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Observer design for interconnected systems with model reduction and unknown inputs

Tim Steiner * **Steven Liu** *

* *Institute of Control Systems, Department of Electrical and Computer Engineering, University of Kaiserslautern, Kaiserslautern, Germany(e-mail: steiner|liu@eit.uni-kl.de).*

Abstract: In this paper a combination of distributed parameter systems and lumped parameter systems is investigated, also known as interconnected system. In particular, the heat distribution and the influence of single chips on a base plate is of interest, here in context of insulated-gate bipolar transistors. Only temperature measurements on the base plate are available. A method is presented with which the temperature inside the chip can be estimated. A combination of model reduction and unknown input observer is utilized.

Keywords: interconnected systems, proper orthogonal decomposition, model reduction, boundary control, method of snapshots, unknown input observer

1. INTRODUCTION

In recent years, the complexity of the systems to be controlled has increased continuously and interconnected systems are one example. In context of this work interconnected systems refers to the connection between lumped parameter system (LPS) and distributed parameter system (DPS) in form of partial differential equations (PDEs). The reason for the shown methods is the heat distribution in insulated-gate bipolar transistors (IGBTs). Here, the junction temperature inside the chips is estimated with the help of measurements on the base plate. The base plate is modeled as DPS and the different stack of chips as LPS. An unknown input observer is necessary, as the junction temperatures of the IGBTs aren't measurable due to a lack of measurements inside the chips.

Some recent investigations into control of interconnected PDE-ODE systems where done by Susto and Krstic (2010) with focus on cascaded systems controlled with backstepping. The PDE is considered either as dynamic of the actuator or sensor. The work of Yilmaz and Basturk (2019) considers more an output feedback control with periodic disturbances. In this special case the PDE is the actuator dynamic. The main goal is in both publications to stabilize or control the LPS. Both approaches handle the PDE in an infinite dimensional way and do not consider model reduction. For the observer design presented by Zobiri et al. (2017) an infinite dimensional approach and information about the input data are considered.

Finite element methods (FEM) are used to solve PDEs for fluid simulation or structural analysis Reddy (2004). However, these solutions are high dimensional and not directly applicable for practical controller and observer design, real time simulation or online optimization. Different methods for controller and observer design of PDEs have been introduced. One approach is the direct use of the PDE for the controller design as presented by Altmüller (2014). Therefore the infinite dimensional system is used to build a model predictive control (MPC) directly or for a state feedback control (see Deutscher (2012)). Model order reduction (MOR) is another approach before designing the controller considered by Hakenberg (2015). Here the reduced model is used for mathematical system analysis as well as for the final design of the controller. The advantage of using a reduced model is the lower dimensionality. In the presented work MOR is considered before the actual controller and observer design.

One commonly used model reduction technique is the proper orthogonal decomposition (POD), also known as Karhunen-Loeve decomposition or principal component analysis. This method is based on finding a reduced set of basis functions for the spacial domain (see Liang et al. (2002)). The spacial part is split up from the temporal part to generate a reduced system of ODEs. One practical adaption is the *method of snapshots*. The system dynamics are reflected by the used data sets at certain time instances. The accuracy of the results are system and data dependent. The POD approach is used in combination of MPC based controller design, for example by Hakenberg and Abel (2013).

The proposed interconnected reduced system has disadvantages for observer design due to missing detectable input in the measurements. The so-called observer matching condition is not fulfilled, if the unknown input is in the LPS and the measurement is in the DPS. Typically the observer design for unknown input observer rely on this condition introduced in Guan and Saif (1991). One method to overcome this was presented by Floquet et al. (2007) and by Zhu (2012) and is used in the presented work.

This paper presents a combination of model reduction and observer design in context of temperature estimation. The method for temperature estimation is based on measurements on the DPS and combines unknown input data at the LPS part. Finally the junction temperature in IGBTs can be estimated without the visibility of the unknown input in the measurements.

