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Balanced Stochastic Optimal Control of Uncertain Linear Systems with Constraints

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Abstract: This paper addresses finite-time horizon optimal control for discrete-time dynamics with additional stochastic disturbances. In contrast to most existing approaches to this problem, we also minimize the uncertainty of future states arising from stochastic disturbances and from an uncertain initial state. Thus, the optimal control strategy balances the minimization of the expected distances to a reference signal, and the minimization of the uncertainty respectively. As opposed to prior work, the optimization is formulated subject to possible disturbance feedback policies. This enables to solve one semi-definite program over H steps, instead of solving H problems over one step, and the resulting reduced complexity allows one to use the scheme in online and predictive control. The proposed method is applicable to time-varying state constraints (in the sense of chance constraints) as well as time-invariant input constraints.

Keywords: Stochastic optimal control, disturbance rejection, constrained systems, robust invariance, uncertain systems.

1. INTRODUCTION

This paper addresses the determination of stochastically robust control strategies over finite horizons for discrete-time linear systems, which are subject to additive disturbances and have uncertain initialization. For underlying Gaussian distributions in modeling both types of uncertainties, stochastic robustness is understood as the satisfaction of constraints with respect to a chosen likelihood, sometimes referred to as chance constraints, see e.g. Calafiore and El Ghaoui (2007) for more details. The main objective of the paper on hand is to provide an efficient computational scheme to obtain a stochastically robust stabilizing feedback control law minimizing both, the expected distances of the states to a reference signal and the uncertainty of the disturbed system.

The optimization over robust state feedback control laws for systems with bounded but not probabilistically modeled disturbances is well-studied, also with applications in online use. Formulating the control law as an affine function of the disturbances often leads to a synthesis problem formulated as quadratic program (QP) involving constraints on the states and inputs, see e.g. Ben-Tal et al. (2004); Goulart et al. (2006). Formulations for systems with stochastically modeled disturbances have been set out in van Hessem and Bosgra (2002) and extended to constrained problems by Oldewurtel et al. (2008). In these optimization-based control problems, optimality is understood as the minimization of a weighted compromise between the convergence of the state to a given reference and the control costs (measured in terms of the inputs) – the uncertainties are included in the optimization problems through the constraints, either to be satisfied strictly or as chance constraints. In approaches to stochastic optimal control, the state is in general probabilistically distributed according to the disturbance model, leading either to concepts of min-max optimization (often computationally unfavorable) or to the consideration of the convergence of expected values. The latter option mostly leads to QP problems, too, and is used in schemes of predictive control, e.g. Cannon et al. (2009). The well-established schemes of H_2- and H_∞-control, which minimize the effect of the disturbances onto the outputs in an infinite-horizon setting, see e.g. Doyle et al. (1988), are of less relevance here, since the focus is on online schemes with consideration of constraints. For a summarizing overview on optimal stochastic control, see Mesbah (2016) and the references therein.

The disturbance rejection in finite-horizon stochastic optimal control by use of stochastic reachable sets (without the formulation of min-max optimization problems) was first proposed in Asselborn and Stursberg (2015). That approach determined time-varying affine control laws to shape the state distribution in order to drive the system eventually into a target set with high probability. In Asselborn and Stursberg (2016), this concept was extended to switched linear systems considering state and input constraints. In both references, the optimization is over affine state feedback policies, and the computation of an H-step optimal control policy is divided into H many one-step conic optimization problems, which are solved iteratively. In these approaches, optimality was understood as the minimization of the uncertainty of the stochastically distributed state, a new aspect in stochastic optimal control, differing from the previously cited approaches.

The paper on hand now aims at linking the methods proposed by Asselborn and Stursberg (2015) and by Goulart et al. (2006), in order to formulate one H-step stochastic optimal control problem, which optimizes over disturbance feedback policies and additionally minimizes the uncertainty of the state.

The paper is organized as follows: Section 2 introduces some notation used later, and Sec. 3 specifies the system class and the control problem. Section 4 proposes the method for controller synthesis, which is then illustrated by a simulation example in Sec. 5, including some comments on the applicability in online schemes, and Sec. 6 concludes the paper.

2. PRELIMINARIES

This section clarifies some notation used throughout the paper, and details different mathematical aspects used in the sequel.

Let s_k denote the discrete-time value of a vector $s(t) \in \mathbb{R}^n$ at time $t = t_0 + k \cdot \Delta t$, with $k \in \mathbb{N}^+$, and a constant time step $\Delta t \in \mathbb{R}^+$. The symbol $\mathbb{N}_H$ denotes the set $\left\{k \in \mathbb{N}^{\geq 0} \,|\, k \leq H-1\right\}$.

The expected value of a signal is denoted by $\bar{s} := \mathrm{E}[s]$.

We use (C_s, b_s) to denote a convex polyhedron for a vector $s \in \mathbb{R}^n$:

$$\mathbb{S} = \{s \,|\, C_s \cdot s \leq b_s\},$$

and n_{C_s} is the number of half-spaces.

