right) \leq \frac{C}{D}, \tag{7}$$

where

$$C := \sum_{j=0}^{d} \sum_{\mu=1}^{n} [\max(0, j_\mu - 1)]^2 |s_j|, \tag{8}$$

and the coefficients s_j are given by (4).

The quadratic convergence with respect to subdivision has been studied in Malan et al. (1992). Repeated bisection of $U^{(0,1)} := U$ in all n coordinate directions results at subdivision level

$1 \leq l$ in subboxes $U^{(l,\nu)}$ of edge length 2^{-l}, $\nu = 1, \dots, 2^{nl}$, see Garloff (1986), Fischer (1990). An n-variate polynomial f can be represented as

$$f(x) = \sum_{i=0}^{d} b_i^{(d,\nu)}(f)\; B_i^{(d,U^{(l,\nu)})}(x), \text{ for } x \in U^{(l,\nu)}, \tag{9}$$

where $b_i^{(d,\nu)}(f)$ are the *Bernstein coefficients* of degree d over $U^{(l,\nu)} = [\underline{q}_{(l,\nu)}, \overline{q}_{(l,\nu)}]$.

Theorem 2. (Malan et al., 1992, Theorem 2) For each $1 \leq l$ it holds

$$H\left(R(U), E^{(D)}(f, U^{(l,\nu)})\right) \leq C'(2^{-l})^2, \tag{10}$$

where C' is a constant which can be given explicitly, and independently of l.

We define the *grid point* of the μ^{th} component of x by $x_i^{(D)}$,

$$x_{i,\mu}^{(D)} = \underline{x}_\mu + \frac{i_\mu}{D_\mu}(\overline{x}_\mu - \underline{x}_\mu), \; \mu = 1, \dots, n. \tag{11}$$

Corollary 3. If $D_\mu \geq 2$, by [Garloff (1986), p. 42] for all i, $0 \leq i \leq D$, the following bound holds $|\min f(x_i^{(D)}) - \min b_i^{(D)}(f)| \leq \frac{C}{D}$, where C is the constant (8).

3. TENSORIAL CERTIFICATES OF POSITIVITY

It may be the case where we have positive polynomials over a box in the monomial form, but they have non-positive Bernstein coefficients.

By denoting with $(O^{(D)}(f))$ the ordered list of Bernstein coefficients of a (multivariate) polynomial f of degree D, we follow the definition of Bernstein CP, $CP(O^{(D)}(f))$, given in Leroy (2008), $\mu = 1, ..., n$,

$$CP(O^{(D)}(f)) := \begin{cases} b_i^{(D)}(f) \geq 0, & 0 \leq i \leq D \\ b_{\hat{i}}^{(D)}(f) > 0, & \hat{i}_\mu \in \{0, D_\mu\}. \end{cases}$$

$CP(O^{(D)}(f))$ implies that f is positive over U, precisely, the expression of f in the Bernstein basis of degree D over U provides CP for f over U.

By raising the degree D of Bernstein high enough, from Theorem 1, the minimum Bernstein coefficient of f converges to the minimum of the range of f, and consequently obtain the following theorem.

Theorem 4. Consider a polynomial function f of degree d defined on the unit interval U. Moreover, consider its Bernstein form of some degree D. Then, f is positive on U if and only if it satisfies CP with

$$D > \frac{C}{\underline{f}},$$

where $\underline{f}$ is the minimum of $f(x_i^{(d)})$ and C is given in (8).

In this paper, we focus on CP by elevating the degree of BPs. Nevertheless, for completeness, we present the analogous approximation result that concerns the subdivision scheme. The proof is omitted as an immediate consequence of Theorem 2.

Theorem 5. Let f be a (monomial form) polynomial of degree d, positive on U. Assume that $2^l > \frac{\sqrt{C'}}{\sqrt{\underline{f}}}$, where $\underline{f}$ is the minimum value of $f(x_i^{(d)})$ over U. Then f satisfies the local CP associated to the subdivision level of $U^{(l,\nu)}$.

