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Parameter preference for the continuous super-twisting-like algorithm based on $\mathcal{H}_\infty$ norm analysis

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Abstract: In variable structured systems, plenty of designs are built to be homogeneous. Such unperturbed homogeneous dynamics with negative homogeneous degree guarantee finite time convergence. Previous studies provide lower bounds for parameters that result in such finite-time convergence property. In this paper, we propose a new perspective on parameter preference, based on $\mathcal{H}_\infty$ norm analysis. Contrary to other studies, which propose such norm non-homogeneous or homogeneous, yet of non-zero degree, we build a homogeneous $\mathcal{H}_\infty$ norm of homogeneous degree zero, thus global and constant. Based on data collected of this norm on the continuous super-twisting-like algorithm, we give recommendations for choosing the parameters.

Keywords: Homogeneity, H-infinity norm, continuous super-twisting-like algorithm

1. INTRODUCTION

Homogeneous dynamics are widely used in variable structure systems. This is particularly the case in sliding mode algorithms, since negative homogeneous degree of a system indicates global finite time convergence (FTC) (Bacciotti and Rosier, 2005). Homogeneous controller and observer designs have been proposed e.g. by Andrieu et al. (2008); Bernuau et al. (2014); Levant (2005); Qian and Lin (2006).

The $\mathcal{H}_\infty$ norm can be interpreted either as the maximum amplitude of a frequency response for linear time-invariant systems or as the maximum $\mathcal{L}_2$ gain from input to output (Başar and Bernhard, 1995). When taking the latter interpretation, concepts based on the $\mathcal{H}_\infty$ norm can also be transferred to nonlinear systems (Khalil, 2003; van der Schaft, 2000).

Hong (2001) applies the $\mathcal{H}_\infty$ norm to a nonlinear homogeneous affine system. In the author's remark, it is noticed that such norm might not be constant. Unnoticed by the author at that time, using traditional Hamilton-Jacobi-Isaacs inequality and forming quadratic terms of control and disturbance in the inequality can result in a constant $\mathcal{H}_\infty$ norm only if the input and output are of the same homogeneous weight.

Zhang and Reger (2018, 2019) employ the state transformation from Moreno and Osorio (2008) and the traditional Hamilton-Jacobi-Isaacs inequality as in (Başar and Bernhard, 1995; Hong, 2001) to suggest a convex parameter set for observer design and a lower bound for controller design by studying the $\mathcal{H}_\infty$ norm for the super-twisting algorithm (STA). Yet, despite the optimal parameter range is global, its corresponding $\mathcal{H}_\infty$ norm is of non-zero homogeneous degree, thus is a local norm for the transformed system.

In this paper, we develop a homogeneous $\mathcal{H}_\infty$ norm of homogeneous degree zero, thus global and constant, and apply it to the continuous super-twisting-like algorithm (CSTLA) in Sánchez et al. (2018). In doing so we can confidently compare the corresponding $\mathcal{H}_\infty$ norm using different parameter sets. In addition we are able to verify the preferred parameter set derived by a similar method in Zhang and Reger (2018) by comparing the corresponding constant $\mathcal{H}_\infty$ norm. Further, by studying figures indicative of the behavior of CSTLA, we find the closed analytical form of this constant $\mathcal{H}_\infty$ norm for the parameter preference range. This means that we provide such worst input that achieves this norm, thus making the constant $\mathcal{H}_\infty$ norm a tight maximum $\mathcal{L}_2$ gain.

2. SUPER-TWISTING-LIKE ALGORITHM

For the continuous super-twisting-like algorithm (CSTLA) we first normalize the homogeneous weight of x_2 to $\tau_2 = 1$ and set the homogeneous degree of the dynamics to $d \in [-1, 0]$. The closed loop system reads (Sánchez et al., 2018)

$$\begin{aligned} \dot{x}_1 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + x_2 \\ \dot{x}_2 &= -k_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} + b\,\phi \end{aligned} \tag{1}$$

where $\lceil x \rfloor^\vartheta$ is the sign preserving power $\lceil \cdot \rfloor^\vartheta = |\cdot|^\vartheta \mathrm{sign}(\cdot)$. The homogeneous weight of x_1 is $\tau_1 = 1 - d$ and that of ϕ is $\tau_\phi = 1 + d$. When $d = 0$, system (1) is linear, and in case $d = -1$, it is in STA form.

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Note that the derivatives of $\lceil x \rfloor^{\vartheta}$ and $|x|^{\vartheta}$ with $\vartheta \in (0, 1]$ and scalar $x \in \mathbb{R} \setminus \{0\}$ are

$$\frac{d\lceil x \rfloor^{\vartheta}}{dx} = \vartheta |x|^{\vartheta-1}, \quad \frac{d|x|^{\vartheta}}{dx} = \vartheta \lceil x \rfloor^{\vartheta-1},$$

resp., and $x = 0$ is a singular point for the derivative when $\vartheta \in (0, 1)$. Let us probe system (1) in a different way and multiply the dynamics by a constant $L > 0$, i.e.