An ellipsoidal set is defined by a center point $\bar{s}$ and a shape matrix $\mathcal{S}$. In some points, it will be useful to define an ellipsoidal set as a shaped unit ball:

$$\begin{aligned}\varepsilon(\bar{s}, \mathcal{S}) &= \left\{s \,\middle|\, (s-\bar{s})^T \mathcal{S}^{-1}(s-\bar{s}) \leq 1\right\} \\ &= \left\{s \,\middle|\, s = \bar{s} + \mathcal{S}^{\frac{1}{2}} \cdot z, \|z\|_2 \leq 1\right\}.\end{aligned}$$

An affine transformation of an ellipsoidal set with matrix M and vector v is again an ellipsoidal set according to:

$$M \cdot \varepsilon(\bar{s}, \mathcal{S}) + v = \varepsilon\left(M\bar{s} + v, M\mathcal{S}M^T\right).$$

The volume of an ellipsoid is proportional to the determinant of the shape matrix, i.e.:

$$vol\left(\varepsilon(\bar{s}, \mathcal{S})\right) \propto \det(\mathcal{S}).$$

If the determinant is close to zero when minimizing the volume of an ellipsoid, a common way is to approximate the minimization of the determinant by the minimization of the trace of the shape matrix, i.e.:

$$\min \det(\mathcal{S}) \approx \min \mathrm{trace}(\mathcal{S}). \tag{1}$$

In Durieu et al. (1996) it is shown, that the minimization of the trace balances the length of the semi axis but leads to a suboptimal volume. In contrast, the minimization of the determinant (which leads to the optimal volume) may result in very narrow ellipsoids, which corresponds to a large uncertainty in one direction, even if the volume of the ellipsoid approaches zero.

A multivariate normal distribution of an n-dimensional random vector s with covariance matrix $\mathcal{S} \in \mathbb{R}^{n\times n}, \mathcal{S} = \mathcal{S}^T \geq 0$, and mean value $\bar{s} \in \mathbb{R}^n$ is denoted by $s \sim \mathcal{N}(\bar{s}, \mathcal{S})$. The sum of two normally distributed random variables $s_1 \sim \mathcal{N}(\bar{s}_1, \mathcal{S}_1)$ and $s_2 \sim \mathcal{N}(\bar{s}_2, \mathcal{S}_2)$ is again normally distributed:

$$s_1 + s_2 \sim \mathcal{N}(\bar{s}_1 + \bar{s}_2, \mathcal{S}_1 + \mathcal{S}_2). \tag{2}$$

The level curves of a Gaussian probability density function are ellipsoidal.

If $c = (F_{\chi^2})^{-1}(\delta, n)$ holds for a probability $\delta \in [0,\ 1]$ and F_{χ^2} being the cumulative distribution function of a χ^2-distribution, then:

$$\mathcal{S}^\delta = \varepsilon(\bar{s}, c \cdot \mathcal{S}) \tag{3}$$

denotes an *confidence ellipsoid*, which contains s with confidence δ and which is scaled by c, see also Asselborn and Stursberg (2015). Throughout the paper, the mean value of a distribution coincides with the center point of the confidence ellipsoid, and the shape matrix is equal to the covariance matrix of the distribution, thus we use the same notation for these quantities.

The symbol $\|s_k\|_Q = s_k^T \cdot Q \cdot s_k$ denotes a weighted 2-norm of a vector s_k with symmetric and positive semi-definite weighting matrix Q.

3. SYSTEM AND PROBLEM DEFINITION

The system class under consideration in this paper is a discrete-time linear system with probabilistically modeled additive disturbances and initialization:

$$\begin{aligned}&x_{k+1} = Ax_k + Bu_k + Ew_k \\ &x_0 \sim \mathcal{N}(\bar{x}_0, \mathcal{X}_0) \\ &w_k \sim \mathcal{N}(0, \mathcal{W}_k),\end{aligned} \tag{4}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ the input vector, and $w_k \in \mathbb{R}^{n_w}$ the disturbance vector. The initial state is normally distributed (n.d.) with covariance matrix $\mathcal{X}_0$ and mean value $\bar{x}_0$. The additive disturbances are also normally distributed, but with zero-mean and covariance $\mathcal{W}_k$. The initial state and disturbances are assumed to be iid.

Assumption 1. The mean value and covariance of the initial state are known. The covariance matrix of the disturbance may vary over time, but is assumed to be known, too.

The state x_k and input u_k are chance-constrained by polytopic sets with a given probability δ_x, and δ_u respectively:

$$\Pr(x_k \in \mathbb{X}_k) \geq \delta_x, \quad \mathbb{X}_k = \{x \,|\, C_{x_k} \cdot x \leq b_{x_k}\}, \tag{5}$$

$$\Pr(u_k \in \mathbb{U}) \geq \delta_u, \quad \mathbb{U} = \{u \,|\, C_u \cdot u \leq b_u\}, \tag{6}$$

i.e., x_k and u_k have to satisfy these constraints at least with the probabilities δ_x, and δ_u respectively. The sets $\mathbb{X}$ and $\mathbb{U}$ are assumed to be compact and contain the origin in their interior.

