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Safe Tracking Control of Euler-Lagrangian Systems Based on A Novel Adaptive Super-twisting Algorithm ⋆

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Abstract: In this paper, a novel adaptive second-order sliding mode controller is designed for Euler-Lagrangian systems with hard safety constraints. Different from the conventional sliding mode controllers, the proposed method provides adaptive controller parameters, such that the robustness of the controller is ensured without bringing up chattering. The controller also guarantees strict compliance to hard state-dependent inequality constraints. The asymptotic convergence of the tracking errors of the proposed controller is proven by a direct Lyapunov method. Finally, the proposed controller is validated by numerical simulation on a three-degree-of-freedom robot platform. The results confirm that the controller ensures strict constraint compliance and precise trajectory tracking, which reveals its potential applicability to the safe control of mechatronic systems.

Keywords: human-robot interaction, adaptive sliding mode control, super-twisting algorithm, parameter self-tuning, robot safety control, state-constrained control, hard safety constraints.

1. INTRODUCTION

Precise tracking control of mechatronic systems in the free space or with equality constraints has been well studied in the past several decades. However, the conventional formulations are mainly concerned with reducing the tracking errors for desired trajectories. Therefore, the conventional control methods and are confronted with challenges when hard safety constraints should be strictly complied with. For example, in human-robot interaction tasks, the robots should be designed under certain proximity or speed limitations to guarantee the safety of humans. The work in (Blanchini, 1999) is among the earliest ones to demonstrate the constrained control problems under the framework of positively invariant set, which inspires the invariance control based methods (Wolff and Buss, 2004). Other popular methods for constrained control include model predictive control (Wilson et al., 2016) and the barrier-function-based methods (Guo et al., 2017). For these methods, however, the safety constraints may be violated when disturbances or unmodeled dynamics exist in the system. Therefore, the robustness of the control methods with inequality constraints is still a challenging problem.

Sliding mode control has been popularly applied to the control of mechatronic systems for its excellent robustness. The super-twisting algorithm, as a second-order sliding mode controller, is popular for robot manipulators (Guendouzi et al., 2013) benefiting the reduced chattering level. Towards a balance between the robustness and chattering, controllers with adaptive gains are proposed (Utkin and Poznyak, 2013) and (Mobayen et al., 2017). As a result, the robustness of the system is ensured without manually assigning the controller parameters, which greatly improves the conventional sliding mode controllers. For the safety of systems, sliding mode controllers with hard state-dependent constraints are also investigated (Bartolini et al., 2000; Incremona et al., 2017). However, robust controllers with both adaptive parameters and hard-constraint compliance are still lacking.

The major contribution of this paper is to propose a novel trajectory tracking controller for Euler-Lagrangian systems, which ensures robustly precise tracking and strict compliance to the hard safety constraints, even with disturbances and system uncertainties. The paper is organized as follows. Sec. 2 formulates the problem to be investigated in this paper. The adaptive super-twisting-based tracking controller is presented in Sec. 3, and improved in Sec. 3.2 for the constraint compliance. In Sec. 5, the proposed method is validated by the simulation on a three-Degree-of-freedom (DoF) robot manipulator. Finally, Sec. 6 concludes the paper.

2. PROBLEM FORMULATION

The dynamic model of an n-DoF Euler-Lagrangian system is formed as follows,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + \boldsymbol{F}(\dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{\mathrm{d}}, \qquad (1)$$

where $\boldsymbol{q}(t)$, $\dot{\boldsymbol{q}}(t)$, $\ddot{\boldsymbol{q}}(t) \in \mathbb{R}^n$ are respectively the vectors of the generalized coordinates, angular velocities and accelerations, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^n$ and $\boldsymbol{F}(\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ are respectively the inertia matrix, Coriolis and centrifugal matrix, gravitational and frictional

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vectors, $\boldsymbol{\tau}(t) \in \mathbb{R}^n$ is the commanded torque input and $\boldsymbol{\tau}_{\mathrm{d}}(t) \in \mathbb{R}^n$ denotes the external disturbance exerted on the actuators. Here, we assume that $\boldsymbol{\tau}_{\mathrm{d}}(t)$ is bounded, i.e., $\|\boldsymbol{\tau}_{\mathrm{d}}(t)\| \leq \epsilon_{\mathrm{d}}$, $\forall\, t > 0$, $\exists\, \epsilon_{\mathrm{d}} \in \mathbb{R}^+$, where $\|\cdot\|$ denotes the 2-norm of vectors. The nominal task of the manipulator is to ensure the precise tracking of a desired trajectory $\boldsymbol{q}_{\mathrm{d}}(t)$, $\dot{\boldsymbol{q}}_{\mathrm{d}}(t)$, $\ddot{\boldsymbol{q}}_{\mathrm{d}}(t) \in \mathbb{R}^n$, such that the tracking errors $\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_{\mathrm{d}}(t)$ converge to zeros.

To ensure safety, the system is confined by a set of hard kinematic constraints which are depicted by state-dependent inequalities $\phi_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) \leq 0$, $i = 1, 2, \cdots, p$, where $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth functions and p is the number of the inequality constraints. The set of the system state in which all the constraints are complied with is referred to as the *admissible set* $\Phi \subseteq \mathbb{R}^n \times \mathbb{R}^n$,

$$\Phi = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) | \phi_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) \leq 0, \forall\, i = 1, 2, \cdots, p\}, \qquad (2)$$

and the system state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Phi$ is called an *admissible state*. Therefore, the system state $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ should be guaranteed admissible for all $t > 0$, by which we formulate the following constrained control problem for system (1).

Problem 1. For system (1), design a state-feedback controller $\boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, such that the following conditions hold for any initial condition $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) \in \Phi$ and bounded disturbance $\boldsymbol{\tau}_{\mathrm{d}}(t)$.

(a). The system state is confined by $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \in \Phi$, $\forall\, t > 0$.

(b). The tracking error $\boldsymbol{e}(t)$ converges to zero, if (a) holds.

3. ROBUST TRACKING CONTROL

In this section, we present the super-twisting-based robust tracking controller with adaptive gains for Problem 2-(b). The stability of the tracking error dynamics is proven by a direct Lyapunov method.