$$\begin{aligned} L\dot{x}_1 &= -Lk_1 \lceil x_1 \rfloor^{\frac{1}{1-d}} + Lx_2 \\ L\dot{x}_2 &= -Lk_2 \lceil x_1 \rfloor^{\frac{1+d}{1-d}} + Lb\,\phi \end{aligned}$$

as in Levant and Alelishvili (2008), and exchange the variables (x_1, x_2, ϕ) with $(\hat{x}_1, \hat{x}_2, \hat{\phi}) = (Lx_1, Lx_2, L\phi)$. Thus we have

$$\begin{aligned} \dot{\hat{x}}_1 &= -L^{\frac{-d}{1-d}} k_1 \lceil \hat{x}_1 \rfloor^{\frac{1}{1-d}} + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -L^{\frac{-2d}{1-d}} k_2 \lceil \hat{x}_1 \rfloor^{\frac{1+d}{1-d}} + b\,\hat{\phi} \end{aligned} \tag{2}$$

and if we initialize system (1) at (x_{1o}, x_{2o}) with input $\phi(t)$, then in system (2) we initialize at (Lx_{1o}, Lx_{2o}) with input $L\phi(t)$. We would end up with the same scaled trajectory $(\hat{x}_1, \hat{x}_2, \hat{\phi})$ in (2) when replacing the parameters as per $(k_1, k_2) \mapsto (L^{\frac{-d}{1-d}} k_1, L^{\frac{-2d}{1-d}} k_2)$. So in this homogeneous system, such parameter set is not unique. It can always be scaled in this manner and will result in the same trajectory by a linear mapping. In view of this, the problem of choosing the parameter set is turned into finding the relationship between k_1 and k_2 when one is fixed, as shown in Zhang and Reger (2018, 2019).

Remark 1. For all CSTLA, also STA, the pairing relationship is $(L^{\frac{1}{2}} k_1, Lk_2)$. That is why in (Zhang and Reger, 2018) the parameter set for the STA observer is $k_1 = 2\sqrt{k_2}$. This means that if k_2 is picked L times larger, then k_1 should be paired with $\sqrt{L}$ times larger, to have the system result in the same linearly scaled trajectory, i.e. the same behavior. Levant (2001) offers $k_1 = 1.5\sqrt{b}$, $k_2 = 1.1b$ for the STA, which is the parameter set widely used in studies. The scaling relationship is then mimicked through b.

3. $\mathcal{H}_\infty$ NORM OF ZERO HOMOGENEOUS DEGREE

When extended to nonlinear systems, the $\mathcal{H}_\infty$ norm can be interpreted as the $\mathcal{L}_2$ gain from input to output (Khalil, 2003; van der Schaft, 2000). Yet we shall first look at the linear system, here for $d = 0$ in (1). We obtain

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 + x_2 \\ \dot{x}_2 &= -k_2 x_1 + b\phi \end{aligned}$$

and by linearity, a linearly transformed input $L\phi$ results in also a linearly transformed state trajectory $Lx_1(t), Lx_2(t)$. Thus, the $\mathcal{H}_\infty$ norm defined for the linear case from input ϕ to the state x, i.e.

$$\gamma^\dagger = \sup_{\phi \neq 0} \frac{\|x\|_2}{\|\phi\|_2}$$

using the truncated $\mathcal{L}_2$ norm $\|x\|_2 = \sqrt{\int_0^T x(t)^\top x(t)\, dt}$ of time signal x, is not changed under this linear mapping.

For cases where $d \in (-1, 0]$, the homogeneous CSTLA (1) allows to construct a norm which is of homogeneous degree 0 which then will retain unchanged under a homogeneous dilation. For example, if the initial value, input and time are scaled for system (1) according to

$$\begin{aligned} x_1(0) &\to \kappa^{1-d} x_1(0) \\ x_2(0) &\to \kappa x_2(0) \\ \phi(t) &\to \kappa^{1+d} \phi(\kappa^{-d} t) \end{aligned} \tag{3}$$

with $\kappa > 0$ and $t \in [0, T]$ leads to the scaled trajectory as

$$\begin{aligned} x_1(t) &\to \kappa^{1-d} x_1(\kappa^{-d} t) \\ x_2(t) &\to \kappa x_2(\kappa^{-d} t). \end{aligned} \tag{4}$$

Using transformation $\xi = \left(\lceil x_1 \rfloor^{\frac{1}{1-d}}, x_2\right)^\top$, similar to the one first introduced by Moreno and Osorio (2008), we shall define a $\mathcal{H}_\infty$ norm of zero homogeneous degree, see (Zhang and Reger, 2018; Hong, 2001; Zhou et al., 1996), s.t. for any $d \in (-1, 0]$

$$\gamma'(k_1, k_2, b) = \sup_{\phi \neq 0} \frac{\|E\xi\|_2}{\|\lceil \phi \rfloor^{\frac{1}{1+d}}\|_2}, \tag{5}$$

where $E = \mathrm{diag}\left(\sqrt{E_1}, \sqrt{E_2}\right), E_1, E_2 > 0$ for emphasis on which state to minimize. Note that $\|E\xi\|_2$ and $\|\lceil \phi \rfloor^{\frac{1}{1+d}}\|_2$ are the truncated $\mathcal{L}_2$ norm of the transformed variable ξ and $\lceil \phi \rfloor^{\frac{1}{1+d}}$ instead of original x and ϕ. The transformation is diffeomorphism for $d \in (-1, 0]$. In this sense the norm shall be called homogeneous $\mathcal{L}_2$ gain. Then γ' will keep its value under the scaled trajectory of (3) and (4). Observe that γ' is of homogeneous degree 0, since both the numerator and denominator are of homogeneous degree 1.