According to Assumption 1, the initial state and disturbance are n.d., which means, that deterministic values are not available at the time-instance the control law is determined. Nevertheless, one can predict the behavior of the state x_1 under the impact of the control input u_0 according to (4), which is with respect to (2) again n.d:

$$x_1 = Ax_0 + Bu_0 + Ew_0 \sim \mathcal{N}(\bar{x}_1, \mathcal{X}_1).$$

With respect to the zero-mean disturbance, the covariance and mean value of the state at time $k = 1$ result to:

$$\bar{x}_1 = A\bar{x}_0 + B\bar{u}_0$$
$$\mathcal{X}_1 = A\mathcal{X}_0 A^T + B\mathcal{U}_0 B^T + E\mathcal{W}_0 E^T.$$

For determining u_0 (and each later input), we refer to the class of time-varying state feedback control laws as in Asselborn and Stursberg (2015). However, rather than choosing an affine structure, we employ the linear equivalent to the deterministic control law in Goulart et al. (2006):

$$u_k = \sum_{r=0}^{k} K_{k,r} x_r. \tag{7}$$

Here, $K_{k,r} \in \mathbb{R}^{n_u \times n_x}$ denotes the feedback matrix used in the control law of time k and which is multiplied by the earlier state at time r. When using such a state feedback law, the control input u_k becomes normally distributed since x_k is n.d., too. We use $\bar{u}_k$ to denote the input obtained by applying the control law to the expected state $\bar{x}_k$. The control law (7) affects the mean value as well as the covariance of the distribution of x_1. This holds, of course, also for each later input u_k and state x_{k+1}, $k \geq 0$, if $K_{k,0} \neq 0$. By applying (4) recursively, one can compute a sequence of n.d. future states for a given input trajectory. Following the idea of Asselborn and Stursberg (2015) and using the fact that the system definition is based on normal distributions, the distribution of state trajectories can be represented by a sequence of confidence ellipsoids $\mathcal{X}_k^\delta$ according to (3). This is an extension of the ellipsoidal reachable set calculus, see Kurzhanskiĭ and Vályi (2009), to linear stochastic systems.

For the given system class, the main control objective in this paper is to steer the mean value of the state to the origin with acceptable costs for the input, and at the same time to minimize the uncertainty of the state. This minimization of the uncertainty is equivalent to minimizing the volume of the confidence ellipsoids (and thus to minimize the stochastic reachable sets). The formalization of this ambition leads to the following optimization problem, formulated for a finite time horizon of H steps:

$$\begin{aligned}
&\min_{K_{k,r}} \sum_{k=1}^{H} \|\bar{x}_k\|_Q + \|\bar{u}_{k-1}\|_R + \|\mathrm{trace}(\mathcal{X}_k)\|_S \\
\text{s.t.:}\quad & x_{k+1} = Ax_k + Bu_k + Ew_k \\
& x_0 \sim \mathcal{N}(\bar{x}_0, \mathcal{X}_0), \quad w_k \sim \mathcal{N}(0, \mathcal{W}_k), \\
& u_k = \textstyle\sum_{r=0}^{k} K_{k,r} x_r, \\
& \Pr(u_k \in \mathbb{U}) \geq \delta_u \quad \forall\, k \in \mathbb{N}_H., \\
& \Pr(x_{k+1} \in \mathbb{X}_{k+1}) \geq \delta_x \quad \forall\, k \in \mathbb{N}_H.
\end{aligned} \tag{8}$$

The solution of this problem aims at bringing the expected states and inputs as closely as possible to the origin, at guaranteeing the satisfaction of chance constraints (5) and (6), and at minimizing the uncertainty of the states by minimizing the semi axis of the confidence ellipsoids according to (1). The matrices Q, R, and S are used to balance between the minimization of the expected values and the minimization of the uncertainty.

If using the methods of Asselborn and Stursberg (2015) to solve problem (8), this results in a convex optimization problem with bi-linear matrix inequalities, which would not be efficiently solvable for many problem instances. The next section describes an alternative path (involving a conic optimization problem with just linear matrix inequalities), leading to a result which also provides the solution to (8).

4. CONTROLLER SYNTHESIS

4.1 Equivalence of State and Disturbance Feedback

In order to consider the states and inputs over the horizon of length H, the following stacked vectors are defined:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_H \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{H-1} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{H-1} \end{bmatrix}.$$

With matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$ of adequate size (given in the appendix A) and the feedback matrix $\mathbf{K} \in \mathbb{R}^{H \cdot n_u \times (H+1) n_x}$, the evolution of states and inputs over the horizon follows to:

$$\mathbf{x} = \mathbf{A}x_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \tag{9}$$
$$\mathbf{u} = \mathbf{K}\mathbf{x} \tag{10}$$

with

$$\mathbf{K} = \begin{bmatrix} K_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ K_{H-1,0} & \cdots & K_{H-1,H-1} & 0 \end{bmatrix}. \tag{11}$$

Lemma 2. Given the state equation (9), the state feedback law (10) can be reformulated into the feedback of the initial state and disturbances:

$$\mathbf{u} = \mathbf{K}\mathbf{x} = \mathbf{V}x_0 + \mathbf{M}\mathbf{w},$$

and vice versa.

