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Trajectory planning for semilinear time-fractional reaction-diffusion systems under Robin boundary conditions ⋆

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Abstract: This paper describes how to design flatness-based distributed feedforward controllers for the solution of the trajectory planning problem of semilinear time-fractional reaction-diffusion systems (TFRDSs), where the first-order time derivative of conventional reaction-diffusion system is extended to a Caputo fractional derivative of order $\alpha \in (0, 1]$. To this end, an implicit system variable and control input parametrization is determined based on the spectral property of system operator with respect to a basic output and its fractional-order derivatives. The convergence of the parametrizations is guaranteed by restricting the basic output to some certain Gevrey classes. With these, we propose two approaches on solving the trajectory planning problem within a prescribed finite time interval. A simulation example is finally included to illustrate our results.

Keywords: Trajectory planning; Differential flatness; Time-fractional reaction-diffusion systems; Distributed control; Robin boundary conditions.

1. INTRODUCTION

Over the last two decades, there is an increasing activity in the study of trajectory planning problems for conventional reaction-diffusion systems, which are used to model the heating or cooling down processes of the metal slabs in steel industry by assuming that the diffusion medium is spatially homogeneous, see, e.g., Meurer and Kugi (2009) and the references therein. However, the situation may not always be this simple. Most of the diffusion environments in practice are spatially inhomogeneous and may be of an 'anomalous' nature due to the complicated interactions between the reactants and their environment (Yamamoto et al., 2014; Ge et al., 2018a,b). Based on this, we here focus on processes in which the reactants exhibit subdiffusive behaviour. It is shown in Metzler and Klafter (2000); Ge et al. (2016a,b,c, 2017) and Luchko (2012) that TFRDSs possess pattern formation for the same sets of model parameters as the subdiffusion processes. This is due to the fact that the fractional-order derivative is defined as a kind of convolution hence representing well the dynamics inheriting subdiffusive properties (see e.g. Kilbas et al. (2006); Ge and Chen (2020, 2019)). Therefore, research on trajectory planning problems for TFRDSs should be both interesting and challenging.

Let $\Omega = \{x \in \mathbf{R}^k : x = (x_1, x_2, \cdots, x_k) \text{ and } 0 < x_i < L_i,\ i \in I_k\}$, $I_k = \{1, 2, \cdots, k\}$ be a $1 \leq n-$dimensional parallelepipedon and let $T > 0$. In this paper, we consider the following semilinear TFRDSs :

$$ {}_0^C D_t^\alpha y(x,t) = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} y(x,t) + f(y(x,t)) + Bu(t) \tag{1} $$

under the Robin boundary conditions (BCs)

$$ \begin{cases} p_i y_{x_i}(x,t) - q_i y(x,t) = 0, x_i = 0, i \in I_k, \\ r_i y_{x_i}(x,t) + s_i y(x,t) = 0, x_i = L_i, i \in I_k, \end{cases} t \in [0,T] \tag{2} $$

and the initial condition

$$ y(x,0) = 0,\ x \in \Omega, \tag{3} $$

where ${}_0^C D_t^\alpha,\ \alpha \in (0,1]$ denotes the Caputo fractional derivative, $f(y(x,t))$ is a continuous function to be specified later and

$$ Bu(t) = g(x)u(t). \tag{4} $$

In addition, $u \in C[0,T]$ is the control input and $g \in L^2(\Omega)$ denotes the spatial distribution of actuators. Here $L^2(\Omega)$ represents the usual square-integrable function space endowed with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$.

Furthermore, since both integer-order and time fractional-order reaction-diffusion systems incorporate an extra position variable in the model, their control can be divided into two parts. The first one is the so-called distributed control, which can be regarded as the distributed energy sources emerging in the right part of the integer-order or time-fractional partial differential equations (PDEs). The second one concerns the so-called boundary control. In this case, the plant is controlled from a bound appearing in the boundary conditions.

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To solve the trajectory planning problems of integer-order reaction-diffusion systems, the flatness-based feedforward tracking control strategy has been proposed to realize that the solution of the considered systems could reach to a final state along the certain pre-planned path. For the boundary control cases, flatness concepts were proposed e.g. in Laroche et al. (2000) for the motion planning problem of one-dimensional linear diffusion systems and in Meurer and Krstic (2011) to investigate the nonlinear motion planning problem of linear multi-agent PDE dynamics. Spectral analysis was discussed in Meurer (2011) for the trajectory planning problem of boundary controlled diffusion-reaction system defined on a $1 \leq r-$dimensional parallelepipedon. There also exist some results on the study of trajectory planning problem via a Volterra-type integral equation for higher-dimensional spatial domains linear parabolic PDEs in Meurer and Kugi (2009) and for semilinear cases in Schörkhuber et al. (2013). In addition, for the case of distributed control, we refer the reader to Kharitonov and Sawodny (2006) where a flatness-based distributed feedforward controller was designed for exact output tracking of the linear parabolic PDEs. All in all, the basic idea of flatness-based technique is to parametrize the system variable and the control inputs by means of a flat output. This is appealing in many applications, where the trajectories under some features can be generated based on the desired motion performance.