The $\mathcal{H}_\infty$ norms in Hong (2001); Zhang and Reger (2018, 2019) are homogeneous, yet of a non-zero degree, and yield a scaled norm under the homogeneous dilation in (3) and (4). For example Zhang and Reger (2018) defined the norm

$$\lambda = \sup_{\phi \in \Phi} \frac{\|E\xi\|_2}{\|\phi\|_2},$$

for (1) with $d = -1$ (STA case); for the definition of the set Φ refer to Zhang and Reger (2018). Such norm for (1) is of homogeneous degree $-d$, thus serves only as a local norm. That is, under the scaled trajectory of (3) and (4), the norm is also scaled by κ^{-d}. As shown by Zhang and Reger (2018, 2019), the derived norm will require x_1 not leaving a neighborhood of equilibrium to remain valid.

Remark 2. For $d = -1$, γ' in (5) is undefined since with $k_2 > |b\phi|$ the system enjoys FTC for non-vanishing input ϕ. Then $\gamma' = 0$. Whenever $k_2 < |b\phi|$, the state may be excited, depending on the magnitude and persistence time of ϕ, and γ' tends to infinity such that no supremum will exist. Therefore, Zhang and Reger (2018, 2019) have to restrict the input $\phi \in \Phi$ to keep the states bounded.

4. METHOD FOR CALCULATING THE $\mathcal{H}_\infty$ NORM

In order to calculate the norm in (5), let us first introduce

Lemma 3. (Cruz-Zavala and Moreno, 2016; Hestenes, 1966) Let $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\omega : \mathbb{R}^n \to \mathbb{R}_+$, that is $\omega(x) \geq 0$, $\forall x \in \mathbb{R}^n$, be two continuous homogeneous functions with the same homogeneous weight $\tau = (\tau_1, \cdots, \tau_n)$ and homogeneous degree m, such that

$$\{x \in \mathbb{R}^n \backslash \{0\} : \omega(x) = 0\} \subseteq \{x \in \mathbb{R}^n \backslash \{0\} : \psi(x) < 0\}.$$

Then there exists a real number $\gamma^\star$ such that for all $\gamma \geq \gamma^\star$ and all $x \in \mathbb{R}^n \backslash \{0\}$, and some $c > 0$, we have $\psi(x) - \gamma\omega(x) < -c\|x\|_{\tau,p}^m$, where the ν^τ-homogeneous norm is defined by Bacciotti and Rosier (2005) such that for $p \geq 1$

$$\|x\|_{\tau,p} \triangleq \left(\sum_{i=1}^n |x_i|^{\frac{p}{\tau_i}}\right)^{\frac{1}{p}}.$$

With such lemma, we may show

Theorem 4. If the unperturbed system (1) with $d \in (-1, 0]$ is locally asymptotically stable for some (k_1, k_2), then it is input to state stable (ISS). There exists a storage function $V \in \mathcal{C}^1$ of homogeneous degree $2-d$ and a constant $\underline{a}_1$, s.t. for all $a_1 > \underline{a}_1$, $V_\gamma = a_1 V$, which for $x \in \mathbb{R}^2 \backslash \{0\}$ satisfies

$$\dot{V}_\gamma\big|_{\phi\equiv 0} + E_1|x_1|^{\frac{2}{1-d}} + E_2|x_2|^2 < 0. \tag{6}$$

For each such V_γ a finite $\gamma^\star$ exists, s.t. for all $\gamma > \gamma^\star$, the value function satisfies

$$J_\gamma \triangleq \dot{V}_\gamma + E_1|x_1|^{\frac{2}{1-d}} + E_2|x_2|^2 - \gamma^2|\phi|^{\frac{2}{1+d}} \le 0. \tag{7}$$

Then $\gamma^\star$ is an upper bound of the $\mathcal{H}_\infty$ norm γ' of system (1). It can be calculated by searching the solutions of

$$\begin{aligned}\gamma^{\star 2} &= \max_{x_1, x_2, \phi} \zeta(V_\gamma, E_1, E_2, x_1, x_2, \phi)\\ \zeta &= \frac{\dot{V}_\gamma + E_1|x_1|^{\frac{2}{1-d}} + E_2|x_2|^2}{|\phi|^{\frac{2}{1+d}}}\end{aligned} \tag{8}$$

on the surface of the unit sphere wrt. x_1, x_2, ϕ.

Proof. With stability of the unperturbed homogeneous system (1) for $d \in (-1, 0]$, the dynamics are continuous. Then there is a storage function $V \in \mathcal{C}^1$ of homogeneous degree $2 - d$, when $\phi \equiv 0$, to serve as a strict Lyapunov function (Th. 5.8 in Bacciotti and Rosier (2005), Rosier (1992)). With the storage function, there is a $c > 0$ s.t.

$$\dot{V}_{\phi\equiv 0} \le -c\|x\|^2_{\tau,1} = -c|x_1|^{\frac{2}{1-d}} - 2c|x_1|^{\frac{1}{1-d}}|x_2| - c|x_2|^2,$$

see Corollary 5.4 in Bacciotti and Rosier (2005). Thus we can always find $\underline{a}_1 > \max(E_1, E_2)/c$, such that $V_\gamma = a_1 V$ satisfies (6). Define the functions

$$\begin{aligned}\omega(x_1, x_2, \phi) &\triangleq |\phi|^{\frac{2}{1+d}}\\ \psi(x_1, x_2, \phi) &\triangleq \dot{V}_\gamma + E_1|x_1|^{\frac{2}{1-d}} + E_2|x_2|^2\end{aligned}$$

and note that both are continuous and of homogeneous degree 2. Clearly $\omega(x_1, x_2, \phi) \ge 0$. Since $\omega(x_1, x_2, \phi) = 0 \Leftrightarrow \phi = 0$, if we can ensure that $\psi(x_1, x_2, 0) < 0$ and according to Lemma 3 some $\gamma^\star$ exists s.t. for all $\gamma > \gamma^\star$, (7) is satisfied. System (1) is ISS (Başar and Bernhard, 1995) since $\psi(x_1, x_2, 0) < 0$ is equivalent to (6), which is shown above to be valid for each homogeneous storage function V_γ.