4. STABILITY ANALYSIS FOR THE TENSORIAL BERNSTEIN CONTROL SYSTEMS

We consider systems whose dynamics is captured by polynomials. We investigate the stabilization problem around an equilibrium point x_0 of the vector field, using polynomial state feedback. Specifically, $x \in \mathbb{R}^n$, and the affine control system is given by

$$\dot{x} =: F_u(x) = p(x) + g(x)u(x), \tag{12}$$

where the vector field $F_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by the drift $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and the control $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with the input matrix function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$.

If there are u and V such that the negative Lie derivative $\mathcal{L}_{F_u}(V)$ is positive semidefinite in the neighborhood of the equilibrium, then we will say there exists a stabilizing control for F_u.

Definition 6. (Khalil (2002)) Let x_0 be an equilibrium point for (12) and let $A \subseteq \mathbb{R}^n$ be a box containing the interior point x_0. Let $V : A \to \mathbb{R}$ be a continuously differentiable function such that $V(x_0) = 0$,

$$V(x) > 0, \; \forall x \in A \setminus \{x_0\},$$

$$\mathcal{L}_{F_u}(V)(x) = -\frac{\partial V}{\partial x}(x)F_u(x) > 0, \; \forall x \in A \setminus \{x_0\}, \tag{13}$$

where $\mathcal{L}$ denotes the (negative) Lie derivative. Then V is a Lyapunov function for F_u.

Suppose all p, g, u, V are given in the monomial form of degrees d_p, d_g, d_u, d_V, respectively. Then $\mathcal{L}_{F_u}(V)(x)$ is also given in the monomial form of degree

$$d_{LF} = \max\{d_p + d_V - 1, d_g + d_u + d_V - 1\}. \tag{14}$$

Without loss of generality, we assume the candidate Lyapunov function is a polynomial expressed in the monomial form of degree d_V and $x_0 = 0$, where $V(x_0) = 0$. Consequently, the Bernstein form of $V(x)$ of degree $D_V \geq d_V$ is given analytically as in (1).

4.1 Bernstein Arithmetic Operations

In this subsection, we provide a way to express the different operations in the Lyapunov decrease conditions (13) in Bernstein basis, namely, differentiation, multiplication and addition.

Remark 7. (Farouki and Rajan (1987), degree elevation) The Bernstein basis of degree D can be expressed in terms of those of degree $D + D^*$, where $D^* = (D_1, ...D_\mu - 1, ...D_n)$, $\mu \in \{1, ..., n\}$, $0 \leq i \leq D$, as

$$B_i^{(D)}(x) = \sum_{j=i}^{i+D^*} \frac{\binom{D}{i}\binom{D^*}{j-i}}{\binom{D+D^*}{j}} B_j^{(D+D^*)}(x). \tag{15}$$

Lemma 8. (Farin (1986)) Consider $V(x)$ of degree D_V is given in the Bernstein form. The Bernstein coefficients of $\frac{\partial V}{\partial x_\mu}(x)$ can be calculated by taking linear combinations of $b_i(V)$ of degree $(D_1, ..., D_\mu - 1, ...D_n) = D^*$, i.e.,

$$V'_\mu(x) := \frac{\partial V}{\partial x_\mu}(x) = \sum_{i=0}^{D} b_i^{(D)}(V)\frac{\partial B_i^{(D)}}{\partial x_\mu}(x) \tag{16}$$

$$= \sum_{i \leq D^*} D_\mu(b_{i_1,...i_\mu+1,...i_n} - b_i) \cdot B_i^{(D^*)}(x).$$

Hence, for $i = 0, ..., D^*$,

$$b_i^{(D^*)}(V'_\mu) = D_\mu(b_{i_1,...i_\mu+1,...i_n} - b_i). \tag{17}$$

Corollary 9. (Farouki and Rajan (1988)) Let $p(x)$ be a Bernstein polynomial of degree D_p, and $f(x)$ be of degree D^*. Then, it holds that

$$p(x)\cdot f(x) = \sum_{i=0}^{D_p+D^*}\Bigg(\sum_{j=\max(0,i-D^*)}^{\min(D_p,i)} \frac{\binom{D_p}{j}.\binom{D^*}{i-j}}{\binom{D_p+D^*}{i}}\cdot b_j^{(D_p)}(p) b_{i-j}^{(D^*)}(f)\Bigg) B_i^{(D_p+D^*)}(x)$$

$$= \sum_{i=0}^{D_p+D^*} b_i(p\cdot f) B_i^{(D_p+D^*)}(x). \tag{18}$$

Remark 10. For $x \in Q$, let the polynomials $p(x)$ and $f(x)$ be in the Bernstein form of the same degree D. Then, we have

$$p(x) + f(x) = \sum_{i=0}^{D}(b_i^{(D)}(p) + b_i^{(D)}(f))B_i^{(D)}(x). \tag{19}$$

Remark 11. The number of Bernstein coefficients of the tensorial BP of degree $D = (D_1, ..., D_n)$ is $N := \prod_{\mu=1}^{n}(D_\mu + 1)$.