Notice that TFRDSs can be regarded as an extension of the conventional reaction-diffusion systems, where the first-order time derivative is generalized to a fractional derivative of order $\alpha \in (0, 1]$. We are able to propose the flatness-based distributed feedforward tracking control strategy for the solution of the trajectory planning problem of semilinear TFRDSs (1). To the best of our knowledge, no result is available on this topic. For this purpose, we first parametrize both the system variable and the control input based on the spectral theory of the system operator with respect to the basic output and its fractional-order derivatives. Then we study the convergence of the parametrizations. With these, two approaches are given to solve the trajectory planning problem of semilinear TFRDSs within a prescribed finite time interval.

The rest of this paper is organized as follows. The problem formulation and some basic results used thereafter are given in section 2. In Section 3, we present our main results on the flatness-based trajectory planning approach and their convergence for semilinear TFRDSs. This is illustrated in Section 4, where a simulation result is presented.

2. PRELIMINARIES

First, we give some basic results on fractional calculus.

Definition 1. (Kilbas et al., 2006) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $\phi : [0, \infty) \to \mathbf{R}$ is given by

$$ {}_0I_t^\alpha \phi(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds, \tag{5}$$

where $\Gamma(\alpha)$ represents the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$ and the right side is pointwise defined on $[0, \infty)$.

Lemma 2. (Kilbas et al., 2006) Given $T > 0$, if ϕ is continuous on $[0, T]$, the Riemann-Liouville fractional integration operators ${}_0I_t^\alpha$ with $\alpha > 0$ is bounded, i.e.,

$$\max_{t \in [0,T]} |{}_0I_t^\alpha \phi(t)| \leq \frac{T^\alpha}{\alpha\Gamma(\alpha)} \max_{t\in[0,T]} |\phi(t)|. \tag{6}$$

Definition 3. (Kilbas et al., 2006) The Caputo fractional derivative of order $\alpha > 0$ for a function $\phi : [0, \infty) \to \mathbf{R}$ is defined as

$$ {}_0^C D_t^\alpha \phi(t) = {}_0I_t^{n-\alpha} \frac{d^n}{dt^n} \phi(t) \tag{7}$$

provided that the right side is pointwise defined on $[0, \infty)$. Here $[-\alpha]$ denotes the greatest integer less than or equal to $-\alpha$ and $n = -[-\alpha]$.

Introduce the Laplace operator $\triangle = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2}$ with the domain $\mathcal{D}(\triangle) = \left\{\phi \in L^2(\Omega) : \phi \text{ satisfies the BCs (2)}\right\}$. Lemma 2 of Meurer (2011) yields that the eigenvalue of $(\triangle, \mathcal{D}(\triangle))$ satisfies $\lambda_n \leq 0$ and the corresponding eigenfunctions satisfy $\xi_n(x) = \prod_{i=1}^k \xi_{n_i}(x_i)$ with $n = (n_1, n_2, \cdots, n_k) \in \mathbf{N}^k$ and each $\xi_{n_i}(x_i)$ determined as the solution of the Sturm-Liouville problem. Moreover, $\{\xi_n\}_{n\in\mathbf{N}^k}$ forms a Riesz basic of space $L^2(\Omega)$.

Based on these, by Luchko (2012), if $y_0 \in L^2(\Omega)$ and $f \in L^\infty\left(C(0, T; L^2(\Omega)), L^2(\Omega)\right)$, there exists a unique weak solution $y \in C\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C\left(0, T; L^2(\Omega)\right)$ to system $(1)-(3)$. To obtain our results, the following further assumption on the function f holds true.

Assumption 1: $f(0) = 0$ and given $r > 0$, y, $y^* \in E := C\left(0, T; L^2(\Omega)\right)$ with $\|y\|_E$, $\|y^*\|_E \leq r$, a constant $c = c(r)$ can be found satisfying

$$|(f(y(\cdot, t)) - f(y^*(\cdot, t)), \xi_n)| \leq c\, |(y(\cdot, t) - y^*(\cdot, t), \xi_n)|$$

for all $t \geq 0$ and $n \in \mathbf{N}^k$. Here $(\cdot, \cdot)$ denotes the inner product of space $L^2(\Omega)$.