Proof. Following the procedure in Goulart et al. (2006), consider first the computation of matrices $(\mathbf{V}, \mathbf{M})$ with given $\mathbf{K}$. By substituting the input (10) into (9), and inserting it again in (10), we have:

$$\begin{aligned}
&\mathbf{x} = (I - \mathbf{B}\mathbf{K})^{-1}(\mathbf{A}x_0 + \mathbf{E}\mathbf{w}) \\
\Rightarrow\ &\mathbf{u} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{A}x_0 + \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{E}\mathbf{w} \\
&\quad =: \mathbf{V}x_0 + \mathbf{M}\mathbf{w},
\end{aligned}$$

i.e., a linear function of the initial state x_0 and the disturbance vector $\mathbf{w}$, parameterized by:

$$\mathbf{V} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{A}, \quad \mathbf{M} = \mathbf{K}(I - \mathbf{B}\mathbf{K})^{-1}\mathbf{E}.$$

In opposite direction, for given matrices $\mathbf{V}$ and $\mathbf{M}$, the state feedback $\mathbf{K}$ follows with the left inverse $\mathbf{E}'$ of matrix $\mathbf{E}$ and with $x_0 = e_1 \cdot \mathbf{x} = [I, 0, \ldots, 0] \cdot \mathbf{x}$, since x_0 is the first element of $\mathbf{x}$:

$$\begin{aligned}
&\mathbf{w} = \mathbf{E}'(\mathbf{x} - \mathbf{A}x_0 - \mathbf{B}\mathbf{u}) \\
\Rightarrow\ &\mathbf{u} = \mathbf{V}x_0 + \mathbf{M}\mathbf{E}'(\mathbf{x} - \mathbf{A}x_0 - \mathbf{B}\mathbf{u}) \\
&\quad = (I + \mathbf{M}\mathbf{E}'\mathbf{B})^{-1}(\mathbf{V}x_0 + \mathbf{M}\mathbf{E}'\mathbf{x} - \mathbf{M}\mathbf{E}'\mathbf{A}x_0) \\
&\quad = (I + \mathbf{M}\mathbf{E}'\mathbf{B})^{-1}((\mathbf{V} - \mathbf{M}\mathbf{E}'\mathbf{A})e_1 + \mathbf{M}\mathbf{E}')\mathbf{x} \quad (12) \\
&\quad =: \mathbf{K}\mathbf{x}.
\end{aligned}$$

□

Remark 3. The left inverse $\mathbf{E}'$ exists, if $\mathrm{rank}(E) = n_w$, and then invertibility of both $(I - \mathbf{B}\mathbf{K})^{-1}$ and $(I + \mathbf{M}\mathbf{E}'\mathbf{B})^{-1}$ is guaranteed by the structure of the bold matrices. Even though the left inverse $\mathbf{E}'$ may not be defined uniquely, and thus uniqueness of $\mathbf{K}$ is not given, each admissible parameterization of $\mathbf{E}'$, and respectively $\mathbf{K}$, leads to the same input trajectory $\mathbf{u}$.

Writing the control law as $\mathbf{u} = \mathbf{V}x_0 + \mathbf{M}\mathbf{w}$ is well motivated, since the uncertainty of $\mathbf{x}$ stems from the uncertainty of the initial state and the disturbances. If $\mathbf{K}$ is structured according to (11), $\mathbf{V} \in \mathbb{R}^{H\cdot n_u \times n_x}$ is full dimensional, and $\mathbf{M} \in \mathbb{R}^{H\cdot n_u \times H\cdot n_w}$ has the following structure:

$$\mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{H-1} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{H-1,0} & \cdots & M_{H-1,H-2} & 0 \end{bmatrix}.$$

A single entry $M_{k,r} \in \mathbb{R}^{n_u \times n_w}$ of $\mathbf{M}$ refers to the feedback matrix used in time-step k to account for the disturbance w_r. The structure of $\mathbf{M}$ implies that for a each time-step k only all prior disturbances w_r with $r < k$ are considered, since they can be determined from state measurements. With a single entry $V_k \in \mathbb{R}^{n_u \times n_x}$ of $\mathbf{V}$, the complete control law in time k can be extracted by using:

$$u_k = V_k \cdot x_0 + \sum_{r=0}^{k-1} M_{k,r} w_r. \tag{13}$$

Remark 4. Since state and disturbance feedback are equivalent in their impact on the state evolution according to Lemma 2, the alternative problem with optimization over disturbance feedback policies is possible. The feedback matrix $\mathbf{K}$ can be computed subsequently from $\mathbf{V}$ and $\mathbf{M}$.

4.2 Closed loop behavior

To prepare the minimization of the state uncertainty, the closed-loop behavior of the system is formulated first. Inserting the control law (13) into (4) results in a state equation given as linear combination of the initial state and all prior disturbances:

$$x_{k+1} = \mathcal{A}_k x_0 + \sum_{r=0}^{k} \mathcal{E}_{k,r} w_r, \tag{14}$$

with closed-loop matrices:

$$\mathcal{A}_k = A^{k+1} + \sum_{i=0}^{k} A^{k-i} B V_i$$
$$\mathcal{E}_{k,r} = A^{k-r} E + \sum_{i=r+1}^{k} A^{k-i} B M_{i,r}.$$

This leads to the following expression for the expected value and the covariance matrix of each state vector:

$$\bar{x}_{k+1} = \mathcal{A}_k \bar{x}_0$$
$$\mathcal{X}_{k+1} = \mathcal{A}_k \mathcal{X}_0 \mathcal{A}_k^T + \sum_{r=0}^{k} \mathcal{E}_{k,r} \mathcal{W}_r \mathcal{E}_{k,r}^T. \tag{15}$$

4.3 Input constraints

The input constraint (6) requires, that each u_k is contained in $\mathbb{U}$ with likelihood δ_u. Whereas this introduces a certain amount of conservatism, it is required that all δ_u-confidence ellipsoids $\mathcal{U}_k^{\delta_u}$ are within $\mathbb{U}$ with respect to the control policy (13). Note that applying the control law to ellipsoidal state sets implies the use of ellipsoidal input sets.