1.1 Outline

The following section explains the POD based model reduction. This is the main part for modelling the PDE of the interconnected system. Some special acknowledgement due to the boundary input handling is done. Also the main idea of lumped modelling is introduced in this section. Section 3 introduces the coupling between the lumped model and the reduced PDE model. A workflow for temperature estimation in interconnected systems is presented, especially with unknown input data. Finally some simulation results are provided for the model reduction as well as for the designed observer.

2. MODEL ORDER REDUCTION

In this section all preliminaries for POD-Galerkin based model reduction are introduced. The following model reduction is considered for a heat transfer problem. The temperature ϑ in a solid material is given by

$$\partial_t \vartheta(x,t) = \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} \vartheta(x,t)) \qquad \text{in} \quad Q = \Omega \times (0,T) \tag{1a}$$

$$\vartheta(x_D,t) = 0 \qquad \text{on} \quad \Sigma_D = \Gamma_D \times (0,T) \tag{1b}$$

$$\partial_x \vartheta(x_N,t) = v(t) \qquad \text{on} \quad \Sigma_N = \Gamma_N \times (0,T) \tag{1c}$$

$$\vartheta(x,0) = \vartheta_0(x) \qquad \text{in} \quad \Omega \tag{1d}$$

with the gradient $\boldsymbol{\nabla}$, the divergence $\boldsymbol{\nabla} \cdot (\cdot)$, the thermal diffusivity α, the homogenous Dirichlet boundary u and the Neumann boundary $v(t)$. The temperature is initialized with $\vartheta_0(x)$. This model holds for the spacial domain Ω and the time domain $(0,T)$. Additionally we assume that the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ of Ω is piecewise differentiable.

In general, the analytical solution for a PDE isn't available, so we want to find the so called *weak solution*. The first step is to multiply (1a) with the test function $\phi \in H_0^1(\Omega)$ and then integrate over Ω. With partial integration on the right sided term, the weak formulation of (1) simplifies to

$$\int_\Omega \frac{\partial \vartheta}{\partial t} \cdot \phi(x)\,\mathrm{d}x = -\int_\Omega \alpha \boldsymbol{\nabla}\vartheta(x,t) \cdot \boldsymbol{\nabla}\phi(x)\,\mathrm{d}x - \int_\Gamma \alpha v(t) \cdot \phi(x)\,\mathrm{d}S\,. \tag{2}$$

The test function $\phi = 0$ on Γ_D due to the homogenous Dirichlet boundary. A function ϑ is called weak solution of (1), if it satisfy (2) for all test functions and the initial condition (1d). Next, we approximate ϑ by

$$\tilde{\vartheta}(x,t) \approx \sum_{i=1}^{N} a_i(t)\psi_i(x) \tag{3}$$

with the free selectable trial function ψ_i from $H_0^1(\Omega)$. Consider $\phi(x) = \psi(x)$ and (3) the approximation is called Galerkin method. The time mode $a_i(t)$ is later used for the reduced state space representation. We roughly introduced the main idea and a detailed mathematical discussion of Galerkin methods can be found in Fletcher (1984).

The previous steps help us to separate the PDE into a temporal and a spacial part. Later we use this separation for the model reduction.

2.1 Proper Orthogonal Decomposition

The POD method is well known in structural mechanics. In contrast to other model reduction methods, POD requires the solution of the differential equation. This can be realized with measurements as well as with high-resolution simulations. In this section the *method of snapshots* is considered to find the orthogonal set of basis functions ψ_i based on a high-resolution simulation. We define $\vartheta(x,t) = \vartheta_j(x)$ as solution of the PDE in the domain $\Omega \times (0,T)$ at some discrete time steps $j\Delta t$. The goal is to minimize the error between ϑ and the approximated model. The minimization problem can be simplified to

$$\begin{aligned} &\min_{\psi_i(x)} \frac{1}{n} \sum_{j=1}^{n} \left\| \vartheta_j(x) - \vartheta_j^\star(x) \right\|^2_{L^2(\Omega)} \\ &\text{s.t.} (\psi_i(x), \psi_i(x))_{L^2(\Omega)} = \delta_{ij} \\ &\vartheta_j^\star(x) = \sum_{i=1}^{N_{POD}} \langle \vartheta_j(x), \psi_i(x) \rangle_{L^2(\Omega)} \psi_i(x) \end{aligned} \tag{4}$$