Now we are ready to give the following definition.

Definition 4. Given $y_T \in L^2(\Omega)$, the considered trajectory planning problem for TFRDSs $(1)-(3)$ concerns how to design the input trajectory $Bu(t)$ such that the function $y(x, t)$ starting from $y_0(x) \equiv 0$ could reach y_T as close as possible along a pre-planned path within $t \in [0, T]$, i.e.,

$$y_0(x) \xrightarrow[t\in[0,T]]{Bu(t)} y_T(x), \quad x \in \overline{\Omega}. \tag{8}$$

3. STATE AND INPUT PARAMETRIZATION

3.1 Operational parametrization

Let $\sum\limits_{n\in\mathbf{N}^k} = \sum\limits_{n_1=1}^\infty \cdots \sum\limits_{n_k=1}^\infty$ and $\prod\limits_{n\in\mathbf{N}^k} = \prod\limits_{n_1=1}^\infty \cdots \prod\limits_{n_k=1}^\infty$. Since $y(\cdot, t), g \in L^2(\Omega)$ and $\{\xi_n\}_{n\in\mathbf{N}^k}$ forms a Riesz basic of $L^2(\Omega)$, one has

$$y(x, t) = \sum_{n\in\mathbf{N}^k} (y(\cdot, t), \xi_n)\xi_n(x). \tag{9}$$

Let $y_n(t) = (y(\cdot, t), \xi_n)$, $(f_n(y_n))(t) = (f(y(\cdot, t)), \xi_n)$ and $u_n(t) = (Bu(t), \xi_n) = (g, \xi_n)\, u(t)$. Then the spectral representation of system $(1)-(3)$ follows as

$$\begin{cases} {}_0^C D_t^{\alpha} y_n(t) = \lambda_n y_n(t) + (f_n(y_n))(t) + u_n(t), n \in \mathbf{N}^k \\ y_n(0) = 0. \end{cases} \tag{10}$$

Consider the property

$$\begin{aligned} \mathcal{L}\left\{{}_0^C D_t^{\alpha} y_n\right\}(s) &= \int_0^{\infty} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{\partial y_n(\tau)}{\partial \tau} d\tau e^{-ts} dt \\ &= s^{\alpha-1} \int_0^{\infty} e^{-s\tau} \frac{\partial y_n(\tau)}{\partial \tau} d\tau = s^{\alpha} \mathcal{L}\left\{y_n\right\}(s) \end{aligned}$$

as a consequence of $y_n(0) = 0$, taking Laplace transform on both sides of (10) yields that

$$\tilde{y}_n(s) = \frac{\mathcal{L}\{f_n(y_n)\}(s) + \tilde{u}_n(s)}{s^{\alpha} - \lambda_n},\ n \in \mathbf{N}^k, \tag{11}$$

where $\tilde{y}_n(s)$, $\tilde{u}_n(s)$ and $\mathcal{L}\{f_n(y_n)\}(s)$ denote respectively, the Laplace transforms of $y_n(t)$, $u_n(t)$ and $(f_n(y_n))(t)$. Moreover, it is worth noting that $\mathcal{L}\{f_n(y_n\}(s)$ is only introduced to present the procedure but of course will and can not be evaluated explicitly. Therefore,

$$\begin{aligned} \tilde{y}_n(s) &= \frac{\mathcal{L}\{f_n(y_n)\}(s) + \tilde{u}_n(s)}{-\lambda_n\left(1 - s^{\alpha}/\lambda_n\right)} \\ &= \frac{-1}{\lambda_n} \frac{\prod\limits_{i \in \mathbf{N}^k, i \neq n} \left(1 - \frac{s^{\alpha}}{\lambda_i}\right)}{\prod\limits_{i \in \mathbf{N}^k} \left(1 - \frac{s^{\alpha}}{\lambda_i}\right)} \left(\mathcal{L}\{f_n(y_n)\}(s) + \tilde{u}_n(s)\right). \end{aligned} \tag{12}$$

Suppose that $\{\tilde{z}_n(s)\}_{n \in \mathbf{N}^k}$ is in the operational domain such that

$$\tilde{y}_n(s) = \frac{-1}{\lambda_n} \prod_{i \in \mathbf{N}^k, i \neq n} \left(1 - \frac{s^{\alpha}}{\lambda_i}\right) \tilde{z}_n(s) \tag{13}$$

and

$$\tilde{u}_n(s) = \prod_{i \in \mathbf{N}^k} \left(1 - \frac{s^{\alpha}}{\lambda_i}\right) \tilde{z}_n(s) - \mathcal{L}\{f_n(y_n)\}(s). \tag{14}$$