Proposition 5. Let $b_u^{(i)}$ and $C_u^{(i)}$ denote the i-th half-space of the polytopic input constraint parameterized by (C_u, b_u). Then, if for all $k \in \mathbb{N}_H$ and all $i \in \{1, ..., n_{C_u}\}$ the set of LMIs:

$$\mathcal{L}u_k^{(i)} = \begin{bmatrix} (b_u^{(i)} - C_u^{(i)} V_k \bar{x}_0) & C_u^{(i)} \cdot c_u^{\frac{1}{2}} r_k \\ \star & (b_u^{(i)} - C_u^{(i)} V_k \bar{x}_0) \cdot I_{n_x + k\cdot n_w} \end{bmatrix} \succcurlyeq 0,$$

is satisfied with:

$$r_k = \left[V_k \mathcal{X}_0^{\frac{1}{2}},\ M_{k,0} \mathcal{W}_0^{\frac{1}{2}},\ \ldots,\ M_{k,k-1} \mathcal{W}_{k-1}^{\frac{1}{2}} \right],$$
$$c_u = (F_\chi^2)^{-1}(\delta_u, n_u),$$

then (6) is satisfied, too.

Proof. According to (13), the δ_u-confidence ellipsoid with $c_u = (F_\chi^2)^{-1}(\delta_u, n_u)$ for a specific time-step k is given by:

$$\mathcal{U}_k^{\delta_u} = V_k \mathcal{X}_0^{\delta_u} + \sum_{r=0}^{k-1} M_{k,r} \mathcal{W}_r^{\delta_u},$$

where:

$$\mathcal{X}_0^{\delta_u} = \varepsilon(\bar{x}_0, c_u \mathcal{X}_0) = \left\{ x \,\middle|\, x = \bar{x}_0 + (c_u \mathcal{X}_0)^{\frac{1}{2}} \cdot z, \|z\|_2 \le 1 \right\},$$
$$\mathcal{W}_k^{\delta_u} = \varepsilon(0, c_u \mathcal{W}_k) = \left\{ w \,\middle|\, w = (c_u \mathcal{W}_k)^{\frac{1}{2}} \cdot z, \|z\|_2 \le 1 \right\}.$$

In order to satisfy $\mathcal{U}_k^{\delta_u} \subseteq \mathbb{U}$, the row-wise maximization of u_k for each half-space with index $i \in \{1, \ldots, n_{C_u}\}$ leads to:

$$\max_{q\in Q} C_u^{(i)} q \le b_u^{(i)},$$
$$\text{s.t.:} \quad Q = \left\{ q \,\middle|\, \begin{array}{l} q = V_k x_0 + \sum_{r=0}^{k-1} M_{k,r} w_r, \\ x_0 \in \mathcal{X}_0^{\delta_u}, w_r \in \mathcal{W}_r^{\delta_u} \end{array} \right\}.$$

When using the representation as shaped unit ball, this is equivalent to:

$$\max_{q\in Q} C_u^{(i)} q \le b_u^{(i)} - C_u^{(i)} V_k \bar{x}_0,$$
$$\text{s.t.:} \ Q = \left\{ q \,\middle|\, \begin{array}{l} q = V_k (c_u \mathcal{X}_0)^{\frac{1}{2}} \cdot z_k + \sum\limits_{r=0}^{k-1} M_{k,r} (c_u \mathcal{W}_r)^{\frac{1}{2}} \cdot z_r, \\ \|z_r\|_2 \le 1 \, \forall \, r \in \{0, \ldots, k\} \end{array} \right\}.$$

The left hand side of the inequality is limited by the 2-norm of the unit ball, i.e. satisfying the following inequality is sufficient:

$$\left\| C_u^{(i)} V_k (c_u \mathcal{X}_0)^{\frac{1}{2}} \right\|_2 + \sum_{r=0}^{k-1} \left\| C_u^{(i)} M_{k,r} (c_u \mathcal{W}_r)^{\frac{1}{2}} \right\|_2 \cdots$$
$$\le b_u^{(i)} - C_u^{(i)} V_k \bar{x}_0 \quad \forall \, i \in \{1, \ldots, n_{C_u}\}.$$

$$\mathcal{L}x_{k+1}^{(i)} = \begin{bmatrix} (b_{x_{k+1}}^{(i)} - C_{x_{k+1}}^{(i)} \mathcal{A}_k \bar{x}_0) & C_{x_{k+1}}^{(i)} c_x^{\frac{1}{2}} \left[\mathcal{A}_k \mathcal{X}_0,\ \mathcal{E}_{k,0} \mathcal{W}_0,\ \ldots,\ \mathcal{E}_{k,k} \mathcal{W}_k \right] \\ \star & \operatorname{blkdiag}(\mathcal{X}_0, \mathcal{W}_0, \ldots, \mathcal{W}_k)(b_{x_{k+1}}^{(i)} - C_{x_{k+1}}^{(i)} \mathcal{A}_k \bar{x}_0) \end{bmatrix} \succcurlyeq 0 \tag{16}$$

Using the Schur complement (see e.g. Boyd et al. (1994)), this inequality is satisfied if the LMI in Proposition 5 is satisfied. □

4.4 State constraints

Similarly to the input constraint, the approach is used that the constraint (5) is satisfied, if all δ_x-confidence ellipsoids lie within the admissible state space, requiring that $\mathcal{X}_k^{\delta_x} \subseteq \mathbb{X}_k$ holds for all time-steps.