for the inner product $\langle f,g \rangle_{L^2(\Omega)} = \int_{L^2(\Omega)} f \cdot g\,\mathrm{d}x$ and for n snapshots. For solving (4) the correlation matrix $K_{ij} = \frac{1}{N} (\vartheta_j(x), \vartheta_i(x))_{L^2(\Omega)}$ is build from the set of solutions of the PDE. Then the corresponding eigenvalues $\lambda_1 \geq ... \geq \lambda_l > 0$ and the eigenvectors $v_1, ..., v_l$ of K_{ij} are computed. Finally the reduced basis functions are

$$\psi_i(x) = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} (v_i)_j \vartheta_j(x).$$

Following the previous POD based model reduction steps combining (2) with (3) the unknown time modes $a_i(t)$ build a state space models with N_{POD} equations. It follows

$$\begin{aligned} \dot{a}(t) = &- \alpha \left\langle \boldsymbol{\nabla}\psi(x), \boldsymbol{\nabla}\psi(x) \right\rangle_{L^2(\Omega)} \cdot a(t) \\ &- \alpha \langle v(t), \tau_\Gamma(\psi(x)) \rangle_{L^2(\Gamma)} \end{aligned} \tag{5}$$

with the continuous trace operator $\tau_\Gamma : H^1(\Omega) \to L^2(\Gamma)$ providing the boundary of the domain Ω. A detailed mathematical description of POD and the *method of snapshots* can be found in Kunisch and Volkwein (2001).

It should be mentioned, that homogeneous boundary conditions are included in the basis functions due to the *method of snapshots*. If we have decomposed a solution with homogeneous boundary conditions, each resulting POD basis meets these boundary conditions either of the Dirichlet or Neumann type.

2.2 Lumped thermal models

In this work, we describe the lumped model as a equivalent current circuits description, which is a well known modeling approach for thermal systems. Therefore the temperature is mapped onto voltage, heat flux onto current as well as thermal resistors and capacitors onto equivalent electric parts. The lumped model is a rough discretization for the PDE introduced in (1a). We get the thermal resistance

$$R = \frac{l}{\kappa \cdot A}$$

with the constant thermal conductivity κ, the length l in heat transfer direction and the area A where the heat is transferred through. For the thermal capacitance of a fixed material we obtain

$$C = c_p \cdot \rho \cdot l \cdot A$$

where c_p is the specific heat capacitance, ρ is the density and $V = l \cdot A$ is the volume of the material. Finally a differential equation for the temperature behaviour between two solid materials is set up. We use this simplified model for 1D heat transfer in a homogenous material, if the distribution inside the material is not necessarily known. Combining the thermal resistance and capacity, the time evolution of the considered temperature ϑ_C is

$$\dot{\vartheta}_C = -\sum_{i=1}^{N} \frac{1}{R_i C}\vartheta_C + \sum_{i=1}^{N} \frac{1}{R_i C}\vartheta_i + f(t)$$

including all N neighbours with the connecting thermal resistors R_i. Additionally $f(t)$ represents possible other heat source inputs. A deeper description and mathematical derivation of lumped thermal modelling can be found for example in Wang (2014).

3. INTERCONNECTED SYSTEM CONTROL

In this section, the connection between the reduced PDE system and the LPS is discussed. This results from the problem of estimating the temperatures within an IGBT. The DPS represents the base plate and the LPS the chips of the IGBT and can be selected problem specific. An two dimensional system for the DPS and a first order LPS are used for simplicity reasons. The goal of this section is to get a reduced order model that enhances the following observer design regarding the amount of necessary calculations. A POD based approach is considered for the DPS part.

Fig. 1 shows an example of a possible interconnection between the DPS and a LPS. A dimensionless square with $l = 1$ is assumed for the DPS. A constant and known Dirichlet boundary is assumed for Γ_D with a constant temperature. The coupling between the DPS and LPS is at Γ_N with a width of 0.5. The remaining boundaries are assumed to be perfectly isolated, the heat flux is zero for Γ_{N0}. The input temperature T_{in} isn't measurable due to the constructional restrictions of IGBT chips.