3.1 Robust Controller for Mechatronic Systems

For robust tracking control of system (1), we design the following controller,

$$\boldsymbol{\tau} = \hat{\boldsymbol{M}}(\boldsymbol{q})(\ddot{\boldsymbol{q}}_{\mathrm{d}} - c\dot{\boldsymbol{e}} + \boldsymbol{u}) + \hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \hat{\boldsymbol{G}}(\boldsymbol{q}) + \hat{\boldsymbol{F}}(\dot{\boldsymbol{q}}), \quad (3)$$

where $\hat{\boldsymbol{M}}(\boldsymbol{q})$, $\hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $\hat{\boldsymbol{G}}(\boldsymbol{q})$ and $\hat{\boldsymbol{F}}(\dot{\boldsymbol{q}})$ are the identified system parameters, $c \in \mathbb{R}^+$ is a convergence coefficient to be determined and $\boldsymbol{u}(t) \in \mathbb{R}^n$ is a second-order sliding mode controller in the form of

$$\boldsymbol{u}(t) = -\boldsymbol{A}|\boldsymbol{\sigma}(t)|^{\frac{1}{2}}\mathrm{sgn}(\boldsymbol{\sigma}(t)) - \boldsymbol{\Gamma}\int \mathrm{sgn}(\boldsymbol{\sigma}(t))\mathrm{d}t, \qquad (4)$$

where $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n)$ are gain parameters to be determined, $\boldsymbol{\sigma}(t) \in \mathbb{R}^n$ is the sliding mode variable defined as

$$\boldsymbol{\sigma}(t) = \dot{\boldsymbol{e}}(t) + c\boldsymbol{e}(t), \qquad (5)$$

and $\mathrm{sgn}(\cdot)$ is the element-wisely defined sign function. Note that the operator $|\cdot|^{\frac{1}{2}}\mathrm{sgn}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is also defined in an element-wise manner, i.e.,

$$\left(|\boldsymbol{\sigma}|^{\frac{1}{2}}\mathrm{sgn}(\boldsymbol{\sigma})\right)_i = |\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i),\ \forall \boldsymbol{\sigma} \in \mathbb{R}^n,$$

where $(\cdot)_i$ denotes the i-th element of a vector. Substituting (3) and (4) to the system model (1), we obtain

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\hat{\boldsymbol{M}}(\ddot{\boldsymbol{q}}_{\mathrm{d}} - c\dot{\boldsymbol{e}} + \boldsymbol{u}) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau}_{\mathrm{d}} - \tilde{\boldsymbol{C}}\dot{\boldsymbol{q}} - \tilde{\boldsymbol{G}} - \tilde{\boldsymbol{F}}\right), \quad (6)$$

where $\tilde{\boldsymbol{M}}(\boldsymbol{q}) = \boldsymbol{M}(\boldsymbol{q}) - \hat{\boldsymbol{M}}(\boldsymbol{q})$, $\tilde{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $\tilde{\boldsymbol{G}}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q}) - \hat{\boldsymbol{G}}(\boldsymbol{q})$ and $\tilde{\boldsymbol{F}}(\dot{\boldsymbol{q}}) = \boldsymbol{F}(\dot{\boldsymbol{q}}) - \hat{\boldsymbol{F}}(\dot{\boldsymbol{q}})$ are the unmodeled dynamics. Then, we take the time derivative of the sliding function $\boldsymbol{\sigma}(t)$ in (5) and obtain

$$\dot{\boldsymbol{\sigma}}(t) = \boldsymbol{u}(t) - \boldsymbol{\eta}(t), \qquad (7)$$

where $\boldsymbol{\eta}(t) = \hat{\boldsymbol{M}}^{-1}(\tilde{\boldsymbol{M}}\ddot{\boldsymbol{q}} + \tilde{\boldsymbol{C}}\dot{\boldsymbol{q}} + \tilde{\boldsymbol{G}} + \tilde{\boldsymbol{F}} - \boldsymbol{\tau}_{\mathrm{d}})$ is the system uncertainty term including the unmodeled dynamics and the external disturbances. For the system uncertainties, we propose the following assumption.

Assumption 1. The time derivative of $\boldsymbol{\eta}(t)$ is bounded, i.e., $\|\boldsymbol{\eta}(t)\| \leq \bar{\eta}$, $\exists\, \bar{\eta} \in \mathbb{R}^+$.

Note that the boundedness of the system uncertainties is a basic assumption popularly used in related works, such as (Youcef-Toumi and Wu, 1991; Jeong et al., 2018).

3.2 Adaptive Super-twisting Algorithm

Deriving from (7), the dynamics of sliding function $\boldsymbol{\sigma}(t)$ in each dimension reads $\dot{\sigma}_i(t) = u_i(t) - \eta_i(t)$, $i = 1, 2, \cdots, n$. Substituting (4) to it, we have

$$\dot{\sigma}_i = -\alpha_i|\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i) - \gamma_i\int \mathrm{sgn}(\sigma_i)\mathrm{d}t - \eta_i. \qquad (8)$$

Although the determination of the parameters α_i and γ_i are theoretically provided by existing framework (Bartolini et al., 2000; Jeong et al., 2018), selecting a proper integral coefficient γ_i, is a challenging work in practical applications. In this paper, we propose a adaptive tuning law for γ_i which is given by the following theorem.

Theorem 1. For the dynamics of $\sigma_i(t)$ in (8), $i = 1, 2, \cdots, r$, if Assumption 1 is ensured and $\alpha_i > 0$, then $\sigma_i(t)$ asymptotically converges to zero within a finite time $t_c < \infty$, with the following adaptive tuning law,

$$\dot{\gamma}_i = \varrho_i\, \mathrm{sgn}(\sigma_i)\int \mathrm{sgn}(\sigma_i)\mathrm{d}t, \qquad (9)$$

where $\varrho_i > 2\bar{\gamma}_i - \alpha_i^2/2$, with $\bar{\gamma}_i \in \mathbb{R}^+$ being a scalar such that

$$\frac{\alpha_i^3}{\epsilon(\bar{\gamma}_i) + \sqrt{2\epsilon(\bar{\gamma}_i)^2 + 4\alpha_i^2}} + \bar{\eta}\sqrt{\alpha_i^2 + 1} = \frac{\alpha_i}{2}, \qquad (10)$$

where

$$\epsilon(\bar{\gamma}_i) = 2\bar{\gamma}_i + \alpha_i^2 - 1. \qquad (11)$$

Proof 1. Let us assume that $\gamma_i^* > 0$ is an ideal parameter selection of γ_i, such that the dynamics of $\sigma_i(t)$ in (8) is stable if $\gamma_i = \gamma_i^*$. Therefore, the sliding dynamics in (8) can be represented as

$$\dot{\sigma}_i = -\alpha_i|\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i) - (\tilde{\gamma}_i + \gamma_i^*)\int \mathrm{sgn}(\sigma_i)\mathrm{d}t - \eta_i, \quad (12)$$

where $\tilde{\gamma}_i = \gamma_i - \gamma_i^*$ denotes the error of the parameter tuning. By defining an auxiliary variable

$$\varepsilon_i = -\gamma_i^*\int \mathrm{sgn}(\sigma_i)\mathrm{d}t - \eta_i, \qquad (13)$$

we formulate the closed-loop dynamics in (12) as follows,

$$\begin{aligned} \dot{\sigma}_i &= -\alpha_i|\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i) - \tilde{\gamma}_i\int \mathrm{sgn}(\sigma_i)\mathrm{d}t + \varepsilon_i, \\ \dot{\varepsilon}_i &= -\gamma_i^*\mathrm{sgn}(\sigma_i) - \dot{\eta}_i. \end{aligned} \qquad (14)$$

Note that similar techniques are also used in (Yu and Efe, 2015) and (Jeong et al., 2018). Let us define $z_1^i =$