Assume that the evolution of state starts at the origin, implying $V_\gamma(0) = 0$. For any $\gamma > \gamma^\star$, (7) holds, yielding

$$\int_0^T J_\gamma dt = V_\gamma(T) - V_\gamma(0) + \|E\xi\|_2^2 - \gamma^2\|\lceil\phi\rfloor^{\frac{1}{1+d}}\|_2^2 \le 0$$

which leads to

$$\|E\xi\|_2^2 \le \gamma^2\|\lceil\phi\rfloor^{\frac{1}{1+d}}\|_2^2.$$

Consequently, the smallest $\gamma^\star$ that renders $J_\gamma \le 0$ for all time is the corresponding constant norm for this V_γ.

Whenever $\phi \ne 0$, the numerator and denominator of ζ in (8) are of homogeneous degree 2. Therefore ζ is a homogeneous function with degree 0. In addition note that the value of $\zeta(x_1, x_2, \phi)$ in the whole space of $(x_1, x_2, \phi) \in \mathbb{R}^3$ can be projected onto the unit sphere, since by definition of homogeneity for all $\kappa > 0$ we have

$$\zeta(\kappa^{1-d}x_1, \kappa x_2, \kappa^{1+d}\phi) = \kappa^0\zeta(x_1, x_2, \phi).$$

Then we can find κ such that

$$\begin{gathered}x_1' = \kappa^{1-d}x_1, x_2' = \kappa x_2, \phi' = \kappa^{1+d}\phi,\\ {x_1'}^2 + {x_2'}^2 + {\phi'}^2 = 1\end{gathered}$$

by solving

$$\kappa^{2-2d}x_1^2 + \kappa^2 x_2^2 + \kappa^{2+2d}\phi^2 = 1. \tag{9}$$

When $\kappa = 0$, the left-hand side of (9) is 0, for $\kappa \to \infty$, it tends to infinity. By continuity of the fractional order polynomial, there exist a solution of κ, which makes the left-hand side equal to 1. Then $\zeta(x_1', x_2', \phi') = \zeta(x_1, x_2, \phi)$, thus we can search only on the unit sphere wrt. x_1, x_2, ϕ to probe the value of ζ in $\mathbb{R}^3$. □

Since the disturbance input ϕ in (1) is affine, we are able to simplify the general search of (8) using the next corollary.

Corollary 5. Let $f(x) = (f_1(x),\ f_2(x))^\top$, $B = (0,\ b)^\top$ with

$$\begin{aligned}f_1(x) &= -k_1\lceil x_1\rfloor^{\frac{1}{1-d}} + x_2\\ f_2(x) &= -k_2\lceil x_1\rfloor^{\frac{1+d}{1-d}}\end{aligned}$$

s.t. (1) becomes $\dot{x} = f(x) + B\phi$. Then with storage function V_γ of Theorem 4 the bound $\gamma^\star$ follows by search wrt.

$$\begin{aligned}\gamma^{\star 2} &= \left|\max_{x_1, x_2} \eta(V_\gamma, E_1, E_2, x_1, x_2)\right|^{\frac{1-d}{1+d}}\\ \eta &= C\frac{\left\lfloor\frac{\partial V_\gamma}{\partial x}B\right\rceil^{\frac{2}{1-d}}}{-J_\gamma\big|_{\phi\equiv 0}}, \quad C = \left|\frac{1+d}{2}\right|^{\frac{1+d}{1-d}} - \left|\frac{1+d}{2}\right|^{\frac{2}{1-d}}\end{aligned} \tag{10}$$

on the curve of the unit circle wrt. x_1, x_2.

Proof. Choose storage function V_γ in Th. 4 and obtain

$$\begin{aligned}J_\gamma &= \frac{\partial V_\gamma}{\partial x}(f(x) + B\phi) + E_1|x_1|^{\frac{2}{1-d}} + E_2|x_2|^2 - \gamma^2|\phi|^{\frac{2}{1+d}}\\ &= J_\gamma\big|_{\phi\equiv 0} + \frac{\partial V_\gamma}{\partial x}B\phi - \gamma^2|\phi|^{\frac{2}{1+d}}\end{aligned}$$

since for any (x_1, x_2) when $\phi \to \pm\infty$, $J_\gamma \to -\infty$. So we can take the partial derivative against ϕ to get

$$\frac{\partial J_\gamma}{\partial \phi} = \frac{\partial V_\gamma}{\partial x}B - \frac{2}{1+d}\gamma^2\lceil\phi\rfloor^{\frac{1-d}{1+d}},$$

and find the maximum of J_γ with respect to ϕ along

$$\phi = \left|\frac{1+d}{2\gamma^2}\right|^{\frac{1+d}{1-d}}\left\lceil\frac{\partial V_\gamma}{\partial x}B\right\rfloor^{\frac{1+d}{1-d}}. \tag{11}$$