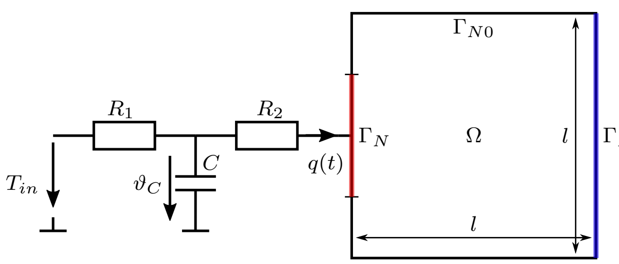

Fig. 1. Coupled DPS-LPS with temperature input at the RC model

For the controller design as well as for the observer design we combine both systems into one state space model. Therefore, the Neumann boundary at the DPS is used as connection, which is proportional to the heat flux density. The relation to the temperature gradient is

$$q(t) = -\kappa \boldsymbol{\nabla}\vartheta(x,t) \tag{6}$$

with the thermal conductivity κ. This is applicable to the reduced model with Neumann boundary. So the state space model (5) becomes

$$\begin{aligned}\dot{a}(t) =& -\alpha \left\langle \boldsymbol{\nabla}\psi(x), \boldsymbol{\nabla}\psi(x)\right\rangle_{L^2(\Omega)} \cdot a(t) \\ &+ \frac{\alpha}{\kappa}\langle q(t), \tau_\Gamma(\psi(x))\rangle_{L^2(\Gamma)}\end{aligned} \tag{7}$$

for the 1D rod shown in Fig. 1. On the other side the presented state space model for the LPS is obviously

$$\dot{\vartheta}_C = -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)\vartheta_C + \frac{1}{R_1 C}T_{in} + \frac{1}{R_2 C}\vartheta(0,t) \tag{8}$$

with the thermal resistances R_1 and R_2 and the thermal capacity C. To connect both models, we reformulate $q(t)$ with respect to ϑ_C as well as $\vartheta(0,t)$ with respect to the reduced model. The heat flux is simply

$$q(t) = \frac{\vartheta_C - \vartheta(0,t)}{R_2} \tag{9}$$

and adding the approximation for the DPS

$$q(t) = \frac{\vartheta_C - \sum_{i=1}^{N_{POD}} a_i(t)\psi_i(0)}{R_2} \tag{10}$$

follows. Similar steps are used for the other coupling, so that (8) becomes

$$\begin{aligned}\dot{\vartheta}_C =& -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)\vartheta_C \\ &+ \frac{1}{R_1 C}T_{in} + \frac{1}{R_2 C}\sum_{i=1}^{N_{POD}} a_i(t)\psi_i(0).\end{aligned} \tag{11}$$

Here the boundary temperature $\vartheta(0,t)$ is replaced by the approximation $\sum_{i=1}^{N_{POD}} a_i(t)\psi_i(0)$. Combining both models a state space model can be defined. The resulting state space model with the state vector

$$x = [a_1, ..., a_{N_{POD}}, \vartheta_C]^T \tag{12}$$

and input $u = T_{in}$ is

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{13}$$

The state matrix and the input matrix are separated into the DPS part with the reduced state variables $[a_1, ..., a_{N_{POD}}]$ and LPS part for the state ϑ_C and the coupling matrices combined to

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{14}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{15}$$

More precise the parts for the state matrix are

$$\begin{aligned}A_{11} =& -\alpha\langle \boldsymbol{\nabla}\psi(x), \boldsymbol{\nabla}\psi(x)\rangle_{L^2(\Omega)} \\ &- \frac{\alpha}{\kappa R_2}\left\langle \tau_\Gamma(\psi(x)), \tau_\Gamma(\psi(x))\right\rangle_{L^2(\Gamma)}\end{aligned} \tag{16a}$$

$$A_{12} = \frac{\alpha}{\kappa R_2}\int_\Gamma \tau_\Gamma(\psi(x))\,\mathrm{d}S \tag{16b}$$

$$A_{21} = \frac{1}{R_2 C}\psi(0) \tag{16c}$$

$$A_{22} = -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) \tag{16d}$$

and the input matrix is

$$B_1 = 0 \tag{17a}$$

$$B_2 = \frac{1}{R_1 C}. \tag{17b}$$

It should be mentioned that the boundary $\vartheta(x_D, t)$ is homogeneous and included in the reduced basis functions of the reduced DPS system.