$|\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i)$, $z_2^i=\varepsilon_i$ and represent (14) as $\dot{z}_1^i=\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}\dot{\sigma}_i$, $\dot{z}_2^i=\dot{\varepsilon}_i$, which then leads to

$$\dot{z}_1^i=|\sigma_i|^{-\frac{1}{2}}\left(-\frac{\alpha_i}{2}|\sigma_i|^{\frac{1}{2}}\mathrm{sgn}(\sigma_i)-\frac{\tilde{\gamma}_i}{2}\int\mathrm{sgn}(\sigma_i)\mathrm{d}t+\frac{z_2^i}{2}\right),$$
$$\dot{z}_2^i=-\gamma_i^*\mathrm{sgn}(\sigma_i)-\dot{\eta}_i.$$

Substituting the functions of σ_i with z_1^i and z_2^i, we obtain

$$\dot{z}_1^i=-\frac{\alpha_i}{2}|\sigma_i|^{-\frac{1}{2}}z_1^i+\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}z_2^i-\frac{\tilde{\gamma}_i}{2}|\sigma_i|^{-\frac{1}{2}}\int\mathrm{sgn}(\sigma_i)\mathrm{d}t,$$
$$\dot{z}_2^i=-\gamma_i^*|\sigma_i|^{-\frac{1}{2}}z_1^i-\dot{\eta}_i,$$

and further we have

$$\dot{\boldsymbol{z}}_i=-|\sigma_i|^{-\frac{1}{2}}\boldsymbol{\Lambda}_i\boldsymbol{z}_i-\boldsymbol{\zeta}_i(\sigma_i)\tilde{\gamma}_i-\dot{\boldsymbol{\eta}}_i, \tag{15}$$

where $\boldsymbol{z}_i=\begin{bmatrix}z_1^i & z_2^i\end{bmatrix}^\top$, $\boldsymbol{\zeta}_i(\sigma_i)=\begin{bmatrix}\zeta_i(\sigma_i) & 0\end{bmatrix}^\top$, $\boldsymbol{\eta}_i=\begin{bmatrix}0 & \eta_i\end{bmatrix}^\top$ and

$$\boldsymbol{\Lambda}_i=\begin{bmatrix}\frac{\alpha_i}{2} & -\frac{1}{2}\\ \gamma_i^* & 0\end{bmatrix},\ \zeta_i(\sigma_i)=\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}\int\mathrm{sgn}(\sigma_i)\mathrm{d}t.$$

To investigate the stability of $\boldsymbol{z}_i$ at the zero equilibrium, we define the following Lyapunov function

$$V_i=\frac{1}{2}\boldsymbol{z}_i^\top\boldsymbol{P}_i\boldsymbol{z}_i+\frac{1}{2}\tilde{\gamma}_i^2,\ \boldsymbol{P}_i=\begin{bmatrix}2\gamma_i^*+\frac{1}{2}\alpha_i^2 & -\frac{1}{2}\alpha_i\\ -\frac{1}{2}\alpha_i & 1\end{bmatrix} \tag{16}$$

where $\boldsymbol{P}_i$ is a positive definite matrix. Taking the derivative of V_i, we have $\dot{V}_i=\boldsymbol{z}_i\boldsymbol{P}_i\dot{\boldsymbol{z}}_i+\tilde{\gamma}_i\dot{\gamma}_i$. Substituting $\dot{\boldsymbol{z}}_i$ in (15) and $\dot{\gamma}_i$ in (9) to it, we obtain

$$\begin{aligned}\dot{V}_i=&-\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}\boldsymbol{z}_i^\top\boldsymbol{Q}_i\boldsymbol{z}_i-\boldsymbol{z}_i^\top\boldsymbol{P}_i\dot{\boldsymbol{\eta}}_i-\boldsymbol{z}_i^\top\boldsymbol{P}_i\boldsymbol{\zeta}_i(\sigma_i)\tilde{\gamma}_i\\&+\tilde{\gamma}_i\varrho_i\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t\\=&-\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}\boldsymbol{z}_i^\top\boldsymbol{Q}\boldsymbol{z}_i-\boldsymbol{z}_i^\top\boldsymbol{P}_i\dot{\boldsymbol{\eta}}_i+\frac{\alpha_i}{2}\varepsilon_i\tilde{\gamma}_i\zeta_i(\sigma_i)\\&+\left(\varrho_i-2\gamma_i^*-\frac{1}{2}\alpha_i^2\right)\tilde{\gamma}_i\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t,\end{aligned} \tag{17}$$

where

$$\boldsymbol{Q}_i=\boldsymbol{P}_i\boldsymbol{\Lambda}_i+\boldsymbol{\Lambda}_i^\top\boldsymbol{P}=\frac{1}{2}\alpha_i\begin{bmatrix}2\gamma_i^*+\alpha_i^2 & -\alpha_i\\ -\alpha_i & 1\end{bmatrix}.$$

Therefore, the eigenvalues of $\boldsymbol{Q}_i$, $\lambda_1(\boldsymbol{Q}_i)$ and $\lambda_2(\boldsymbol{Q}_i)$, satisfy that $\lambda_1(\boldsymbol{Q}_i)+\lambda_2(\boldsymbol{Q}_i)=\alpha_i\gamma_i^*+\frac{1}{2}\alpha_i^3+\frac{1}{2}\alpha_i$, $\lambda_1(\boldsymbol{Q}_i)\lambda_2(\boldsymbol{Q}_i)=\frac{1}{2}\alpha_i^2\gamma_i^*$, which indicates that $\boldsymbol{Q}_i$ is positive definite if $\alpha_i,\gamma_i^*>0$ holds. Meanwhile, since γ_i^* is an ideal parameter selection, (13) denotes an ideal sliding mode and we have $\varepsilon_i=0$ in the sense of Filippov. Substituting $\varepsilon_i=0$ to (17), we obtain

$$\begin{aligned}\dot{V}_i\le&-\frac{1}{2}|\sigma_i|^{-\frac{1}{2}}\lambda_{\min}(\boldsymbol{Q}_i)\|\boldsymbol{z}_i\|^2+\|\boldsymbol{z}_i\|\|\boldsymbol{P}_i\dot{\boldsymbol{\eta}}_i\|\\&+\left(\varrho_i-2\gamma_i^*-\frac{1}{2}\alpha_i^2\right)\tilde{\gamma}_i\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t,\end{aligned} \tag{18}$$

where $\|\boldsymbol{P}_i\dot{\boldsymbol{\eta}}_i\|=\left\|[-\alpha_i/2\ 1]^\top\dot{\eta}_i\right\|\le\bar{\eta}\sqrt{\alpha_i^2/4+1}$ and $\lambda_{\min}(\boldsymbol{Q}_i)$ is the minimal eigenvalue of $\boldsymbol{Q}_i$,