Example 12. Let $p(x_1, x_2) = 4x_1^2x_2^2$ and $f(x_1, x_2) = 10x_1x_2^2$ be of degree $d := (2, 2)$ given over the box $Q = [2, 5] \times [1, 3]$. By the representation (3), we can compute the ordered list of Bernstein coefficients for both p and f. For $i_1, i_2 = (0, 0), ..., (2, 2)$, we put the list of coefficients in two $3 \times 3-$matrices of Bernstein coefficients. It follows that the summation of the corresponding Bernstein coefficients of degree d gives the Bernstein coefficients of $p(x) + f(x)$, where each b_{i_1,i_2} in p is added to the corresponding b_{i_1,i_2} in f,

$$\{b_{i_1,i_2}(p) + b_{i_1,i_2}(f)\} = \begin{pmatrix} -4 & -12 & -36 \\ 5 & 15 & 45 \\ 50 & 150 & 450 \end{pmatrix}.$$

Remark 13. Let $p(x)$ and $V'_\mu(x)$ be of degree D_p and D^*, respectively. Then, we have

$$p + V'_\mu = \sum_{i=0}^{D_p}\Bigg(b_i^{(D_p)}(p) + \sum_{j=\max(0,i-D_p+D^*)}^{\min(D^*,i)} \frac{\binom{D^*}{j}.\binom{D_p-D^*}{i-j}}{\binom{D_p}{i}} b_j^{D^*}(V'_\mu)\Bigg)B_i^{(D_p)}. \tag{20}$$

Theorem 14. (Smith, 2012, Theorem 3.3) Let $f(x)$ be a polynomial in Bernstein form of any degree D. Then its monomial form is

$$f(x) = \sum_{i=0}^{D} a_i x^i,$$

where

$$a_i = \sum_{j=0}^{i}(-1)^{i-j}\binom{D}{i}\binom{i}{j} b_j^{(D)}, \;\; 0 \le i \le D.$$

4.2 *Bernstein Coefficients for Lyapunov and Controller*

Let us assume that there exist polynomials $u(x)$ and $V(x)$ so that a stabilizing control for F_u exists. In this section, we provide a method for finding $V(x)$ within a finite number of Bernstein coefficients, and computing the control law $u(x)$. We refer to d_g and D_g to be the degrees of g in the monomial form and the Bernstein form respectively. By partition (subdivision of intervals) of the box Q around the equilibrium such that the equilibrium point x_0 is equal zero, then we assume that $b_0^{D_V}(V) = 0$ is the corresponding Bernstein coefficient of x_0.

Our goal is to find the Bernstein coefficients such that the Lyapunov decrease conditions, i.e., the negative Lie derivative is positive semi-definite. We define $D^* = D_V - 1$ and define the product $\sum_{\mu=1}^{n} V'_\mu g_{\mu r}(x) =: VG_r(x), \forall r \in \{1, ..., m\}$, and let it be in the Bernstein form of degree $D_G + D^*$. Following (17), (18) and (20), the polynomial $\mathcal{L}_{F_u}(V)(x)$ can be rearranged with unknown Bernstein coefficients for u as

$$\mathcal{L}_{F_u}(V)(x) = \sum_{\mu=1}^{n}\sum_{i=0}^{D_p+D^*} b_i(V'_\mu p_\mu)B_i^{(D_p+D^*)}(x) + \sum_{j=0}^{D_u} b_j(u_1)B_j^{(D_u)}(x) \sum_{i=0}^{D^*+D_G} b_i(VG_1)B_i^{(D^*+D_G)}(x) + ... + \sum_{j=0}^{D_u} b_j(u_m)B_j^{(D_u)}(x) \sum_{i=0}^{D^*+D_G} b_i(VG_m)B_i^{(D^*+D_G)}(x). \tag{21}$$

For any $j_0 = 0, ..., D_u$ and $i_0 = 0, ..., D^* + D_G$, we define the dot product between $b_{j_0}(u)$ and $b_{i_0}(VG)$ as

$b_{j_0}(u).b_{i_0}(VG) := b_{j_0}(u_1)b_{i_0}(VG_1)+...+b_{j_0}(u_m)b_{i_0}(VG_m)$.