Plugging back this homogeneous worst input ϕ, we have

$$J_\gamma = J_\gamma\big|_{\phi\equiv 0} + \gamma^{2\frac{-1-d}{1-d}}C\left|\frac{\partial V_\gamma}{\partial x}B\right|^{\frac{2}{1-d}}.$$

Note that (6) in Th. 4 indicates $J_\gamma\big|_{\phi\equiv 0} < 0$. So solving for the smallest $\gamma^\star$ s.t. $J_\gamma \le 0$, we have (10). With η of homogeneous degree 0, its value with $(x_1, x_2) \in \mathbb{R}^2$ can be projected onto the unit circle curve $x_1^2 + x_2^2 = 1$. □

Eventually we shall search for

$$\gamma' = \min_{V_\gamma} \gamma^\star(V_\gamma)$$

where γ' as in (5) is the $\mathcal{L}_2$ gain for given b, k_1, k_2, E_1, E_2. For $d = -1$ (STA case) it is not defined. However, we may study the STA case for the limit $d \to -1$.

Remark 6. Since J_γ in (7) is homogeneous, the scaling in (2) can be applied to J_γ. Thus we can scale some parameter in J_γ to reduce the study by one dimension, i.e. without loss of generality, we can fix b and allow changes to all the other parameters and learn the behavior of the gain γ'.

5. LYAPUNOV-FUNCTION FAMILY AND SEARCH

5.1 Polynomial Lyapunov Function

Along with Sánchez and Moreno (2014) we are selecting a family of homogeneous storage functions of homogeneous degree $2-d$ with stabilizing (k_1, k_2) as per

$$V_\gamma = a_1 V = a_1 \left(\tfrac{1-d}{2-d}|x_1|^{\frac{2-d}{1-d}} - a_{12}x_1x_2 + \tfrac{a_2}{2-d}|x_2|^{2-d} \right) \quad (12)$$

with $a_1, a_{12}, a_2 > 0$. The time derivative expands to

$$\begin{aligned}\dot{V} &= (k_2a_{12} - k_1)|x_1|^{\frac{2}{1-d}} - a_{12}|x_2|^2 \\ &+ (1 + k_1a_{12})\lceil x_1 \rfloor^{\frac{1}{1-d}} x_2 - k_2a_2\lceil x_1 \rfloor^{\frac{1+d}{1-d}} \lceil x_2 \rfloor^{1-d} \\ &- ba_{12}x_1\phi + ba_2\lceil x_2 \rfloor^{1-d}\phi. \end{aligned} \quad (13)$$

With such V_γ equation (11) becomes

$$\phi = \left| \frac{ba_1(1+d)}{2\gamma^2} \right|^{\frac{1+d}{1-d}} \left\lceil -a_{12}x_1 + a_2\lceil x_2 \rfloor^{1-d} \right\rfloor^{\frac{1+d}{1-d}}.$$

5.2 Rendering $V_{\phi\equiv 0}$ a Lyapunov function

In order to ensure positive definiteness of V we use Young's inequality (Moreno, 2012): For any real numbers $a > 0$, $b > 0$, $c > 0$, $p > 1$ and $q > 1$, with $p^{-1} + q^{-1} = 1$ holds

$$ab \le \frac{c^p}{p}a^p + \frac{c^{-q}}{q}b^q.$$

First of all, when $x_1x_2 < 0$, V is surely positive definite. So let us look at the case $x_1x_2 > 0$. We may then rewrite

$$V = \tfrac{1-d}{2-d}|x_1|^{\frac{2-d}{1-d}} - a_{12}|x_1||x_2| + \tfrac{a_2}{2-d}|x_2|^{2-d}. \quad (14)$$

Now we use Young's inequality, letting $p = \frac{2-d}{1-d}$, $q = 2-d$, which satisfies $p^{-1} + q^{-1} = 1$. Consequently, we have

$$|x_1||x_2| \le \tfrac{(1-d)c^{\frac{2-d}{1-d}}}{2-d}|x_1|^{\frac{2-d}{1-d}} + \tfrac{c^{-2+d}}{2-d}|x_2|^{2-d}$$

which yields

$$V \ge \tfrac{1-d}{2-d}(1 - a_{12}c^{\frac{2-d}{1-d}})|x_1|^{\frac{2-d}{1-d}} + \tfrac{a_2 - a_{12}c^{d-2}}{2-d}|x_2|^{2-d}$$

and for positive definite V leads to

$$1 \ge a_{12}c^{\frac{2-d}{1-d}} \quad \text{and} \quad a_2 \ge a_{12}c^{d-2}.$$

So finally we require

$$(1/a_{12})^{1-d} \ge a_{12}/a_2 \quad \Leftrightarrow \quad a_2 \ge a_{12}^{2-d}$$

for positive definite V.

Since the CSTLA is ISS, as shown in Theorem 4, in (13) we are asking for that $\dot{V}_{\phi\equiv 0} < 0$, thus

$$\begin{aligned}\dot{V}_{\phi\equiv 0} &= (k_2a_{12} - k_1)|x_1|^{\frac{2}{1-d}} + (k_1a_{12} + 1)\lceil x_1 \rfloor^{\frac{1}{1-d}} x_2 \\ &- k_2a_2\lceil x_1 \rfloor^{\frac{1+d}{1-d}} \lceil x_2 \rfloor^{1-d} - a_{12}|x_2|^2 < 0\end{aligned}$$

and obviously we need $k_2a_{12} \le k_1$. We may only derive the analytical region of a_{12}, a_2 where $V_{\phi\equiv 0}$ is a Lyapunov function for special cases $d = 0$ (linear) and $d = -1$ (STA) by using both Young's inequality and homogeneity. Thus, the region of a_{12}, a_2 that renders (14) a Lyapunov function will be found by carrying out search wrt. x_1, x_2.