Define $M_0 = N_0 = 1$ and $M_j = \sum\limits_{i_r \in \mathbf{N}^k \backslash \{n\}, r \in I_j} \frac{(-1)^j}{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}}$, $N_j = \sum\limits_{i_r \in \mathbf{N}^k, r \in I_j} \frac{(-1)^j}{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}}$, when $j \geq 1$, we obtain that

$$\begin{aligned} \tilde{y}_n(s) &= \frac{-1}{\lambda_n} \sum_{j=0}^{\infty} M_j s^{\alpha j} \tilde{z}_n(s), \\ \tilde{u}_n(s) &= \sum_{j=0}^{\infty} N_j s^{\alpha j} \tilde{z}_n(s) - \mathcal{L}\{f_n(y_n)\}(s). \end{aligned} \tag{15}$$

As a result, this provides a way to explicitly parametrize $\tilde{y}_n(s)$ and $\tilde{u}_n(s)$. Here $z_n(t)$ is called to be the flat output.

Taking into account the definition of high-order Caputo fractional derivative and $\mathcal{L}\left\{{}_0 I_t^{\alpha} \rho\right\}(s) = s^{-\alpha} \tilde{\rho}(s),\ \alpha \geq 0$, one has

$$\mathcal{L}\left\{{}_0^C D_t^{\alpha} \rho\right\}(s) = s^{\alpha} \tilde{\rho}(s) - \sum_{k=0}^{n-1} s^{\alpha-n+k} \rho^{(k)}(0+). \tag{16}$$

Then if the flat output $z_n(t)$ is chosen such that

$$z_n^{(k)}(0+) \equiv 0,\ \forall k = 0, 1, 2, \cdots, \tag{17}$$

Eq. (16) leads to $\mathcal{L}\left\{{}_0^C D_t^{\alpha j} z_n\right\}(s) = s^{\alpha j} \tilde{z}_n(s)$. With this, making use of the inverse Laplace transform on both sides of (15), it formally yields that

$$y_n(t) = \frac{-1}{\lambda_n} \sum_{j=0}^{\infty} M_j\ {}_0^C D_t^{\alpha j} z_n(t) \tag{18}$$

and

$$u_n(t) = \sum_{j=0}^{\infty} N_j\ {}_0^C D_t^{\alpha j} z_n(t) - (f_n(y_n))(t). \tag{19}$$

3.2 Convergence analysis

To this end, the notion of a Gevrey class as introduced, e.g. in Rodino (1993); Rabbani et al. (2010) is required, which can be regarded as an extension of analytic functions with convergent Taylor expansion.

Definition 5. (Gevrey class) Given $T > 0$, the function $\varphi(t)$ is of Gevrey class $\beta > 0$ in $(0, T)$, denoted by $G_{K,\beta}(0, T)$, if $\varphi \in C^{\infty}[0, T]$ and for any compact subsect $I \subseteq (0, T)$, a positive constant L can be found such that

$$\max_{t \in I} \left|\varphi^{(n)}(t)\right| \leq L \frac{(n!)^{\beta}}{K^n} \quad \text{for all } n = \mathbf{N}^{+} \cup \{0\}. \tag{20}$$

Definition 5 yields that a Gevrey function of order β_1 is obviously also of order β_2 if $\beta_2 \geq \beta_1$. A classical result states that if $\beta < 1$, the function is entire, while it is analytic for $\beta = 1$. Besides, for $\beta > 1$, there are compactly supported functions in the class that are not identically zero (Laroche et al., 2000). To further illustrate the Gevrey class, we consider the following bump function

$$\varphi_{\gamma,T}(t) = \begin{cases} 0 \quad \text{if } t < 0, \\ \dfrac{\int_0^t \phi_{\gamma,T}(s) ds}{\int_0^T \phi_{\gamma,T}(s) ds} \quad \text{if } t \in [0, T), \\ 1 \quad \text{if } t \geq T \end{cases} \tag{21}$$

with $\gamma > 1$,

$$\phi_{\gamma,T}(t) = \begin{cases} \exp\left(-\left[\left(1 - \dfrac{t}{T}\right)\dfrac{t}{T}\right]^{-\gamma}\right), t \in [0, T), \\ 0,\ \text{else} \end{cases} \tag{22}$$

and get that $\psi_{\gamma,T}(t)$ is a Gevrey function of order $1 + \frac{1}{\gamma}$. Moreover, choose $T = 0.6$, we plot the bump function with $\gamma = 1, 1.5, 2, 10$ (i.e., $\beta = 2, 5/3, 1.5, 1.1$ respectively) in Fig.1, which shows that the larger β, the faster the transition slope.