By noting the tensor product of Bernstein basis between u, V and G, we can rewrite $\mathcal{L}_{F_u}(V)(x)$ in (21) as

$$\begin{pmatrix} b_0(u).b_0(VG) & ... & b_0(u)b_{D^*+D_G}(VG) \\ b_1(u).b_0(VG) & ... & b_1(u)b_{D^*+D_G}(VG) \\ . & & . \\ . & & . \\ . & & . \\ b_{D_u}(u).b_0(G) & ... & b_{D_u}(u)b_{D^*+D_G}(VG) \end{pmatrix}.$$

$$\begin{pmatrix} B_0^{D_u}(x).B_0^{D^*+D_G}(x) & ... & B_{D_u}^{D_u}(x)\cdot B_0^{D^*+D_G}(x) \\ . & & . \\ . & & . \\ . & & . \\ B_0^{D_u}(x).B_{D^*+D_G}^{D^*+D_G}(x) & .... & B_{D_u}^{D_u}(x)\cdot B_{D^*+D_G}^{D^*+D_G}(x) \end{pmatrix}.$$

For $i_0 \in \{0, ..., D_p+D^*\}$, we set $b_{i_0}(V'_1p_1)+...+b_{i_0}(V'_np_n) =: b_{i_0}(Vp)$. Hence, we can write the corresponding Bernstein coefficients of $\mathcal{L}_{F_u}(V)(x)$ as follows.

$$b_i(\mathcal{L}_{F_u}(V)) = \begin{pmatrix} b_0(Vp) \\ . \\ . \\ . \\ b_{D^*+D_p}(Vp) \end{pmatrix} + \begin{pmatrix} b_0(u) & ... & b_{D_u}(u) \\ b_0(u) & ... & b_{D_u}(u) \\ . & & \\ . & & \\ . & & \\ b_0(u) & ... & b_{D_u}(u) \end{pmatrix}. \begin{pmatrix} b_0(VG) & ... & b_{D^*+D_G}(VG) \\ . & & . \\ . & & . \\ . & & . \\ b_0(VG) & ... & b_{D^*+D_G}(VG) \end{pmatrix} \tag{22}$$

where for all i_0, j_0, $b_{j_0}(u) = (b_{j_0}(u_1), ..., b_{j_0}(u_m)$ and $b_{i_0}(VG) = (b_{i_0}(VG_1), ..., b_{i_0}(VG_m))^T$. By (18), the Bernstein degree of $\mathcal{L}_{F_u}(V)(x)$ is given as

$$D_{LF} = \max\{D_p + D^*, D_G + D_u + D^*\}. \tag{23}$$

Define

$$Y = \begin{pmatrix} b_0(u) \;...\; b_{D_u}(u) \\ b_0(u) \;...\; b_{D_u}(u) \\ . \\ . \\ b_0(u) \;...\; b_{D_u}(u) \end{pmatrix}, B =: \begin{pmatrix} b_0(Vp) \\ . \\ . \\ . \\ b_{D_p+D^*}(Vp) \end{pmatrix} \quad (24)$$

and

$$J = \begin{pmatrix} b_0(VG) \;...\; b_{D^*+D_G}(VG) \\ . \qquad\qquad . \\ . \qquad\qquad . \\ . \qquad\qquad . \\ b_0(VG) \;...\; b_{D^*+D_G}(VG) \end{pmatrix}. \quad (25)$$

Let C^{LF}, C^V, be the constants derived from Theorem 4 by application of (8) to $\mathcal{L}_{F_u}(V)(x)$ (of degree $d_{LF} \leq D_{LF}$) and $V(x)$, respectively. By considering (21) and (22), we are in the position to present the following result, which is the main contribution of the paper. Roughly, the theorem associates the existence of polynomial controllers and Lyapunov functions for polynomial dynamical systems, to the existence of Bernstein polynomial controllers and Lyapunov functions. The third condition of Theorem (15) associates the conditions with the corresponding algebraic inequalities, paving the way for constructive methods.