$$\begin{aligned}\lambda_{\min}(\boldsymbol{Q}_i)&=\min(\lambda_1(\boldsymbol{Q}_i),\lambda_2(\boldsymbol{Q}_i))\\&=\frac{\alpha_i}{2}\left(1-\frac{\alpha_i^2}{\epsilon(\gamma_i^*)+\sqrt{2\epsilon(\gamma_i^*)^2+4\alpha_i^2}}\right),\end{aligned}$$

where the scalar function $\epsilon(\cdot)$ is defined as in (11). Note that $\lambda_{\min}(\boldsymbol{Q}_i)$ is a function of γ_i^* for given α_i. Therefore, we represent $\lambda_{\min}(\boldsymbol{Q}_i)$ as $\bar{\lambda}(\gamma_i^*)$. Considering $|\sigma_i|^{\frac{1}{2}}=|z_1^i|\le\|\boldsymbol{z}_i\|$, we have $-|\sigma_i|^{-\frac{1}{2}}\le-\|\boldsymbol{z}_i\|^{-1}$, which leads (18) to

$$\begin{aligned}\dot{V}_i\le&-\frac{1}{2}\bar{\lambda}(\gamma_i^*)\|\boldsymbol{z}_i\|+\|\boldsymbol{z}_i\|\|\boldsymbol{P}_i\dot{\boldsymbol{\eta}}_i\|\\&+\left(\varrho_i-2\gamma_i^*-\frac{1}{2}\alpha_i^2\right)\tilde{\gamma}_i\,\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t\\=&-\frac{1}{2}\|\boldsymbol{z}_i\|\left(\bar{\lambda}(\gamma_i^*)-\bar{\eta}\sqrt{\alpha_i^2+4}\right)\\&+\left(\varrho_i-2\gamma_i^*-\frac{1}{2}\alpha_i^2\right)\tilde{\gamma}_i\,\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t.\end{aligned} \tag{19}$$

Note that (19) also holds in the Filippov sense. From (10), it is known that $\bar{\gamma}_i$ satisfies $\bar{\lambda}(\bar{\gamma}_i)-\bar{\eta}\sqrt{\alpha_i^2+4}=0$, and for any $\gamma_i^*>\bar{\gamma}_i$, we have

$$\bar{\lambda}(\gamma_i^*)-\bar{\eta}\sqrt{\alpha_i^2+4}>0. \tag{20}$$

Note that for any $\varrho_i>2\gamma_i^*-\alpha_i^2/2$, there exists $\gamma_i^*>\bar{\gamma}_i$, such that (20) and $\varrho_i-2\gamma_i^*-\frac{1}{2}\alpha_i^2=0$ holds, which leads to

$$\dot{V}_i\le-\frac{1}{2}\|\boldsymbol{z}_i\|\left(\bar{\lambda}(\gamma_i^*)-\bar{\eta}\sqrt{\alpha_i^2+4}\right)<0.$$

Therefore, $\boldsymbol{z}_i$ converges to zero and γ_i converges to γ_i^* asymptotically. It is worth mentioning that such an ideal value γ_i^* is not unique but belongs to a half-closed set $\gamma_i^*>\bar{\gamma}_i$. Therefore, γ_i will finally reaches an ideal value γ_i^*, such that $\tilde{\gamma}_i=0$ holds. In this sense, from (16), we obtain $V_i=\boldsymbol{z}_i^\top\boldsymbol{P}_i\boldsymbol{z}_i/2$, which leads to $\|z_i\|\ge\sqrt{2V_i/\lambda_{\max}(\boldsymbol{P}_i)}$, where $\lambda_{\max}(\boldsymbol{P}_i)$ is the minimum eigenvalue of $\boldsymbol{P}_i$. For any positive scalar $\beta\in\mathbb{R}^+$ satisfying $\bar{\lambda}(\gamma_i^*)-\bar{\eta}\sqrt{\alpha_i^2+4}>\beta$, we have

$$\dot{V}_i\le-\beta\sqrt{V_i/2\lambda_{\max}(\boldsymbol{P}_i)}. \tag{21}$$

According to the finite-time convergence property of sliding mode (Utkin et al., 1999), (21) indicates that the convergence of $\boldsymbol{z}_i$ is within a finite time

$$t_c=\frac{2}{\beta}\sqrt{\lambda_{\max}(\boldsymbol{P}_i)\boldsymbol{z}_{i,0}^\top\boldsymbol{P}_i\boldsymbol{z}_{i,0}}=\frac{2}{\beta}\lambda_{\max}(\boldsymbol{P}_i)\|\boldsymbol{z}_{i,0}\|,$$

where $\boldsymbol{z}_{i,0}$ is the initial value of $\boldsymbol{z}_i$ when γ_i reaches γ_i^*. According to the definition of $\boldsymbol{z}_i$, $\boldsymbol{\sigma}_i(t)$ and $\dot{\boldsymbol{\sigma}}_i(t)$ also converge to zeros within finite time t_c.

Remark 1. Theorem 1 indicates that $\boldsymbol{\sigma}_i(t)$, $\dot{\boldsymbol{\sigma}}_i(t)$ converge to zero within a finite time for all $i=1,2,\cdots,n$, which ensures that the tracking error $\boldsymbol{e}(t)$ asymptotically converge to zero. Therefore, the controller (3) designed for system (1) guarantees robust trajectory tracking for the system uncertainty $\boldsymbol{\eta}(t)$.

It is noticed that the update of $\boldsymbol{\gamma}$ in (9) involves twice-integration which may produce an over-large control gain $\boldsymbol{\gamma}$, which leads chattering to the control input $\boldsymbol{u}(t)$. To avoid this, we modify the tuning law in (9) as follows

$$\dot{\gamma}_i=\begin{cases}\varrho_i\,\mathrm{sgn}(\sigma_i)\int\mathrm{sgn}(\sigma_i)\mathrm{d}t, & \|\boldsymbol{\sigma}\|\ge\sigma_0,\\ -\kappa\gamma_i, & \|\boldsymbol{\sigma}\|<\sigma_0,\end{cases} \tag{22}$$

where $\kappa>0$ is a decaying factor for γ_i and $\sigma_0>0$ is a boundary layer scalar for $\boldsymbol{\sigma}(t)$. Therefore, the self-tuning of γ_i is only activated when $\boldsymbol{\sigma}(t)$ exceeds the boundary layer, and decays when $\boldsymbol{\sigma}(t)$ is within the boundary layer. As a result, the unlimited growing of γ_i is avoided, and $\boldsymbol{\sigma}(t)$ is confined within the boundary layer, such that the robustness of the closed-loop system is ensured without bringing up chattering.

4. ROBUST INVARIANT CONTROL

In this section, we solve Problem 1-(a) by improving the robust tracking controller (3) to comply with the state-dependent inequality constraints $(\boldsymbol{q}(t)\ \dot{\boldsymbol{q}}(t))\in\Phi$, $\forall\, t>0$.

4.1 Control with Inequality Constraints

Here, we give a brief interpretation of the invariant-set theory.