Proposition 6. The constraint (5) is satisfied, if for all $i \in \{1, \ldots, n_{C_{x_{k+1}}}\}$ and all $k \in \mathbb{N}_H$ the LMIs (16) hold.

Proof. With $c_x = (F_{\chi^2})^{-1}(\delta_x, n_x)$, the confidence ellipsoid for a specific time-step $k+1$ (again represented as a shaped unit ball) is:

$$\begin{aligned}\mathcal{X}_{k+1}^{\delta_x} &= \varepsilon\left(\bar{x}_{k+1}, c_x \cdot \mathcal{X}_{k+1}\right) \qquad (17)\\ &= \left\{x \,\middle|\, x = \mathcal{A}_k \bar{x}_0 + (c_x \cdot \mathcal{X}_{k+1})^{\frac{1}{2}} \cdot z, \|z\|_2 \le 1\right\}.\end{aligned}$$

According to the admissible polyhedral state space $\mathbb{X}_k$, the following row-wise inequality is obtained for each $i \in \{1, \ldots, n_{C_{x_{k+1}}}\}$:

$$\max_{q \in Q} C_{x_{k+1}}^{(i)} \cdot q \le b_{x_{k+1}}^{(i)} - C_{x_{k+1}}^{(i)} \mathcal{A}_k \bar{x}_0,$$

$$\text{s.t.:} \quad Q = \left\{q \,\middle|\, q = (c_x \mathcal{X}_{k+1})^{\frac{1}{2}} \cdot z, \|z\|_2 \le 1\right\}.$$

This is equivalent to:

$$\begin{aligned}&\left\| C_{x_{k+1}}^{(i)} (c_x \cdot \mathcal{X}_{k+1})^{\frac{1}{2}} \right\|_2 \le b_{x_{k+1}}^{(i)} - C_{x_{k+1}}^{(i)} \mathcal{A}_k \bar{x}_0\\ \Leftrightarrow\ &(C_{x_{k+1}}^{(i)} c_x^{\frac{1}{2}} \mathcal{X}_{k+1} c_x^{\frac{1}{2}} C_{x_{k+1}}^{(i)^T}) \le (b_{x_{k+1}}^{(i)} - C_{x_{k+1}}^{(i)} \mathcal{A}_k \bar{x}_0)^2.\end{aligned}$$

With the covariance matrix (15) and the Schur complement, it follows that this inequality is satisfied, if the LMIs referred to in proposition 6 hold. □

4.5 Determination of feedback matrices by SDP

For minimizing the state covariance matrices, and motivated by the scheme in Asselborn and Stursberg (2015), an artificial matrix $\mathcal{S}_{k+1} = \mathcal{S}_{k+1}^T \succcurlyeq 0$ is introduced for all $k \in \mathbb{N}_H$ as an upper bound for each covariance matrix:

$$\begin{aligned}&\mathcal{S}_{k+1} \succcurlyeq \mathcal{X}_{k+1}\\ \Leftrightarrow\ &\mathcal{S}_{k+1} - \mathcal{A}_k \mathcal{X}_0 \mathcal{A}_k^T - \sum_{r=0}^{k} \mathcal{E}_{k,r} \mathcal{W}_r \mathcal{E}_{k,r}^T \succcurlyeq 0.\end{aligned}$$

This inequality can also be transferred into an LMI using the Schur complement:

$$\mathcal{L}s_{k+1} = \begin{bmatrix} \mathcal{S}_{k+1} & [\mathcal{A}_k \mathcal{X}_0,\, \mathcal{E}_{k,0} \mathcal{W}_0,\, \ldots,\, \mathcal{E}_{k,k} \mathcal{W}_k] \\ \star & \text{blkdiag}\,(\mathcal{X}_0, \mathcal{W}_0, \ldots, \mathcal{W}_k) \end{bmatrix} \succcurlyeq 0.$$

With all requirements formulated as matrix inequalities, the SDP problem to be solved results as follows:

$$\begin{aligned}\min_{\mathbf{V},\mathbf{M}} &\sum_{k=1}^{H} \|\bar{x}_k\|_Q + \|\bar{u}_{k-1}\|_R + \|\text{trace}\,(\mathcal{S}_k)\|_S\\ \text{s.t.:}\quad &\bar{x}_{k+1} = \mathcal{A}_k \cdot \bar{x}_0 \quad \forall\, k \in \mathbb{N}_H \qquad (18)\\ &\bar{u}_k = V_k \cdot x_0 \quad \forall\, k \in \mathbb{N}_H\\ &\mathcal{L}s_{k+1} \succcurlyeq 0 \quad \forall\, k \in \mathbb{N}_H\\ &\mathcal{L}x_{k+1}^{(i)} \succcurlyeq 0 \quad \forall\, i \in \{1, \ldots, n_{C_{x_{k+1}}}\}, \forall\, k \in \mathbb{N}_H\\ &\mathcal{L}u_k^{(i)} \succcurlyeq 0 \quad \forall\, i \in \{1, \ldots, n_{C_u}\}, \forall\, k \in \mathbb{N}_H.\end{aligned}$$

Let the optimal solution of (18) be denoted by $(\mathbf{V}^\star, \mathbf{M}^\star)$.