Theorem 15. Consider the system (12) with some feedback law $u(x)$ be defined on the unit box U. Let d_V and d_u be two integers and d_{LF} is given in (14). The following are equivalent:

(1) There exists a polynomial controller $u(x)$ of degree d_u and a polynomial Lyapunov function $V(x)$ of degree d_V (both in the monomial form) such that the equilibrium is asymptotically stable with this control.
(2) There exists a polynomial controller $u(x)$ in the Bernstein form of degree $D_u \geq d_u$ and a polynomial Lyapunov function $V(x)$ in the Bernstein form of degree $D_V \geq d_V$ with $D_V > \frac{C^V}{\min V(x_i^{(d_v)})}$ such that the closed-loop system is asymptotically stable over U and $D_{LF} > \frac{C^{LF}}{\min \mathcal{L}_{F_u}(V)(x_i^{(d_{LF})})}$.
(3) Let Y and J, B be defined in (24), (25). Then, the set of inequalities $Y \cdot J + B > 0$ defines a non-empty set with respect to Bernstein coefficients, where $Y, J \in \mathbb{R}^{D_{LF} \times D_{LF}}$ and $B \in \mathbb{R}^{D_{LF}}$, and D_{LF} is given in (23).

Proof. (1) $\Leftrightarrow$ (2). Suppose there exists a positive Lyapunov function $V(x)$ be of degree d_V and there exists a stabilizing control of degree d_{F_u} over U. Then, by Theorem 4 there exists a Lyapunov function $V(x)$ in the Bernstein basis of degree $D_V > \frac{C^V}{\min V(x_i^{(d_V)})}$ such that $V(x)$ of degree D_V is positive. We can compute coefficients for u of degree $d_u \leq D_u$, and then $u(x)$ is of the same degree in the Bernstein form. By multiplying the Bernstein form of $V'(x)$ with the corresponding Bernstein form of $g(x)$, using (18), the degree of $V'G$ is $D^* + D_G$, hence $(Y.J)$ is of degree $D_u + D^* + D_G$. Similarly, for $V'p$ the degree is $D^* + D_p$. It follows that $(\mathcal{L}_{F_u}(V))(x)$ is of degree D_{LF} (23). Again, applying Theorem 4 with its grid points to $(\mathcal{L}_{F_u}(V))(x_i^{(d_{LF})})$ certifies the positivity of $(\mathcal{L}_{F_u}(V))$ with degree $D_{LF} > \frac{C^{LF}}{\min \mathcal{L}_{F_u}(V)(x_i^{(d_{LF})})}$. The converse follows by Theorem 14.

(2) $\Leftrightarrow$ (3). Note that, if all Bernstein coefficients of $\mathcal{L}_{F_u}(V)(x)$ are positive, then $\mathcal{L}_{F_u}(V)(x)$ is positive for all $x \in U$. By (24) and (25), the problem of computing Bernstein coefficients for $\mathcal{L}_{F_u}(V)(x)$ is reduced to the following system:

$$Y \cdot J + B > 0. \quad (26)$$

Note that adding the corresponding Bernstein coefficients of two terms to each other requires the same number of coefficients. Hence, we elevate the number of coefficients of $Y.J$ with the corresponding number of coefficients of B using (15). By solving the system (26), we can then build $u_r(x)$, $r = 1, ..., m$, in the Bernstein form and $V(x)$ of degree D_V, whereas Theorem 14 can be used to transform the controllers to their monomial form. □

Remark 16. The relations of Theorem 15-(3) are bilinear algebraic inequalities that are clearly nonlinear. There are several possibilities to relax these problems to a set of solvable conditions, using, e.g., bilinear programming approaches Gronski (2019) and results on weak solvability of interval inequalities Hladı et al. (2013). These extensions are definitely not trivial, and are not explored in the current article, however they are in our future research plans.