Definition 1. (adapted from (Blanchini, 1999)): The set $\mathcal{S} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is said controlled positively invariant (CPI) for system (1), if there exists a continuous feedback control law $\boldsymbol{u}(\boldsymbol{q},\dot{\boldsymbol{q}})$, such that for any initial condition $(\boldsymbol{q}(0),\, \dot{\boldsymbol{q}}(0)) \in \mathcal{S}$, $(\boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t)) \in \mathcal{S}$ holds for all $t>0$.

The CPI-set is an important concept for control problems with state-dependent inequality constraints. The admissible-state set in (2) is not violated for all $t>0$, if there exists a CPI set $\mathcal{S}\subseteq\Phi$, where $(\boldsymbol{q}(0),\dot{\boldsymbol{q}}(0))\in\mathcal{S}$. Therefore, the target of Problem 1-(a) is to define a subset $\mathcal{S}$ of the admissible state set Φ and seek for a corresponding feedback control law $\boldsymbol{u}(\boldsymbol{q},\dot{\boldsymbol{q}})$, such that $\mathcal{S}$ is a CPI set. For $\mathcal{S}$, we assume that the following conditions hold.

Assumption 2. The CPI set $\mathcal{S}$ is the convex intersection of r unilateral constraints represented by the following inequalities,

$$s_i(\boldsymbol{q},\dot{\boldsymbol{q}}) \le 0,\ i=1,2,\cdots,r, \tag{23}$$

where $s_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable function.

We define $\partial\mathcal{S}$ as the boundary of $\mathcal{S}$ and $\mathrm{int}(\mathcal{S})$ as the interior of $\mathcal{S}$. Under the conditions in Assumption 2, $\partial\mathcal{S}$ is continuous and piece-wisely continuously differentiable. For any system state $(\boldsymbol{q},\, \dot{\boldsymbol{q}})\in\partial\mathcal{S}$, there exists at least one i, $1\le i\le r$, such that

$$s_i(\boldsymbol{q},\dot{\boldsymbol{q}})=0,\ 1\le i\le r. \tag{24}$$

Although the CPI set $\mathcal{S}$ can be selected the same as Φ, the boundary of Φ is not necessarily piece-wise continuously differentiable. Therefore, $\mathcal{S}$ is usually determined as a conservative subset of the admissible-state set Φ.

In the conventional invariant-set based methods (Wollherr et al., 2001; Kimmel and Hirche, 2017), an invariant controller is designed to confine the system state within $\inf(\mathcal{S})$ by triggering a switching-law when the system attempts to cross the invariant-set boundary $\partial\mathcal{S}$. For sliding-mode-based control methods, however, switching on the boundary $\partial\mathcal{S}$ may break the sliding mode and affect the robust tracking performance of the controller. Therefore, in this paper, we propose a novel sliding-mode based invariant controller by modifying the sliding mode manifold, such that the safety constraints are satisfied while the robustness is retained. Note that, by *invariance*, we mean the controller ensures $\mathcal{S}$ to be a CPI set for the closed-loop system.

4.2 The Sliding-mode-based Invariance Controllers

Before we propose the improved invariant controller, we define the admissible trajectory for the CPI set $\mathcal{S}$.

Definition 2. For a continuous function $\boldsymbol{q}(t)$, if there exist $t_0>0$, such that $(\boldsymbol{q}(t_0),\,\dot{\boldsymbol{q}}(t_0))\in\partial\mathcal{S}$, and its right-hand derivatives $\dot{\boldsymbol{q}}(t_0^+)$ and $\ddot{\boldsymbol{q}}(t_0^+)$ exist, then $\boldsymbol{q}(t)$ is an admissible trajectory, if for all $i=1,2,\cdots,r$,

$$\dot{s}_i(t_0^+)=\frac{\partial s_i}{\partial\boldsymbol{q}^\top}\dot{\boldsymbol{q}}(t_0^+)+\frac{\partial s_i}{\partial\dot{\boldsymbol{q}}^\top}\ddot{\boldsymbol{q}}(t_0^+)\le 0,\ \text{if}\ s_i(t_0)=0. \tag{25}$$

The set of all admissible trajectories for t_0 is represented as $\mathcal{Q}_0$.

Remark 2. An admissible trajectory $\boldsymbol{q}(t)$ moves along the direction to which the functions $s_i(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))$ decrease for all active constraints, in the right neighborhood of t_0, $[t_0,t_0+\Delta t)$, where $\Delta t>\mathbb{R}^+$ is a sufficiently small interval, such that the system shows a tendency to move into $\mathrm{int}(\mathcal{S})$.

From the definition of the sliding mode variable $\boldsymbol{\sigma}(t)$ in (5), we have

$$\boldsymbol{\sigma}(t)=(\boldsymbol{q}(t)+c\dot{\boldsymbol{q}}(t))-(\boldsymbol{q}_\mathrm{d}(t)+c\dot{\boldsymbol{q}}_\mathrm{d}(t)),$$

and precise tracking of $\boldsymbol{q}_\mathrm{d}(t)$ is achieved by forcing $\boldsymbol{\sigma}(t)$ to zero. Therefore, constraints $(\boldsymbol{q},\dot{\boldsymbol{q}})\in\mathcal{S}$ are violated if the desired trajectory $\boldsymbol{q}_\mathrm{d}(t)\notin\mathcal{Q}_0$. This problem can be solved by seeking for a modified reference trajectory $\boldsymbol{q}_\mathrm{r}(t)$, such that $\boldsymbol{q}_\mathrm{r}(t)\in\mathcal{Q}_0$. The robustness of the tracking controller ensures $\boldsymbol{q}(t)$ to converge to $\boldsymbol{q}_\mathrm{r}(t)$ within a finite time and thus also becomes admissible. Note that such admissible trajectory solutions are not unique, since (25) is confined by inequalities instead of equations. Nevertheless, it is straight-forward to select a $\boldsymbol{q}_\mathrm{r}(t)$ that is closest to the original desired trajectory $\boldsymbol{q}_\mathrm{d}(t)$. Based on this idea, we formulate a specified equivalent of Problem 2-(a).

Problem 2. For a piece-wise continuously differential function $\boldsymbol{q}_\mathrm{d}(t)$, if there exists $t_0>0$, such that $(\boldsymbol{q}_\mathrm{d}(t_0),\dot{\boldsymbol{q}}_\mathrm{d}(t_0))\in\partial\mathcal{S}$ and $\boldsymbol{q}_\mathrm{d}(t)\notin\mathcal{Q}_0$, solve the following minimization problem,

$$\min_{t\in[t_0,t_0+\Delta t)}\|\boldsymbol{q}_\mathrm{r}(t)-\boldsymbol{q}_\mathrm{d}(t)\|,\ \text{s.t.}\ \boldsymbol{q}_\mathrm{r}(t_0)=\boldsymbol{q}_\mathrm{d}(t_0), \tag{26a}$$

$$\dot{s}_i(t_0^+)\le 0,\ \text{if}\ s_i(t_0)=0,\ 1\le i\le r. \tag{26b}$$

Remark 3. In Problem 2, (26a) aims to find the closest solution of an admissible trajectory $\boldsymbol{q}_\mathrm{r}(t)$ to the original inadmissible trajectory $\boldsymbol{q}_\mathrm{d}(t)$. The terminal condition confines that the system trajectory $\boldsymbol{q}(t)$ is continuous. The constraints in (26b) require that $\boldsymbol{q}_\mathrm{r}(t)$ is admissible, corresponding to the condition (25).