Remark 7. Conditions $a_2 \ge a_{12}^{2-d}$ and $k_2a_{12} \le k_1$ are necessary conditions for $V_{\phi\equiv 0}$ to be a Lyapunov function. They are always satisfied when searching the region of $J_\gamma|_{\phi\equiv 0} \le 0$ in the following section. At the same time, the analytical region for (k_1, k_2) that renders (12) a Lyapunov function is not within the scope of this paper.

5.3 Description of simulation process

The search for single $\gamma'(k_1, k_2, b, E_1, E_2)$ is performed as:

1. First find an $\underline{a}_1 \ge 0$ small enough, yet under which the range of (a_{12}, a_2) satisfying (6) still exist. Use such $\underline{a}_1$ as the first a_1.
2. Fix an a_1 from Step 1 or Step 5. Search the region of (a_{12}, a_2) satisfying (6), by carrying out a search on the unit circle $x_1^2 + x_2^2 = 1$.
3. Within the region of (a_{12}, a_2) in Step 2, carrying out a maximum search for $\zeta(a_1, a_{12}, a_2)$ in (8) on the unit sphere $x_1^2 + x_2^2 + \phi^2 = 1$. Or $\eta(a_1, a_{12}, a_2)$ in (10) on the unit sphere $x_1^2 + x_2^2 = 1$.
4. Conduct refined searches around the pair (a_{12}, a_2) with the smallest $\gamma^\star(a_1, a_{12}, a_2)$ collected in Step 3. Record the smallest $\gamma^\star$ among this search as $\gamma^\star(a_1)$.
5. Compare between $\gamma^\star(a_1)$ and choose the next a_1 to return to Step 2 for next loop from Step 2 to Step 4; until the gain $\gamma^\star(a_1)$ cannot be improved noticeably.
6. The smallest of such $\gamma^\star(a_1)$ will be recorded as γ' of this k_1, k_2, b, E_1, E_2.

Here $\gamma^{\star 2}(a_1) \triangleq \min_{a_{12}, a_2} \max_{x_1, x_2, \phi} \zeta$ means to take minimum and maximum wrt. any parameter with fixed a_1. Finally, we can change k_1, k_2 to observe its influence on γ', as shown in the next section.

The first subfigure in Fig. 1 shows convergence of $\gamma^{\star 2}(a_1)$ formed by the outermost iteration. The second subfigure shows the range of a_{12} and a_2 and the optimal pair in red cross by the refined search as described in Step 2, 3, and 4. The last subfigure shows $\gamma^{\star 2}(a_1, a_{12}, a_2)$ calculated for each point in the second subfigure. To highlight the convexity of $\gamma^{\star 2}(a_1, a_{12}, a_2)$ wrt. a_{12}, a_2, in this figure we set a maximum at 500. Since the suboptimal values of $\gamma^{\star 2}(a_1, a_{12}, a_2)$ are too big, the clear convexity is clouded.

6. ANALYSIS BASED ON COLLECTED DATA

6.1 Figure and analysis of γ' after search

The qualitative result is probed extending the range of preferred parameters to CSTLA as in Zhang and Reger (2018), i.e.

$$\underline{k}_1^\star \triangleq \sqrt{\frac{3}{2}(1-d)k_2}, \quad \overline{k}_1^\star \triangleq \sqrt{2(1-d)k_2}.$$

In Fig. 2, γ' is plotted for different d and k_2. The left subfigure shows γ' with $E_1 = 1, E_2 = 0$, thus only to channel ξ_1, likewise the right subfigure with $E_1 = 0, E_2 = 1$ to channel ξ_2. Cases $d = -0.5, -0.75, -0.9, -0.99$ are plotted with different color, and for each d three sets of k_2 are adopted, which are $k_2 = 0.99b, k_2 = b, k_2 = 1.01b$. The full tables of optimal a_1, a_{12}, a_2 shall be provided in another paper for lack of space. The findings are as follows:

(1) Similar to the finding of Zhang and Reger (2018, 2019), a bigger k_2 will always reduce the gain γ', so all three lines from above to below are correspondingly $k_2 = 0.99b$, $k_2 = b$ and $k_2 = 1.01b$.
(2) When $k_2 = b, k_1 > \overline{k}_1^\star(d, k_2)$, all four gains γ' of different d coincide. The closed expression for γ' in this region is revealed in (15).
(3) ${\gamma'}_{\xi_1}$ stops decreasing at latest after $k_1 > \overline{k}_1^\star$ (left plot) and there is a clear convexity for ${\gamma'}_{\xi_2}$ (right plot).