Remark 17. In Theorem 15, if the control strategy is fixed, the conditions correspond to the analogous result stated in Sloth and Wisniewski (2014), however for systems in the current article that are defined over boxes. In this special case, the conditions in Theorem 15-(3) become linear, i.e., if we input Bernstein coefficients for V, then B in (24) and J (25) can be computed. Therefore, we solve the system $Y \cdot J + B > 0$ in order to find the Bernstein coefficients of u in Y. Verifying whether a nonempty set is possible, e.g. by an application of Farkas' lemma is given in Leth et al. (2017).

Remark 18. Similarly, when the candidate Lyapunov function is fixed, the existence conditions of Theorem 15 become sufficient only, however, conditions in Theorem 15-(3) become linear and can be readily used to retrieve a stabilizing polynomial controller.

Example 19. Let, after the linear transformation to $U = [0, 1]^2$, $x = (x_1, x_2) \in U$ and

$$\dot{x} = \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \end{pmatrix} + \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \\ g_{31}(x) & g_{32}(x) \end{pmatrix} \cdot \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

be of degree $D = (1, 1)$, with $p_1(x) = x_1$, $p_2(x) = x_1 - 2x_2$, $p_3(x) = -x_1 x_2$, and $g_{11}(x) = g_{32}(x) = 1$, $g_{12}(x) = g_{21}(x) = g_{22}(x) = g_{31}(x) = 0$.

In case of unknown coefficients for both V and u, we compute the Bernstein coefficients of p by (6),

$$\{b_{i_1 i_2}(p_1)\} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \{b_{i_1 i_2}(p_2)\} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix},$$
$$\{b_{i_1 i_2}(p_3)\} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly, we compute the coefficients for all $g(x)$ of (elevated) degree 1. We follow $b_i(\mathcal{L}_{F_u}(V))$ with unknown coefficients for V and u, and to reach finally to Y, B, J, given in (24), (25). $Y \cdot J + B > 0$ is bilinear with respect to Bernstein coefficients of u and V. Following Remark 18, we can input Bernstein coefficients for the candidate Lypunov function V of degree one,

$$\{b_{i_1 i_2}(V)\} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

Therefore, $V(x) = (1 - x_1)x_2 + x_1(1 - x_2) + 2x_1x_2$. Note that the set of coefficients $\{b_{i_1i_2}(g_{12})\} = \{b_{i_1i_2}(g_{21})\} = \{b_{i_1i_2}(g_{22})\} = \{b_{i_1i_2}(g_{31})\} = (0)$, and

$$\{b_{i_1i_2}(g_{11})\} = \{b_{i_1i_2}(g_{32})\} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Computing $b_{i_1i_2}(\mathcal{L}_{F_u}(V))$ with Y, B and J, it follows that the problem is reduced to finding Y such that $Y \cdot J + B > 0$, where

$$J = \begin{pmatrix} (0,0)^T & (1,1)^T \\ (1,1)^T & (2,2)^T \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 4 & -2 \end{pmatrix}.$$

The set $Y \cdot J + B > 0$ is nonempty, so any element of the set can be used to construct a stabilizing controller. For example, by choosing

$$Y = \begin{pmatrix} (1,-1) & (-1,-1) \\ (0,1) & (1,1) \end{pmatrix},$$

we can build $u(x)$ in the Bernstein basis as follows:

$$u_1(x) = (1 - x_1)(1 - x_2) - (1 - x_1)x_2 + x_1x_2.$$
$$u_2(x) = -(1-x_1)(1-x_2) - (1-x_1)x_2 + x_1(1-x_2) + x_1x_2.$$

5. CONCLUSIONS

In this article, we presented a method for finding a Lyapunov function and the associated polynomial controller over boxes. In particular, we developed new certificates of positivity for the Lyapunov decrease condition using Bernstein basis expansions. We provided existence conditions of polynomial Lyapunov functions and controllers, *that are of a maximum degree in the monomial base*. These conditions are only on the Bernstein coefficients and not the state space variables. To reach our result, we used the fact that multiplication, addition, degree elevation and differentiation of functions expressed in Bernstein basis leads to functions still expressed in the Bernstein basis. Last, we provided algebraic inequalities, whose solvability implies existence of a stabilizing controller. These conditions are in principle nonlinear, however they become linear when either the Lyapunov candidate or the controller are fixed.

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