We take the Taylor expansion of $\boldsymbol{q}_\mathrm{r}(t)$ and $\boldsymbol{q}_\mathrm{d}(t)$ in the neighborhood of t_0, and (26a) is equivalent to

$$\min_{\Delta t\to 0}\sum_{i=0}^{\infty}\frac{(\Delta t)^i}{i!}\left\|\boldsymbol{q}_\mathrm{r}^{(i)}(t_0^+)-\boldsymbol{q}_\mathrm{d}^{(i)}(t_0^+)\right\|, \tag{27}$$

where $(\cdot)^{(i)}$ is the i-th order derivative of $(\cdot)$. Since the higher-order derivatives of $\boldsymbol{q}_\mathrm{r}(t)$ for $i>2$ do not show up in the formulation (26), they can be neglected. Then, the formulated problem in (26a) is simplified as

$$\min\left\|\dot{\boldsymbol{q}}_\mathrm{r}^{(i)}(t_0)-\dot{\boldsymbol{q}}_\mathrm{d}^{(i)}(t_0)\right\|^2,\ \forall\, i=0,1,2. \tag{28}$$

Here, we formulate (28) as a quadratic form. Note that solving Problem 2 only ensures that the constraints are complied with in the neighborhood $[t_0,t_0+\Delta t)$. To guarantee safety in a continuous period of time $T\gg\Delta t$, Problem 2 should be solved for all $t\in(t_0,t_0+T)$. Therefore, the sliding-mode-based invariance controller is similar to (3), with $\boldsymbol{q}_\mathrm{d}(t)$ substituted by $\boldsymbol{q}_\mathrm{r}(t)$, i.e.,

$$\boldsymbol{\tau} = \hat{\boldsymbol{M}}(\boldsymbol{q})(\ddot{\boldsymbol{q}}_{\rm r} - c\dot{\boldsymbol{e}} + \boldsymbol{u}) + \hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \hat{\boldsymbol{G}}(\boldsymbol{q}) + \hat{\boldsymbol{F}}(\dot{\boldsymbol{q}}) \quad (29)$$

which ensures asymptotic convergence of the tracking error $\boldsymbol{\varepsilon}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_{\rm r}(t)$ to zero.

It is worth mentioning that, when a modified reference trajectory $\boldsymbol{q}_{\rm r}(t)$ is solved, the continuity of $\boldsymbol{q}_{\rm r}(t)$ is always guaranteed by the terminal conditions (26b), but not necessarily for $\dot{\boldsymbol{q}}_{\rm r}(t)$ and $\ddot{\boldsymbol{q}}_{\rm r}(t)$, which leads to a new transient stage to the sliding mode. Nevertheless, since $\boldsymbol{q}_{\rm r}(t)$ lies on $\partial\mathcal{S}$ and $\partial\mathcal{S}$ is piece-wisely continuously differentiable, a new sliding mode can still be achieved if the finite convergence time of the sliding mode is sufficiently small, such that the robust stability of the system is not affected. However, due to the transient stages, the constraints $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \in \mathcal{S}$ may be violated for a short period of time. Therefore, a more conservative CPI set $\mathcal{S}$ should be determined for the hard constraint set Φ to tolerate possible violations.

4.3 Control with Linear Holonomic Constraints

Problem 2 formulates a minimization problem (28) with constraints (26b), which usually requires numerical methods for solutions. However, analytical solutions can be obtained for linear holonomic constraints which are frequently used in practice. Consider system (1) with the following position-dependent holonomic constraints,

$$\phi_i(\boldsymbol{q}) = \boldsymbol{\omega}_i^\top \boldsymbol{q} + \bar{\omega}_i \leq 0, \ \forall 1 \leq i \leq r,$$

where $\boldsymbol{\omega}_i \in \mathbb{R}^n$ is the constant coefficient and $\bar{\omega}_i \in \mathrm{R}$ is a constant bias. We determine a CPI set $\mathcal{S}$ as follows,

$$s_i(\boldsymbol{q}) = \boldsymbol{\omega}_i^\top \boldsymbol{q} + \bar{\omega}_i + \delta_i^\omega \leq 0, \ \forall 1 \leq i \leq r, \quad (30)$$

where $\delta_i^\omega \geq 0, i = 1, 2, \cdots, r$, is the scalar for the tolerance of the constraint violation due to the transient stages. For given $t_0 > 0$, such that $(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)) \in \partial\mathcal{S}$, there exist $1 \leq l_1 < l_2 < \cdots < l_m \leq r$, such that

$$\boldsymbol{\omega}_i^\top \boldsymbol{q} + \bar{\omega}_i + \delta_i^\omega = 0, \ \forall i = l_1, l_2, \cdots, l_m, \quad (31)$$

where m is the number of active constraints. We represent (31) as the following matrix form

$$\boldsymbol{s}(\boldsymbol{q}) = \boldsymbol{\Omega}^\top \boldsymbol{q} + \bar{\boldsymbol{\omega}} + \boldsymbol{\delta}^\omega = 0,$$

where $\boldsymbol{s} = [\, s_{l_1} \ s_{l_2} \ \cdots \ s_{l_m}]^\top$, $\boldsymbol{\Omega} = [\boldsymbol{\omega}_{l_1} \ \boldsymbol{\omega}_{l_2} \ \cdots \ \boldsymbol{\omega}_{l_m}]$, $\bar{\boldsymbol{\omega}} = [\bar{\omega}_{l_1} \ \bar{\omega}_{l_2} \ \cdots \ \bar{\omega}_{l_m}]^\top$ and $\boldsymbol{\delta}^\omega = [\, \delta_{l_1}^\omega \ \delta_{l_2}^\omega \ \cdots \ \delta_{l_m}^\omega]^\top$. Then, the solution to Problem 1 of $\boldsymbol{q}_{\rm r}(t)$ at the neighborhood $[t_0, t_0 + \Delta t)$ is

$$\boldsymbol{q}_{\rm r}(t) = \begin{cases} \boldsymbol{\Omega}_1 \boldsymbol{q}_{\rm d}(t) + \bar{\boldsymbol{\omega}}_1, & \text{if } \boldsymbol{q}_{\rm d}(t) \notin \mathcal{Q}_0, \\ \boldsymbol{q}_{\rm d}(t), & \text{if } \boldsymbol{q}_{\rm d}(t) \in \mathcal{Q}_0. \end{cases} \quad (32)$$

where $\boldsymbol{\Omega}_1 = \boldsymbol{I} - \boldsymbol{\Omega}\left(\boldsymbol{\Omega}^\top \boldsymbol{\Omega}\right)^{-1} \boldsymbol{\Omega}^\top$, $\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\left(\boldsymbol{\Omega}^\top \boldsymbol{\Omega}\right)^{-1} (\bar{\boldsymbol{\omega}} + \boldsymbol{\delta}^\omega)$, and $\boldsymbol{I}$ is the identity matrix. It is straight forward to verify that, for $\boldsymbol{q}_{\rm r}(t)$,

$$\dot{\boldsymbol{s}}(\boldsymbol{q}_{\rm r}(t)) = \boldsymbol{\Omega}^\top \dot{\boldsymbol{q}}_{\rm r}(t) = 0, \ \text{if } \boldsymbol{q}_{\rm d}(t) \notin \mathcal{Q}_0, \quad (33)$$

which satisfies the admissible condition (26b). Therefore, $\boldsymbol{q}_{\rm r}(t)$ is an admissible trajectory for t_0.