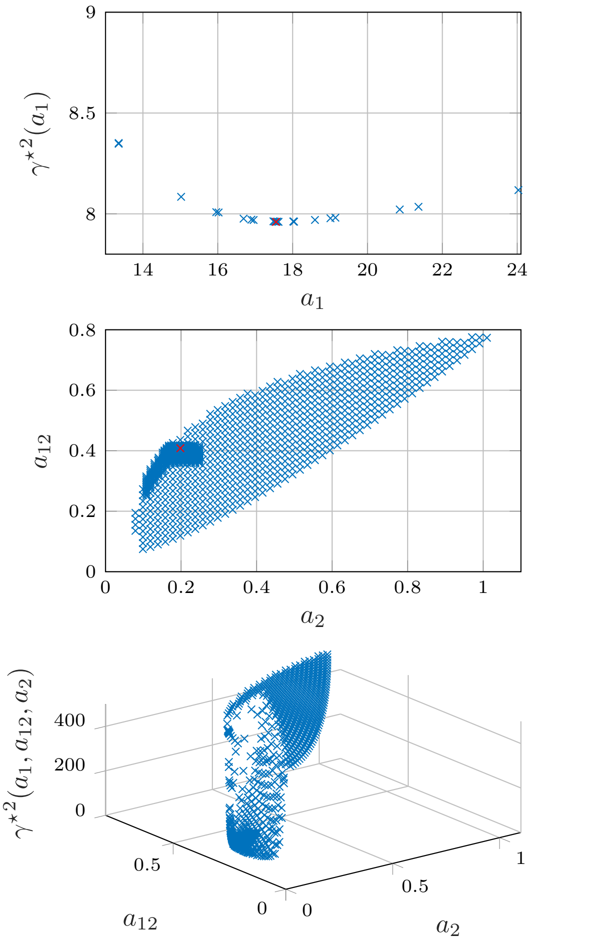

Fig. 1. Convergence of $\gamma^\star(a_1)$: $d = -0.50$, $k_1 = \frac{1}{3}(2\underline{k}_1^\star + \overline{k}_1^\star)$, $k_2 = b = 3$, $E_1 = 0$, $E_2 = 1$.

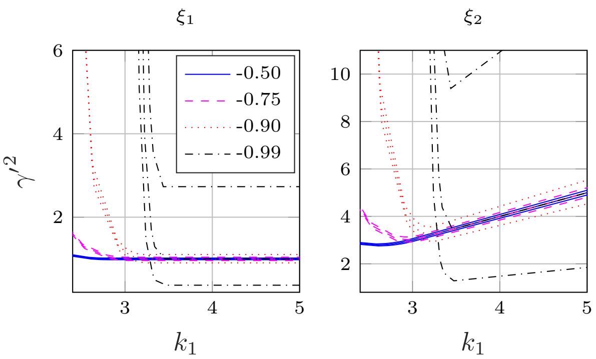

Fig. 2. γ' for different d and k_2.

However, the convex shape is shifted from $\underline{k}_1^\star$ to $\overline{k}_1^\star$ with d decreasing from 0 to -1.

(4) Decreasing d from 0 to -1, the difference between γ' under $k_2 = 0.99b$ and $k_2 = 1.01b$ gets more pronounced in both channels ξ_1 and ξ_2. This shows the tendency of the STA case, where $k_2 > |b\phi|$ will lead to zero gain, while $k_2 < |b\phi|$ to infinite gain.

6.2 Intuition from graphical representation

In Fig. 3, we show the worst ζ that can be achieved at each (ξ_{1o}, ξ_{2o}) for worst ϕ. This may be done by finding a dilation $\overline{\kappa}$ such that $(\overline{\kappa}^{1-d}x_{1o})^2 + (\overline{\kappa}x_{2o})^2 = 1$. Then with $\kappa \in [0, \overline{\kappa}]$, let $\tilde{\phi} = \sqrt{1 - (\kappa^{1-d}x_{1o})^2 - (\kappa x_{2o})^2}$. Now we can project a curve onto the unit sphere and search for ζ along that curve. The worst ϕ that achieves such ζ can be projected back to the original coordinate via $\phi = \kappa^{-1-d}\tilde{\phi}$. The figure resorts to the collected coefficients a_1, a_{12}, a_2 of the optimal Lyapunov function as shown in Fig. 1. Likewise, other cases can lead to such a graph. For saving space, we provide only one figure here.

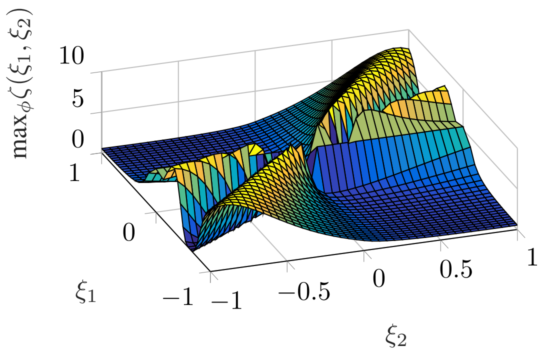

Fig. 3. Simulation results for $\max_\phi \zeta(\xi_1, \xi_2)$ with optimal a_1, a_{12}, a_2, $d = -0.5$, $E_1 = 0, E_2 = 1$, $k_2 = b = 3$, $k_1 = \frac{1}{3}(2\underline{k}_1^\star + \overline{k}_1^\star)$.

Fig. 3 reveals that for nonlinear system (1) the gain γ' is not achievable everywhere, and the input should maintain the trajectory along the peak of Fig. 3 in order to achieve a higher gain. The behavior will be shown in Section 6.4. Note that in the linear case, Fig. 3 will yield a uniformly achievable γ' (Başar and Bernhard, 1995).