Theorem 7. If there exists an optimal solution to (18), an admissible solution to (8) is given by:

$$\mathbf{K} = (I + \mathbf{M}^\star \mathbf{E}' \mathbf{B})^{-1}((\mathbf{V}^\star - \mathbf{M}^\star \mathbf{E}' \mathbf{A})[I, 0, \ldots, 0] + \mathbf{M}^\star \mathbf{E}').$$

If no state and input constraints are active, $\mathbf{K}$ is the optimal solution to (8), i.e. $\mathbf{K}^\star = \mathbf{K}$. Otherwise, $\mathbf{K}$ is sub-optimal due to the inner approximations used for the state and input constraints.

Proof. If $(\mathbf{V}^\star, \mathbf{M}^\star)$ solves (18), there exists a matrix $\mathbf{K}$ according to (12). The sequence of u_k is obtained with $\mathbf{K}$ from (10) and satisfies the input constraints (6) due to Proposition 5. Likewise, with the sequence of $\bar{x}_k$ from the first constraint in (18), the states x_k in (14) are contained in the confidence ellipsoids $\mathcal{X}_k^{\delta_x}$ as defined in (17). In consequence of Proposition 6, the constraint (5) is satisfied, and hence, the state and input sequences obtained from $\mathbf{K}$ are feasible for problem (8).

If, when using $\mathbf{K}$, any state x_{k+1} and any input u_k for $k \in \mathbb{N}_H$ lies in the interior of the sets satisfying the constraints $\mathcal{L}x_{k+1}^{(i)} \succcurlyeq 0$, $i \in \{1, \ldots, n_{C_x}\}$ and $\mathcal{L}u_k^{(i)} \succcurlyeq 0$, $i \in \{1, ..., n_{C_u}\}$ in (18), then there exist no different state and input sequences with lower costs for the less strict constraints defined for (8), i.e., $\mathbf{K}^\star = \mathbf{K}$ optimally solves also (8). If one of the constraints in (18) is active, a solution with (slightly) lower costs may exist for (8). □

5. NUMERICAL EXAMPLE

To illustrate the proposed method, it is applied to the dynamics according to (4) with arbitrarily chosen:

$$A = \begin{bmatrix} 0.87 & 0.06 & -0.03 \\ 0 & 1.09 & 1.20 \\ 0 & 0.06 & 0.91 \end{bmatrix}, \quad B = \begin{bmatrix} 0.64 & 0 \\ -0.24 & 0.34 \\ 0.62 & 0.61 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix},$$

and with the initial state and disturbance distributions:

$$\bar{x}_0 = \begin{bmatrix} 5 \\ 35 \\ 2 \end{bmatrix}, \ \mathcal{X}_0 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.15 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}, \mathcal{W}_k = \mathcal{X}_0\, \forall k \in \mathbb{N}_H.$$

The input constraint is defined by $\mathbb{U} = \{u \,|\, \|u\|_\infty \le 4\}$, and the chance constraint for this range is selected to:

$$\Pr(u_k \in \mathbb{U}) \ge \delta_u = 0.95.$$

A time-invariant state constraint is defined to:

$$\mathbb{X}_{k+1} = \left\{ x \,\middle|\, \begin{matrix} -6 \le x_1 \le 7, \\ -9 \le x_2 \le 55, \\ -8 \le x_3 \le 7 \end{matrix} \right\} \ \forall k \in \mathbb{N}_H,$$

with the probability $\Pr(x_k \in \mathbb{X}_k) \ge \delta_x = 0.95$.

The time horizon is chosen to $H = 20$. To demonstrate the minimization of the covariance matrices, two slightly different parameterizations of the cost functional are used:

i) $Q^{(i)} = I_3$, $R^{(i)} = 0.1$, and $S = 50$
ii) $Q^{(ii)} = I_3$, $R^{(ii)} = 0.1$, and $S = 0$

While version i) conforms to the objective of costs balancing proposed in this paper, the second version represents the case without minimization of the covariance matrices (the latter is comparable to stochastic optimal control without tuning the evolution of the state uncertainty).

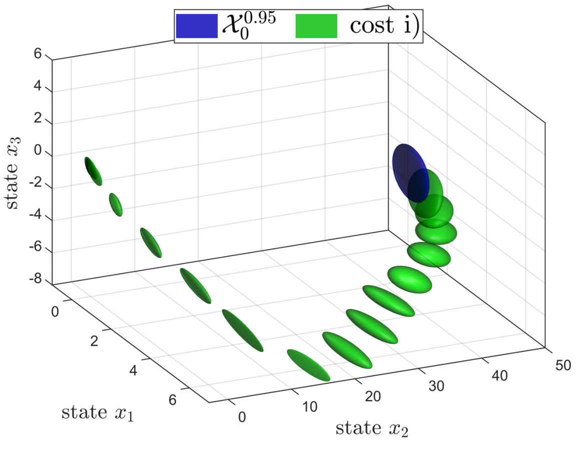
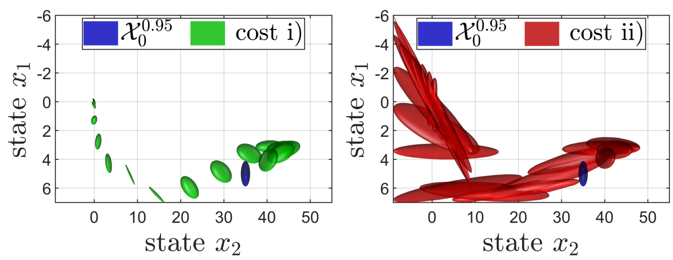