At this point it should be mentioned, that a special treatment of resistor R_2 is necessary due to the connection between both systems. Because of the discretization for the *method of snapshots*, the resistance must also be adapted to the spatial distribution of Γ_N. For the simple case of an equidistant grid on the boundary the resistor R_2 can be replaced by

$$R_2^\star = \Delta\Gamma_N \cdot R_2 \tag{18}$$

for $\Delta\Gamma_N$ elements on the boundary. The thermal conduction between the capacitance C of the LPS and each point on the boundary Γ_N is the new $R_2^\star$.

3.1 Reduced Model Simulation

The accuracy of the reduced model determined with the POD-Galerkin method for the PDE is limited to the basis functions. The resulting shape of the basis function and therefore the reduced model depends on the data set used for the *method of snapshots*. By considering this constraint we want to use as much information as possible for the simulation data set. So for the model reduction we choose the following input for the DPS system

$$q(t) = \begin{cases} 1 & \text{for } 10 < t < 70 \\ 0 & \text{else.} \end{cases} \tag{19}$$

So most of the frequencies are included in this steplike input signal. The reduced basis is selected after the model reduction. Therefore the most significant basis functions are selected. The number N_{POD} of selected basis functions is determined by

$$\frac{\sum_{i=1}^{N_{POD}} \lambda_i}{\sum_{j=1}^{l} \lambda_j} > tol \tag{20}$$

where $\lambda_1 \geq ... \geq \lambda_l > 0$ are the eigenvalues of the correlation matrix from the *method of snapshots* in decreasing order of significance. The threshold $tol \in [0, 1]$ should be very close to 1. The selection of the reduced basis depends on the thermal diffusivity of (1a). For the following studies α is set to 0.1, κ is set to 1 and the right Dirichlet boundary is $\vartheta(x_D, t) = 0$.

For solving the initial DPS for the POD based model reduction, the *pde toolbox* provided by Matlab is used. The 2D plate with unit length is discretised into 122 spatial steps in each dimension. The simulation takes 100 s into account sampling time of 0.1 s . The simulations results are given in Fig. 2 for t = 50s. For the reduced model two POD modes are selected, so more than 99.9 % of the "information" defined in (20) is restored.

Next the DPS-LPS system is verified including the previously calculated reduced PDE system. Therefore we set $R_1 = R_2 = 10$ and $C = 1$ for the LPS system shown in Fig. 1. The temperature input T_{in} at the LPS is

$$T_{in} = 0.5 + 0.25 \sin(0.02\pi t). \tag{21}$$

The simulation time and sampling time are the same like before. For a better comparison the first three N_{POD} temporal modes of the simulated system are shown with the reduced model in Fig. 3. Each mode is weighted with its own eigenvalue for visualization reasons. The deviation between the reduced model and the simulated system is small with some small deviation for the higher modes. The relative error for the temporal modes is less than 1 %. The accuracy of the reduced model depends on the discretization grid used for the *method of snapshots*.

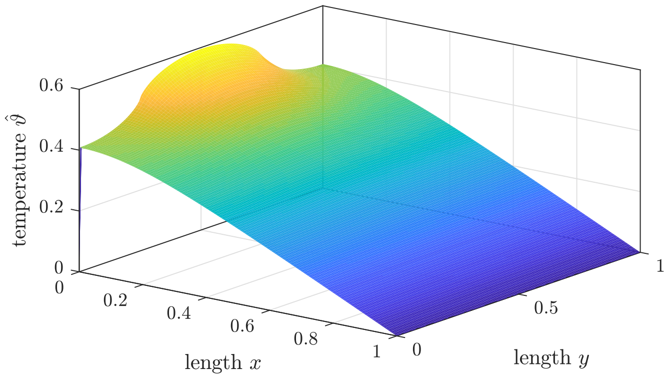

Fig. 2. Numerical solution of the DPS (1) with Neumann boundary input at t= 50s.