In analogy to the proof of Proposition 6-7 in Meurer (2011), we obtain the following proposition and omit its detailed proof.

Proposition 6. The Weierstrass canonical products $\tilde{y}_n(s)$ and $\tilde{u}_n(s) + \mathcal{L}\{f_n(y_n)\}(s)$ given by (15) are two entire functions of finite order $d \in [1/2, n/2]$.

Theorem 7. Given $T > 0$, let *Assumption* 1 hold true and let $z_n(t)$ be a Gevrey function of order $\beta \in (1, 2]$ satisfying (17). Set $n_j = -[-\alpha j]$ for any $j \in \mathbf{N}^{+} \cup \{0\}$, then

$$\sup_{t \in [0,T]} |y_n(t)| \leq \frac{-\left[\frac{-1}{\alpha}\right] L}{|\lambda_n|} \sum_{j=0}^{\infty} C_j \frac{(ead)^{j/d} (j!)^{\beta-1/d}}{K^j} \tag{23}$$

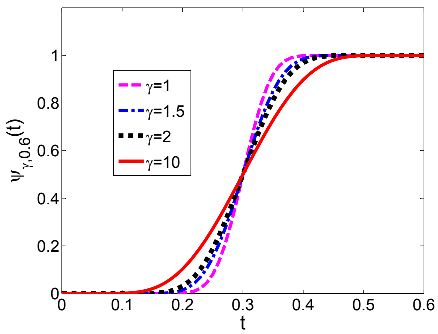


Fig. 1. The bump function $\psi_{\gamma,T}(t)$ with $\gamma = 1, 1.5, 2, 10$ and $T = 0.6$.

and

$$\sup_{t\in[0,T]} |u_n(t)| \leq -\left[\frac{-1}{\alpha}\right] L \left(1+\frac{c}{|\lambda_n|}\right) \sum_{j=0}^{\infty} C_j \frac{(ead)^{j/d}(j!)^{\beta-1/d}}{K^j} \quad (24)$$

hold true for some $a > 0$, where e is the base of the natural logarithm, $[-1/\alpha]$ denotes the greatest integer less than or equal to $-1/\alpha$, c is defined in *Assumption* 1 and

$$C_j = \begin{cases} 1, & \text{if } n_j - \alpha j = 0, \\ \dfrac{T^{n_j-\alpha j}}{(n_j-\alpha j)\Gamma(n_j-\alpha j)} & \text{if } n_j - \alpha j \neq 0. \end{cases} \quad (25)$$

Moreover, the series $\sum_{n\in\mathbf{N}^k} y_n(t)\xi_n(x)$ and $\sum_{n\in\mathbf{N}^k} u_n(t)\xi_n(x)$ with $y_n(t)$ and $u_n(t)$ parameterized according to (18) and (19) converge uniformly in $C\left(0,T;L^2(\Omega)\right)$ for $K < \infty$ if $\beta < 1/d$ and for $K < (ead)^{1/d}$ if $\beta = 1/d$.

Proof. Given $T > 0$, since $z_n(t) \in G_{K,\beta}(0,t)$ for $\beta \in (1, 2]$ satisfies (17), based on the definition of n_j and Lemma 2, it follows that

$$\begin{aligned} \sup_{t\in[0,T]} |y_n(t)| &\leq \sum_{j=0}^{\infty} \left|\frac{M_j}{\lambda_n}\right| \sup_{t\in[0,T]} \left| {}_0I_t^{n_j-\alpha j} \frac{d^{n_j}}{dt^{n_j}} z_n(t)\right| \\ &\leq \sum_{j=0}^{\infty} \left|\frac{M_j}{\lambda_n}\right| \left(-\left[\frac{-1}{\alpha}\right]\right) C_j \sup_{t\in[0,T]} \left|\frac{d^j}{dt^j} z_n(t)\right| \\ &\leq \frac{-\left[\frac{-1}{\alpha}\right] L}{|\lambda_n|} \sum_{j=0}^{\infty} C_j |M_j| \frac{(j!)^{\beta}}{K^j}, \end{aligned} \quad (26)$$

where C_j is defined in (25) and $-[-1/\alpha]$ denotes the number of the derivative order j. By Proposition 6, we obtain that $\tilde{y}_n(s)$ is an entire function of finite order $d \in [1/2, n/2]$. According to Levin (1996), introduce the function $M(\varrho) = \max_{|s|=\varrho^{\alpha}} |\tilde{y}_n(s)|$, it yields that the asymptotic inequality $M(\varrho) \leq \exp(a\varrho^d)$ is fulfilled for some $a > 0$. With this, Lemma 1 of Levin (1996) implies

$$|M_j| \leq \left(\frac{ead}{j}\right)^{j/d} \text{ for all } j \geq 0. \quad (27)$$

It then follows from $(j)^j \geq j!$ that (26) can lead to (23).