Fig. 1. Control result for the 0.95%-confidence ellipsoids $\mathcal{X}_k^{0.95}$ with version i) of the cost functional.



(a) cost i)

(b) cost ii)

Fig. 2. Control result in terms of confidence ellipsoids projected onto the x_1-x_2-space.

Figure 1 shows the control result for version i) of the cost functional: From the initial 95%-confidence ellipsoid $\mathcal{X}_0^{0.95}$ (blue), the state is steered to the origin without violating the state constraints. The uncertainty of the initial state is reduced significantly, as can be seen from the reduced size of the ellipsoids over k.

In addition, Fig. 2 part (a) illustrates the projection of the state evolution onto the first two states. Part (b) shows for comparison the same for version ii) of the cost functional: without minimizing the uncertainty, the size of the ellipsoids increases over the control horizon, while the expected values are almost the same. In both cases, state and input constraints are active.

Table 1 contains for the two versions the corresponding cost measures as the volumes of the confidence ellipsoids in 3-D aggregated over the time horizon, as well as the weighted distances of the expected states to the origin. While the cost for the expected states increases by approximately 25.5% (when changing from version ii) to i)), the occupied state space is reduced by 87%. Note that for a single ellipsoid, the overall 25.5% lower costs for ii) are practically irrelevant, if the confidence ellipsoids are as large as for ii).

Note in addition, that if the test is repeated without the state constraint, the use of version ii) leads to an increase of the aggregated volume of the ellipsoids to 1138, and the use of the version i) reduces the value by even 97 %.

Table 1. Comparison of cost function values.

	cost i)	cost ii)	relative for i)
$\sum_k vol\left(\mathcal{X}_k^{0.95}\right)$	38.18	293.35	−87 %
$\sum_k \|\bar{x}_k\|_Q$	13728	10937	+25.5 %

The optimization problems were solved on a 3.4 GHz Quad-Core CPU using *Matlab2018a* and the solver *Mosek*. With randomized initial mean value $\bar{x}_0$, the optimization problem is solved in an average computation time of around 20 s.

This value deserves comments with respect to the use of the proposed technique in online schemes such as predictive control: If the solution of (18) is to be carried out in any time step k, for many applications much lower computation times than the value provided above are necessary. To this account, note that for problems with a horizon of 8 steps and $n_x = 2$ computation times of below 50 ms are obtained, i.e., the online use for low-dimensional systems appears possible for some applications. In addition, the use in online schemes will often allow to measure the current state, i.e. the setting simplifies to the case in which uncertainties arise only from the disturbances, not the uncertain state initialization.

6. CONCLUSION

The paper has proposed a method for solving a stochastic optimal control problem which accounts for uncertainties due to imprecise knowledge of the initial state and arising from uncertainties in balanced manner. The idea is to employ stochastic reachable sets in terms of confidence ellipsoids, which contain the state with high probability. In contrast to previous work, the shaping of these ellipsoids through linear disturbance feedback control laws is achieved by a single optimization over a given control horizon, rather than solving H many single-step problems. In addition to reducing the overall computational time, this leads to the possibility to balance the convergence towards the goal with the state distribution over the complete horizon. As demonstrated for a simulation example, the state uncertainty can be reduced drastically, while the convergence of the expected values towards the goal is only slightly worsened. The proposed method is applicable to time-invariant input constraints, time-varying state constraints, and time-varying covariance matrices of the disturbance. In particular with respect to the satisfaction of state constraints, the method can significantly reduce the conservatism arising from the state uncertainty in other schemes, which do not explicitly aim at achieving small state distributions.

Future work will investigate the use of the findings of this paper within networked predictive control.

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Appendix A. VECTOR NOTATION

The matrices $\mathbf{A} \in \mathbb{R}^{(H+1)n_x \times n_x}$ and $\mathbf{C} \in \mathbb{R}^{(H+1)n_x \times H \cdot n_x}$ have the following structure:

$$\mathbf{A} = \begin{bmatrix} I_{n_x} \\ A \\ \vdots \\ A^H \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_{n_x} & 0 & \cdots & 0 \\ A & I_{n_x} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{H-1} & A^{H-2} & \cdots & I_{n_x} \end{bmatrix}.$$

The matrices $\mathbf{B} \in \mathbb{R}^{(H+1)n_x \times H \cdot n_u}$ and $\mathbf{E} \in \mathbb{R}^{(H+1)n_x \times H \cdot n_w}$ are chosen to $\mathbf{B} = \mathbf{C}(I_H \otimes B)$, and $\mathbf{E} = \mathbf{C}(I_H \otimes E)$ respectively, where $\otimes$ denotes the Kronecker product.