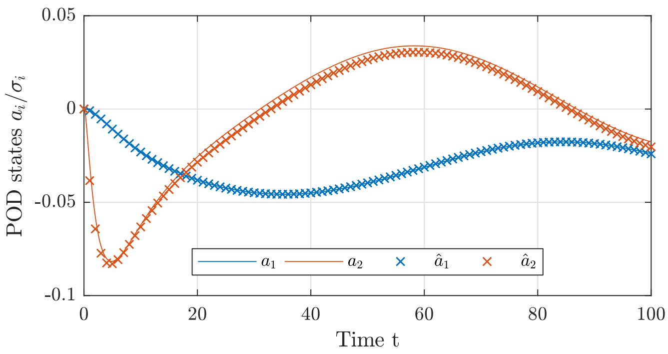

Fig. 3. Sequence of the first two temporal modes of the full model a_i and the reduced model $\hat{a}_i$.

The previous section introduced a thermal model combining a 2D DPS with a LPS. The resulting reduced order model can be used for online applications like control or observation tasks.

4. OBSERVER DESIGN

The previous derived interconnected model is used for the following observer design. The input is assumed to be unknown. Beside the state estimation the unknown input can be reconstructed with a reduced-order observer and a high-order sliding mode observer. Regarding the example introduced in Fig. 1 sensors are placed only at the DPS. The investigated system is

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{22a}$$
$$y(t) = Cx(t) \tag{22b}$$

with $u(t)$ assumed to be unknown. Additionally, system (22) is observable and controllable regarding the unknown input. For common unknown input observers the unknown input needs to be detected in the measurements. This is also known as the observer matching condition

$$\text{rank}(B) = \text{rank}(CB). \tag{23}$$

Due to the additional LPS system connected to the DPS, assumption (23) doesn't hold for our interconnected system. A reduced order observer with auxiliary outputs is introduced and later a sliding mode observer is used to estimate unmeasured auxiliary outputs. This part is based on Zhu (2012). Unlike in other publications, we only consider unknown inputs.

The auxiliary outputs should extend the output matrix in such a way that it has full rank regarding (23). For $\text{rank}(C) = p$ find the smallest integer $r_i (i = 1, 2, \cdots, p)$ such that

$$\begin{cases} C_i A^k B = 0, \text{ for } k = 0, 1, \cdots, r_i - 2 \\ C_i A^{r_i - 1} B \neq 0 \end{cases} \tag{24}$$

holds. Let $r_i \in \mathbb{N}$ be the relative degree of (24) and $(r_1, r_2, \cdots, r_p) \in \mathbb{N}^{1 \times p}$ be the vector relative degree of the system with respect to the unknown input. The new output matrix C_a is with some selected γ_i

$$C_a = \begin{bmatrix} C_1 \\ \vdots \\ C_1 A^{\gamma_1 - 1} \\ \vdots \\ C_p \\ \vdots \\ C_p A^{\gamma_p - 1} \end{bmatrix} = \begin{bmatrix} C_{a1} \\ \vdots \\ C_{ai} \\ \vdots \\ C_{ap} \end{bmatrix} \in \mathbb{R}^{\gamma \times n} \tag{25}$$

and $1 \leq \gamma_i \leq r_i$ holds, so it has full rank and

$$\text{rank}(B) = \text{rank}(C_a B)$$

holds. Also $\gamma = \sum_{i=1}^{p} \gamma_i$ needs to be fulfilled. A reduced order observer for the unknown input system can be set up for the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{26a}$$
$$y_a(t) = C_a x(t). \tag{26b}$$

It should be mentioned, that the invariant zeros of the triplets $\{A, B, C\}$ and $\{A, B, C_a\}$ are identical (cf. Floquet et al. (2007)). The system (22) is minimum phase and so the system with auxiliary outputs is minimum phase, i.e. with all invariant zeros in the left half plane for a continuous system.