Furthermore, based on the Cauchy-Hadamard theorem in Hadamard (2014), the radius r of convergence of the series $\sum_{j=0}^{\infty} \varsigma_j$ is $1/r = \limsup_{j\to\infty} |\varsigma_j|^{1/j}$. It follows from the property of the Gamma function that there exists a constant c_1 satisfying $\alpha\Gamma(\alpha) \geq c_1$ for all $\alpha \in (0, 1]$. Then, $|C_j| \leq T^{n_j-\alpha j}/\varepsilon$ and the radius r of convergence for the power series (23) is

$$1/r = \frac{(ead)^{1/d}}{K} \limsup_{j\to\infty} (j!)^{\frac{\beta-1/d}{j}}. \quad (28)$$

Moreover, if $\beta < 1/d$, a constant $c_2 > 0$ can be found such that $\sum_{n\in\mathbf{N}^k} |y_n(t)|^2 \leq \sum_{n\in\mathbf{N}^k} \frac{c_2}{|\lambda_n|^2}$. Hence, the series $\sum_{n\in\mathbf{N}^k} y_n(t)\xi_n(x)$ is uniformly convergent in $C\left(0,T;L^2(\Omega)\right)$. Besides, if $\beta = 1/d$, the condition $K < (ead)^{1/d}$ has to be imposed to guarantee the finite radius of convergence of the series.

Next, we discuss the convergence of the control input parametrization $u_n(t)$. Since $\tilde{u}_n(s) + \mathcal{L}\{f_n(y_n)\}(s)$ given by (15) is an entire function of finite order $d \in [1/2, n/2]$ as a consequence of Proposition 6, one has

$$\begin{aligned} &\sup_{t\in[0,T]} |u_n(t) + (f_n(y_n))(t)| \\ &\leq \sum_{j=0}^{\infty} |N_j| \sup_{t\in[0,T]} \left| {}_0I_t^{n_j-\alpha j} \frac{d^{n_j}}{dt^{n_j}} z_n(t)\right| \\ &\leq -\left[\frac{-1}{\alpha}\right] L \sum_{j=0}^{\infty} C_j \frac{(ead)^{j/d}(j!)^{\beta-1/d}}{K^j}. \end{aligned} \quad (29)$$

This, together with Eq.(23) and *Assumption* 1, leads to

$$\begin{aligned} &\sup_{t\in[0,T]} |u_n(t)| \\ &\leq -\left[\frac{-1}{\alpha}\right] L \left(1+\frac{c}{|\lambda_n|}\right) \sum_{j=0}^{\infty} C_j \frac{(ead)^{j/d}(j!)^{\beta-1/d}}{K^j}. \end{aligned}$$

Similarly, its convergence radius r satisfies

$$1/r = \frac{(ead)^{1/d}}{K} \limsup_{j\to\infty} (j!)^{\frac{\beta-1/d}{j}}. \quad (30)$$

Then, the series $\sum_{n\in\mathbf{N}^k} u_n(t)\xi_n(x)$ parameterized by (19) is uniformly convergent in $C\left(0,T;L^2(\Omega)\right)$ for $K < \infty$ if $\beta < 1/d$ and for $K < (ead)^{1/d}$ if in addition $\beta = 1/d$. This completes the proof. □

Based on Theorem 7, we obtain that the system variable $y(x,t)$ and the controller $Bu(t)$ satisfy

$$\begin{aligned} y(x,t) &= \sum_{n\in\mathbf{N}^k} \left(\sum_{j=0}^{\infty} \frac{-M_j}{\lambda_n} {}_0^C D_t^{\alpha j} z_n(t)\right) \xi_n(x), \\ Bu(t) &= \sum_{n\in\mathbf{N}^k} \left(\sum_{j=0}^{\infty} N_j \, {}_0^C D_t^{\alpha j} z_n(t) - (f_n(y_n))(t)\right) \xi_n(x), \end{aligned} \quad (31)$$

where $(f_n(y_n))(t) = (f(y(\cdot,t)), \xi_n)$. Then, the trajectory planning problem for system (1) under control input $Bu(t)$ within a prescribed finite time interval can be solved.

4. TRAJECTORY PLANNING AND FEEDFORWARD TRACKING CONTROL

The objective of this section is to propose two approaches for the solution of the trajectory planning problem of system (1).