6.3 Gain under constant input

Reflecting on the linear case, when $k_1 \geq \overline{k}_1$, the worst input for both channels is the constant input. For (1) a constant ϕ will lead to a new equilibrium at $(\bar{x}_1, \bar{x}_2)$ with

$$\bar{x}_1 = \left(\frac{b}{k_2}\phi\right)^{\frac{1-d}{1+d}}, \quad \bar{x}_2 = k_1\left(\frac{b}{k_2}\phi\right)^{\frac{1}{1+d}}.$$

Thus the $\mathcal{L}_2$ gains from $\lceil\phi\rfloor^{\frac{1}{1+d}}$ to both channels are

$$\gamma_{\xi_1}^2 = \left(\frac{b}{k_2}\right)^{\frac{2}{1+d}}, \quad \gamma_{\xi_2}^2 = k_1^2\left(\frac{b}{k_2}\right)^{\frac{2}{1+d}}. \tag{15}$$

These closed expressions agree with the collected data γ' in Fig. 2 in the region $k_1 \geq \overline{k}_1^\star$.

6.4 Worst input that achieves γ' in time simulation

Other than a constant input, we look for a different worst input that achieves γ'. Fig. 4 shows that with larger ω in $\phi(t) = \sin(\omega t)$, the trajectory of ξ_1/ξ_2 takes more an "S shape". Lower ω leads to a trajectory of ξ_1/ξ_2 more aligned to a line, more specifically, aligned to the peak of ζ in Fig. 3 when $k_1 \geq \overline{k}_1^\star$. In order to avoid chattering from discretization, we construct such input as

$$\phi(t) = W(D\,\mathrm{sign}\,(\sin(\omega t)) + (1 - D)\sin(\omega t)) \tag{16}$$

where W is the magnitude of ϕ and ω is the frequency in rad/s of the sine component. D proportionates the ratio between the signum function and sine function. The

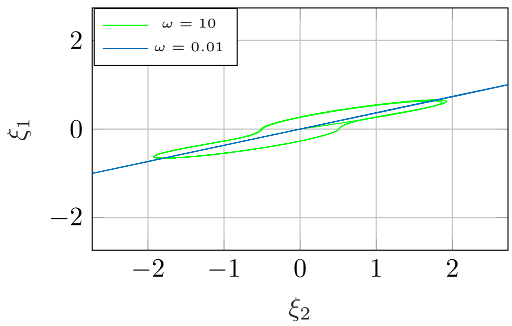

Fig. 4. State trajectory under sine input $\phi = \sin(\omega t)$ for $d = -0.5$, $k_2 = b = 3$, $k_1 = \frac{1}{3}(2\underline{k}_1^\star + \overline{k}_1^\star)$.

Table 1. Achieved $\mathcal{L}_2$ gain for $k_2 = b = 3$, $T = 10^{-4}s$ with ϕ as (16).

d	k_1	W	D	f	$\gamma^2_{\xi_1}$	$\gamma^2_{\xi_2}$
-0.75	$10\overline{k}_1^\star$	0.5	0.45	0.01	0.9975	1047.3
-0.90	$10\overline{k}_1^\star$	0.7	0.65	0.01	0.9974	1137.1
-0.99	$10\overline{k}_1^\star$	0.98	0.96	0.002	0.9989	1192.6
-0.999	$10\overline{k}_1^\star$	0.999	0.999	0.0001	0.9364	1123.1

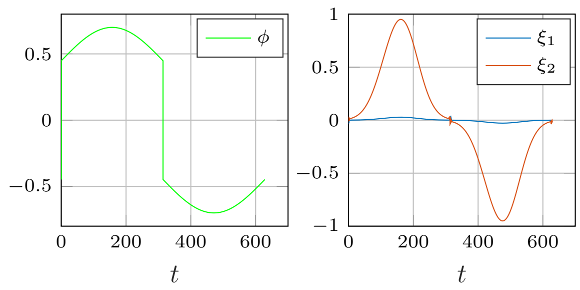

Fig. 5. Time simulation of the second case in Table 1.

resulting $\mathcal{L}_2$ gains are listed in Table 1, which again agree with all the γ' collected for all $k_1 \geq \overline{k}_1^\star$. Since γ' is the upper bound and the actual achieved gain γ is the lower bound of the $\mathcal{H}_\infty$ norm, thus we can say that under the range of $k_1 \geq \overline{k}_1^\star$, we reach the true $\mathcal{H}_\infty$ norm $\gamma' = \gamma$.

7. CONCLUSION

The CSTLA is studied in view of a homogeneous $\mathcal{L}_2$ gain ($\mathcal{H}_\infty$ norm) of zero degree, thus is constant and global. We have verified the $\mathcal{H}_\infty$ norm optimal parameter range derived, similar to Zhang and Reger (2018), by calculating its corresponding global and constant gain. Furthermore we have provided the closed form of such a γ in (15) for the recommended region $k_1 \geq \overline{k}_1^\star = \sqrt{2(1-d)k_2}$.

For fixed k_2 and controller design, we recommend $k_1 \geq \overline{k}_1^\star$. Moreover, even though with larger k_1 the worst γ is constant, yet the worst ϕ needs to be much slower to reach such gain. So larger k_1 practically renders x_1 smaller. For observer design, we notice optimality for k_1 shifting from $\underline{k}_1 = \sqrt{3(1-d)k_2/2}$ to $\overline{k}_1^\star$ from the linear case to the STA, which again is in accordance with Zhang and Reger (2018). Thus, we recommend using $k_1 = \overline{k}_1^\star$ in this case.

Larger k_2 coupled with $k_1 \geq \overline{k}_1^\star$ as recommended above will reduce γ remarkably.

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