An observer can be established for some symmetric positive definite matrix Q_a, if there exits a matrix L_a, a matrix F_a and a symmetric positive definite matrix P_a such that

$$(A - L_a C_a)^T P_a^T + P_a (A - L_a C_a) = -Q_a \tag{27a}$$
$$B^T P_a = F_a C_a \tag{27b}$$

holds. More information are given by Corless and Tu (1998).

The reduced observer is obtained from (Zhu, 2012, Theorem 1). For the available auxiliary output y_a the reduced observer is

$$\begin{aligned} \dot{\hat{z}}_2 =& (\bar{A}_{22} + \bar{K}_a \bar{A}_{12}) \hat{z}_2 + [\bar{K}_a (\bar{A}_{11} - \bar{A}_{12} \bar{K}_a) \\ &+ \bar{A}_{21} - \bar{A}_{22} \bar{K}_a] S_a^{-1} y_a \end{aligned} \tag{28a}$$

$$\hat{x} = W_a^T \begin{bmatrix} S_a^{-1} y_a \\ \hat{z}_2 - \bar{K}_a S_a^{-1} y_a \end{bmatrix} \tag{28b}$$

for (22) with the estimated state vector $\hat{x}$. Here $\bar{A}$ is the transformed state matrix obtained by the Smith orthogonal procedure. Further $\bar{K}_a = \bar{P}_{a3}^{-1} \bar{P}_{a2}^T$ and S_a is a invertible matrix so that $C_a = S_a \hat{C}_a$ and $\hat{C}_a \hat{C}_a^{\,T} = I$ holds. W_a is the orthogonal extended matrix of $\hat{C}_a$. The index numbers denote block decomposition.

Next we use a sliding mode observer to estimate the not measurable auxiliary outputs and enable the reconstruction of the unknown inputs. Therefore Zhu (2012) proposed the sliding mode observer

$$\begin{aligned} \dot{\hat{y}}_{ai,1} =& \hat{y}_{ai,1} - w_{i,1} \\ &\vdots \\ \dot{\hat{y}}_{ai,\gamma_i - 1} =& \hat{y}_{ai,\gamma_i} - w_{i,\gamma_i - 1} \\ \dot{\hat{y}}_{ai,\gamma_i} =& \hat{y}_{ai,\gamma_i + 1} - w_{i,\gamma_i} \\ \dot{\hat{y}}_{ai,\gamma_i + 1} =& - w_{i,\gamma_i + 1} \end{aligned} \tag{29}$$

where

$$\begin{aligned} w_{i,0} =& \hat{y}_{ai,1} - y_{i,1} \\ w_{i,j} =& \lambda_{i,j} |w_{i,j-1}|^{(\gamma_i - j + 1)/(\gamma_i - j + 2)} \\ & \text{sign}(w_{i,j-1}) \,, j = 1, 2, \cdots, \gamma_i, \gamma_i + 1 \end{aligned}$$

and with $\lambda_{i,j} (i = 1, \cdots, p; j = 1, \cdots, \gamma_i + 1)$ are all positive scalar numbers.

Finally the unknown input can be reconstructed. Here (Zhu, 2012, Theorem 3) is used. The reconstruction is

$$\hat{u} = (G^T G)^{-1} G^T [\hat{\xi}_a - \tilde{C}_a (A \hat{x})] \tag{30}$$

for the unknown input u with

$\tilde{C}_a = [(c_1 A^{\gamma_1 - 1})^T \; (c_2 A^{\gamma_2 - 1})^T \cdots (c_p A^{\gamma_p - 1})^T]^T, G = \tilde{C}_a B$ and $\hat{\xi}_a = [\hat{y}_{a1,\gamma_1 + 1} \cdots \hat{y}_{ap,\gamma_p + 1}]^T$.

The unknown input observer is introduced in this section. The complete mathematical derivation and proof can be found in Zhu (2012).