4.1 Direct approach

Motivated by the above consideration, especially the introduction of the bump function in (21), the desired trajectory for the basic output can be designed according to

$$z_n(t) = c_z^n \varphi_{\gamma,T}(t), \tag{32}$$

where $\{c_z^n\}_{n\in\mathbf{N}^k}$ is a sequence of constants satisfying $\sum\limits_{n\in\mathbf{N}^k} (c_z^n)^2 < \infty$, i.e., $c_z^n \in l^2$. Based on the smooth compact support property of bump function $\varphi_{\gamma,T}(t)$, we obtain that $z_n(t)$ is a Gevrey function of order $\beta = 1 + \frac{1}{\gamma} \in (1,2]$ and satisfies (17). Then the uniform convergence of the system variable and input parameterizations (31) holds in $C\left(0,T;L^2(\Omega)\right)$ as a consequence of the Theorem 7. Therefore, given $y_T \in L^2(\Omega)$, the choice of the trajectory $z(x,t)$ guarantees that the solution of system $(1)-(3)$ is transfered from $y_0(x) = 0$ to $y_T \in L^2(\Omega)$ within $t \in [0,T]$.

4.2 Indirect approach

The direct approach essentially relies on the solution of (1) and a priori knowledge of the corresponding input, which imposes certain structural limitations. To weaken this, we consider an indirect approach.

Taking into account that the spatial distribution $g_i(x)$ of an actuator satisfies $g_i \in L^2(\Omega)$, the control $Bu(t)$ in (31) can be parameterized as follows

$$Bu(t) = \sum_{n\in\mathbf{N}^k} \left\{c_z^n \omega_1(t) - (f_n(y_n))(t)\right\} \xi_n(x). \tag{33}$$

where

$$(f_n(y_n))(t) = \left(f\left(\sum_{n\in\mathbf{N}^k} \frac{-c_z^n \omega_2(t)}{\lambda_n} \xi_n \right), \xi_n \right)$$

and

$$\omega_1(t) = \sum_{j=0}^{\infty} N_j \, {}_0^C D_t^{\alpha j} \varphi_{\gamma,T}(t), \; \omega_2(t) = \sum_{j=0}^{\infty} M_j \, {}_0^C D_t^{\alpha j} \varphi_{\gamma,T}(t).$$

Given $y_T \in L^2(\Omega)$, the control inputs that render $\|y(x,Bu(T)) - y_T\|$ minimal can be reduced to the following minimization problem

$$\min_{c_z^n \in l^2} \|y(x,Bu(T)) - y_T\|^2 \tag{34}$$

subject to (33). Obviously, only a finite set of constants c_z^n can be determined for the flat output. Then the trajectory planning problems is solved for system $(1)-(3)$ starting from $y_0 = 0$ to approximate y_T within $t \in [0,T]$.

5. NUMERICAL EXAMPLE

Consider system (1) with $k = 2$, $\alpha = 0.3$, $L_1 = L_2 = 1$, $T = 0.6$, $x = (x_1, x_2)$ and

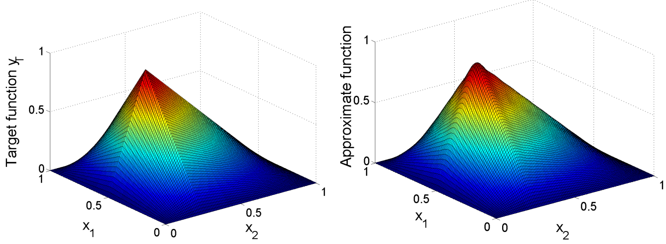


Fig. 2. The comparison between the target function $y_{0.6}$ (left) and its approximate $y(x, Bu(0.6))$ (right).

$$f(y) = 9y + \frac{y}{1+y^2}. \tag{35}$$

Given any $R > 0$ satisfying $\|y\|_E \le R$, it yields that Assumption 1 holds true for $c = 10$. The coefficients of the BCs can be chosen as $p_i = r_i = 0$ and $q_i = s_i = 1$, $i \in I_2$, i.e., Dirichlet BCs. However, Neumann BCs and Robin BCs can be considered in a similar way. In what follows, set the target function $y_{0.6}\,(x_1,x_2) = \varphi(x_1)\psi(x_2) \in L^2(\Omega)$ with $\Omega = (0,1)\times(0,1)$ and