It is worth mentioning that (33) indicates $\boldsymbol{q}_{\rm r}(t) \subseteq \partial\mathcal{S}$ for $\boldsymbol{q}_{\rm d}(t) \notin \mathcal{Q}_0$, i.e., $\boldsymbol{q}_{\rm r}(t)$ lies on the boundary of the CPI set $\mathcal{S}$ when $\boldsymbol{q}_{\rm d}(t)$ violates the constraints. Additionally, (32) shows that $\boldsymbol{q}_{\rm r}(t) \equiv \boldsymbol{q}_{\rm d}(t)$, if $\boldsymbol{q}_{\rm d}(t) \in \mathcal{Q}_0$. Therefore, $\boldsymbol{q}_{\rm r}(t)$ is also piece-wise continuously differentiable, which means that a new sliding mode is achievable after every switching of (32), if the convergence time of the sliding mode is sufficiently small. Thus, both robustly precise tracking and hard constraint compliance are guaranteed.

5. NUMERICAL SIMULATION

In this section, we evaluate the proposed controller by a numerical simulation on a 3-DoF manipulator in MATLAB 2019a. The detailed model information of the robot can be referred to (Zhang et al., 2019). The simulation runs from 0s to 70s at a sampling rate 1kHz, and starts at the zero initial condition $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$. The desired trajectory $\boldsymbol{q}_{\rm d}(t)$ for the system is designed as,

$$\boldsymbol{q}_{\rm d}(t) = \begin{cases} -2\cos\left(\frac{\pi}{8}(t-3)\right)\boldsymbol{q}_0, & \text{if } 11 < t \leq 59, \\ 0, & \text{if } t \leq 3 \text{ or } t > 67, \\ \left(1 - \cos\left(\frac{\pi}{8}(t-3)\right)\right)\boldsymbol{q}_0, & \text{else}, \end{cases}$$

where $\boldsymbol{q}_0 = [\,1.5 \ \ 0.6 \ \ 0.9\,]^\top$. A tracking controller as in (3) and (4) is implemented with strict compliance with the following constraints,

$$-3 \leq s(\boldsymbol{q}(t)) \leq 3, \quad (34)$$

where $s(\boldsymbol{q}(t)) = q_1(t) + q_2(t) + q_3(t)$ is the invariant function, $q_1(t)$, $q_2(t)$ and $q_3(t)$ are respectively the angular positions of the three joints. We define the CPI set $\mathcal{S}$ as

$$\begin{aligned} s_1(\boldsymbol{q}) &= \boldsymbol{\omega}_1^\top \boldsymbol{q}(t) - 3 + \delta_1 \leq 0, \\ s_2(\boldsymbol{q}) &= \boldsymbol{\omega}_2^\top \boldsymbol{q}(t) - 3 + \delta_2 \leq 0, \end{aligned} \quad (35)$$

where $\boldsymbol{\omega}_1 = [\,1\ 1\ 1\,]^\top$, $\boldsymbol{\omega}_2 = [\,-1\ -1\ -1\,]^\top$, and $\delta_1 = \delta_2 = 0.1$ denotes the violation tolerance. The controller parameters are selected as $c = 50$, $\kappa = 0.01$, $\alpha_i = 2$, $\varrho_i = 50$, $\forall i = 1, 2, 3$, $\sigma_0 = 0.2$. For the CPI set $\mathcal{S}$ in (35), the admissible trajectory $\boldsymbol{q}_{\rm r}(t)$ is determined as

$$\boldsymbol{q}_{\rm r}(t) = \begin{cases} \boldsymbol{\Omega}_1 \boldsymbol{q}_{\rm d}(t) + \bar{\boldsymbol{\omega}}_1, & \text{if } \boldsymbol{\omega}_1^\top \boldsymbol{q}_{\rm d}(t) \geq 2.9, \\ \boldsymbol{\Omega}_2 \boldsymbol{q}_{\rm d}(t) + \bar{\boldsymbol{\omega}}_2, & \text{if } \boldsymbol{\omega}_2^\top \boldsymbol{q}_{\rm d}(t) \geq 2.9, \\ \boldsymbol{q}_{\rm d}(t), & \text{if } \boldsymbol{q}_{\rm d}(t), \dot{\boldsymbol{q}}_{\rm d}(t) \in \mathrm{int}(\mathcal{S}). \end{cases}$$

where $\boldsymbol{\Omega}_1 = \boldsymbol{\omega}_1(\boldsymbol{\omega}_1^\top \boldsymbol{\omega}_1)^{-1} \boldsymbol{\omega}_1^\top + \boldsymbol{I}$, $\boldsymbol{\Omega}_2 = \boldsymbol{\omega}_2(\boldsymbol{\omega}_2^\top \boldsymbol{\omega}_2)^{-1} \boldsymbol{\omega}_2^\top + \boldsymbol{I}$, $\bar{\boldsymbol{\omega}}_1 = -2.9\boldsymbol{\omega}_1(\boldsymbol{\omega}_1^\top \boldsymbol{\omega}_1)^{-1}$, and $\bar{\boldsymbol{\omega}}_2 = -2.9\boldsymbol{\omega}_2(\boldsymbol{\omega}_2^\top \boldsymbol{\omega}_2)^{-1}$.

The reference trajectory $\boldsymbol{q}_{\rm r}(t)$ on the three robot joints is shown in Fig. 1 and is compared with the original desired trajectory $\boldsymbol{q}_{\rm d}(t)$. It is noticed that $\boldsymbol{q}_{\rm r}(t)$ deviates from $\boldsymbol{q}_{\rm d}(t)$ when $\boldsymbol{q}_{\rm d}(t)$ violates the constraint (gray area). Fig. 2 clearly shows that the system trajectory $\boldsymbol{q}(t)$ is confined within the constraints (34), even though the desired trajectory $\boldsymbol{q}_{\rm d}(t)$ exceeds the constraints. The trajectory tracking error $\boldsymbol{\varepsilon}(t)$ is illustrated in Fig 3, which indicates that the proposed controller provides precise tracking performance for the reference trajectory $\boldsymbol{q}_{\rm r}(t)$. The value of the adaptive gain $\boldsymbol{\gamma}$ of the proposed controller (29), shown in Fig. 4, reveals the success of the adaptive parameter tuning law. Therefore, the simulation results confirm that the proposed controller provides robustly precise tracking of the reference trajectories and complies with hard safety constraints.