4.1 Simulation Results

In this section a simulation example is provided combining the interconnected DPS-LPS with the unknown input observer. We consider the following system

$$A = \begin{bmatrix} -0.6922 & -1.4210 & -0.1795 \\ -1.4210 & -6.7178 & -0.3715 \\ -0.1980 & -0.4022 & -0.2000 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$$
$$C = [-2.0050 \;\; -4.2085 \;\; 0]$$

being the solution from the previously reduced model simulation 3.1. For this system condition (23) isn't satisfied. So the output matrix including the auxiliary outputs is

$$C_a = \begin{bmatrix} -2.0050 & -4.2085 & 0 \\ 7.3681 & 31.1207 & 1.9233 \end{bmatrix},$$

if we choose $\gamma_1 = r_1 = 2$. For a certain

$$Q_a = \begin{bmatrix} 45.1190 & -0.0235 & 0.0006 \\ -0.0235 & 45.1195 & 0.0012 \\ 0.0006 & 0.0012 & 45.1215 \end{bmatrix}$$

L_a, F_a and P_a are available from Corless and Tu (1998). Further for the resulting C_a we selected S_a, $\hat{C}_a$ and W_a as described by Zhu (2012). Finally for the sliding mode observer the parameter $\lambda = [2.5, 20, 0.5]^T$ is chosen.

In Fig. 4 on the left, the error of the auxiliary output is given. Therefore the sliding mode observer was used to estimate the unmeasurable output $y_{a1,2}$. The estimated output is initialized with $\hat{y}_{a1,2}(0) = 0.3$. After some time, the error is small enough and the convergence is satisfied.

With the estimation of the auxiliary outputs, the state estimation is possible. Fig. 4 shows in the right plot all errors between the actual states and the corresponding estimation. The estimation of the state observer works as desired.

Finally the reconstruction of the unknown input is shown in Fig. 5. Therefore the estimation of $\dot{y}_{a1,3}$ and the state estimation is used. It should be mentioned that $\dot{y}_{a1,3}$ is provided by the sliding mode observer. Also the unknown input reconstruction is satisfactory.

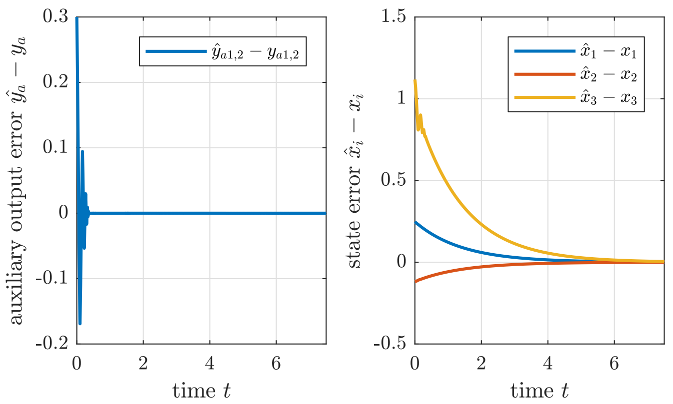

Fig. 4. Auxiliary output error of the unmeasured output (left); error between the estimation and actual states(right)

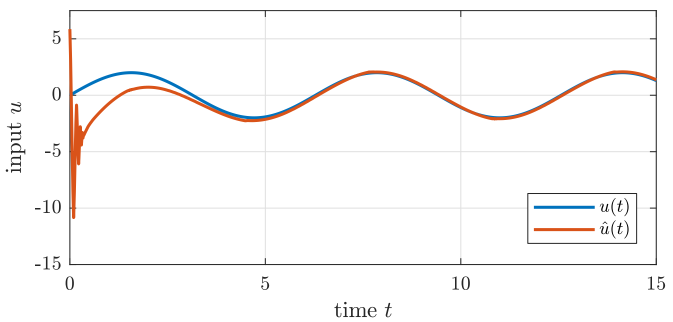

Fig. 5. Comparison of actual input and estimated input

5. CONCLUSION

In the presented work we considered the structure of interconnected DPS and LPS systems for an unknown input observer design. First the reduced model for the interconnected system is derived. Here the connection was realized on basis of the Neumann condition. The resulting structure was utilized for an unknown input observer design. Due to the reduced model the observer needs less auxiliary outputs regarding other methods. The shown framework is applicable for the estimation of internal temperatures in IGBTs, like the junction temperature.

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