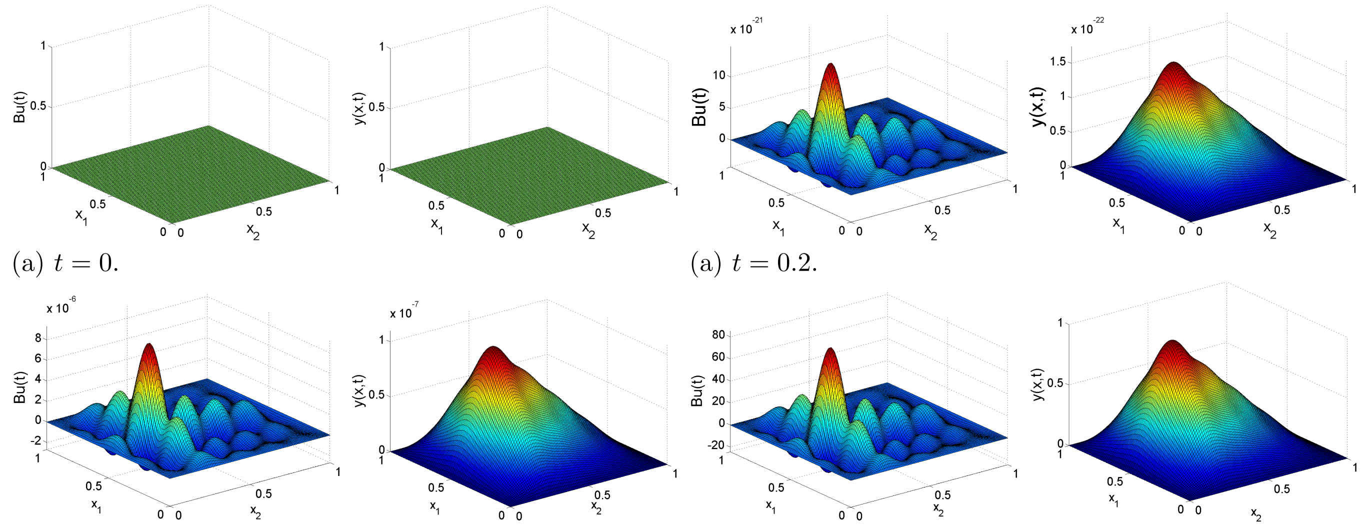
$$\begin{aligned} \varphi(x_1) &= \begin{cases} 2x_1, \; 0 \le x_1 < 0.5, \\ 2 - 2x_1, \; 0.5 \le x_1 \le 1, \end{cases} \\ \psi(x_2) &= \begin{cases} 4x_2, \; 0 \le x < 0.25, \\ (4 - 4x_2)/3, \; 0.25 \le x \le 1 \end{cases} \end{aligned} \tag{36}$$

depicted as the left one of Fig.2, based on the implicit and absolutely stable Crank-Nicholson approach, we solve the trajectory planning problem starting from $y_0 = 0$ to approximate $y_{0.6}$ as close as possible along a pre-planned path within $t \in [0, 0.6]$ following the indirect distributed tracking control approach introduced in Section 4.2.

Indeed, from Curtain and Zwart (2012), one has $\lambda_{n_1n_2} = -(n_1^2 + n_2^2)\pi^2$, $\xi_{n_1n_2}(x_1,x_2) = 2\sin(n_1\pi x_1)\sin(n_2\pi x_2)$ and $\{\xi_{n_1n_2}\}_{n_1,n_2\ge1}$ forms a orthonormal basis of $L^2(\Omega)$. Let $\gamma = 2$. According to Section 4.2, the minimization problem yields a solution $y(x, Bu(0.6))$ with the error $\|y(x,Bu(0.6)) - y_{0.6}\|^2 = 1.327 \times 10^{-5}$. This is shown as the right one of Fig.2. Notice that only a finite number of c_z^n are needed to be determined, by Eq.(33), the feedforward control $Bu(t)$ at different instants of time $t \in \{0, 0.2, 0.4, 0.6\}$ is provided in Fig.3.

6. CONCLUSIONS

In this paper, the trajectory planning problem for a class of semilinear TFRDSs is studied via the flatness-based distributed feedforward control strategic. The conditions under which the studied system could reach to a prescribed desired stationary profile within a prescribed finite time interval are obtained by parameterizing both the system variable and the control input based on the spectral property of system operator with respect to a basic output and its fractional order derivatives. All results here can be extended to more complex nonlinear fractional distributed parameter systems. The trajectory planning problem for nonlinear space-time fractional diffusion systems as well as more new approaches is of great interest.



(a) $t = 0$. (a) $t = 0.2$.

(a) $t = 0.4$. (a) $t = 0.6$.

Fig. 3. Feedforward control Bu(t) and the system vector at different time $t \in \{0, 0.2, 0.4, 0.6\}$.

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