6. CONCLUSION

In this paper, we propose a novel adaptive second-order sliding mode controller for Euler-Lagrangian systems with inequality constraints. Different from the conventional tracking control methods, the proposed controller ensures both precise trajectory tracking and invariance to the

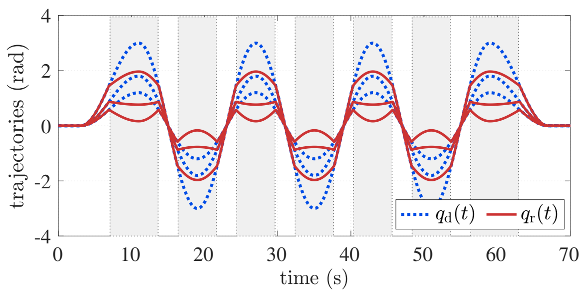

Fig. 1. The modified reference trajectory $\boldsymbol{q}_r(t)$ on the three robot joints, compared to the desired trajectory $\boldsymbol{q}_d(t)$. The time intervals when $\boldsymbol{q}_d(t)$ violates the constraints are marked as light gray.

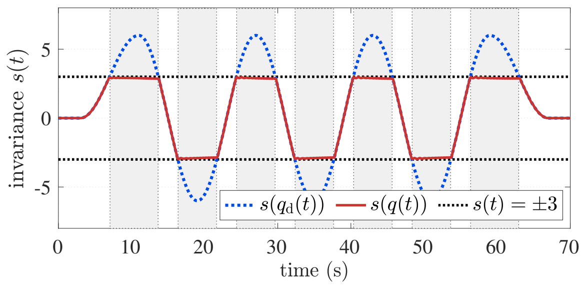

Fig. 2. The invariance functions $s(\boldsymbol{q}_d(t))$ and $s(\boldsymbol{q}(t))$. The invariance set $\mathcal{S}$ is marked by the dashed lines $s = \pm 3$.

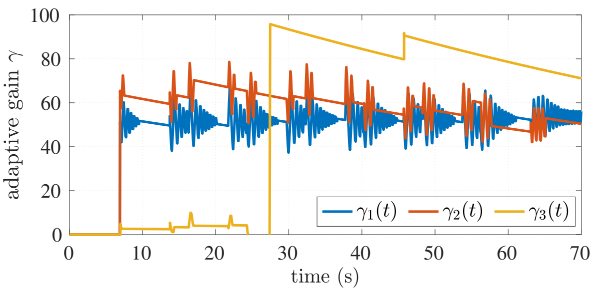

Fig. 3. The adaptive gains γ_i, $i = 1, 2, 3$.

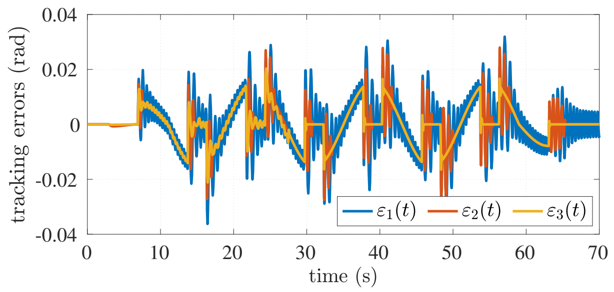

Fig. 4. The tracking error $\boldsymbol{\varepsilon}(t)$ of the three joints.

safety constraint set. By applying the adaptive tuning law for the controller gain, manual parameter assignment is avoided and robustness to system uncertainties is ensured without causing chattering. The convergence of the tracking error is guaranteed by a rigorous Lyapunov-based stability proof. The simulation validation indicates that the method has promising potential in the application to safe control of mechatronic systems, such as in safe human-robot interaction. Future work will be focused on eliminating the transient phases of the sliding modes.

REFERENCES

Bartolini, G., Ferrara, A., and Punta, E. (2000). Multi-input second-order sliding-mode hybrid control of constrained manipulators. *Dynamics and Control*, 10(3), 277–296.

Blanchini, F. (1999). Set invariance in control. *Automatica*, 35(11), 1747–1767.

Guendouzi, A., Boubakir, A., and Hamerlain, M. (2013). Higher order sliding mode control of robot manipulator. In *Proceedings of the 9th International Conference on Autonomic and Autonomous Systems*.

Guo, F., Liu, Y., and Luo, F. (2017). Adaptive stabilisation of a flexible riser by using the lyapunov-based barrier backstepping technique. *IET Control Theory & Applications*, 11(14), 2252–2260.

Incremona, G.P., Rubagotti, M., and Ferrara, A. (2017). Sliding mode control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 62(6), 2965–2972.

Jeong, C.S., Kim, J.S., and Han, S.I. (2018). Tracking error constrained super-twisting sliding mode control for robotic systems. *International Journal of Control, Automation and Systems*, 16(2), 804–814.

Kimmel, M. and Hirche, S. (2017). Invariance control for safe human–robot interaction in dynamic environments. *IEEE Transactions on Robotics*, 33(6), 1327–1342.

Mobayen, S., Tchier, F., and Ragoub, L. (2017). Design of an adaptive tracker for n-link rigid robotic manipulators based on super-twisting global nonlinear sliding mode control. *International Journal of Systems Science*, 48(9), 1990–2002.

Utkin, V., Guldner, J., and Shi, J. (1999). *Sliding mode control in electro-mechanical systems*. Taylor & Francis.

Utkin, V.I. and Poznyak, A.S. (2013). Adaptive sliding mode control with application to super-twist algorithm: Equivalent control method. *Automatica*, 49(1), 39 – 47.

Wilson, J., Charest, M., and Dubay, R. (2016). Non-linear model predictive control schemes with application on a 2 link vertical robot manipulator. *Robotics and Computer-Integrated Manufacturing*, 41, 23–30.

Wolff, J. and Buss, M. (2004). Invariance control design for nonlinear control affine systems under hard state constraints. *IFAC Proceedings Volumes*, 37(13), 555 – 560. 6th IFAC Symposium on NOLCOS 2004, Stuttgart, Germany, 1-3 September, 2004.

Wollherr, D., Mareczek, J., Buss, M., and Schmidt, G. (2001). Rollover avoidance for steerable vehicles by invariance control. In *2001 European Control Conference (ECC)*, 3522–3527.

Youcef-Toumi, K. and Wu, S. (1991). Input/output linearization using time delay control. In *1991 American Control Conference*, 2601–2606. IEEE.

Yu, X. and Efe, M.Ö. (2015). *Recent advances in sliding modes: from control to intelligent mechatronics*, volume 24. Springer.

Zhang, Z., Leibold, M., and Wollherr, D. (2019). Integral sliding-mode observer-based disturbance estimation for euler-lagrangian systems. *IEEE Transactions on Control Systems Technology*, 1–13.