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Data-based guarantees of set invariance properties ⋆

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Abstract: For a discrete-time linear system, we use data from an open-loop experiment to design directly a linear feedback controller enforcing that a given (polyhedral) set of the state is invariant and given (polyhedral) constraints on the control are satisfied. By building on classical results from model-based set invariance and a fundamental result from Willems et al., the controller designed from data has the following desirable features. The satisfaction of the above properties is guaranteed only from data, it can be assessed by solving a numerically-efficient linear program, and, under a certain rank condition, a data-based solution is feasible if and only if a model-based solution is feasible.

Keywords: Data-based control; Control of constrained systems; Constrained control; Linear Systems; Linear programming; Convex optimization.

1. INTRODUCTION

Direct data-driven control design is an approach that aims at designing control laws based on input-output data collected from a system through an experiment, and bypasses completely the identification of a model of the system from the input-output data. Recent direct data-driven control techniques addressing model reference and tracking problems include iterative feedback tuning (Hjalmarsson et al., 1998), virtual reference feedback tuning (Campi et al., 2002), iterative correlation-based tuning (Karimi et al., 2004; Formentin et al., 2013), and unfalsified control (Battistelli et al., 2018). Direct data-driven methods have been considered also for other control problems, including nonlinear (Novara et al., 2013), predictive (Salvador et al., 2018), robust (Dai and Sznaier, 2018) and optimal control (Mukherjee et al., 2018; Baggio et al., 2019).

Most recently, a fundamental result from Willems et al. (2005) has been given new attention because of its deep implications for data-driven control. Namely, Willems et al. (2005) claims in broad terms that the whole set of trajectories of a linear system can be represented by a finite set of trajectories as long as those arise from sufficiently excited dynamics. This result has been exploited in Coulson et al. (2019) for data-based predictive control, and in De Persis and Tesi (2020) for data-driven stabilization and optimal control. De Persis and Tesi (2020) shows in particular that the result by Willems et al. can be used to achieve a data-based parametrization of feedback systems, enabling the design of (optimal) controllers directly via data-dependent linear matrix inequalities, also in the presence of noisy data. This idea has been further developed in van Waarde et al. (2020) to show that data-driven stabilization is possible even when data are not sufficiently rich to enable system identification, and in Berberich et al. (2019b) where – by formulating the data-based parametrization of closed-loop systems with noisy data obtained in De Persis and Tesi (2020) as a linear fractional transformation – data-driven H_∞ control is investigated, thus providing further evidence for developing a theory of data-driven control.

Except for contributions in the area of predictive control such as Salvador et al. (2018) and Berberich et al. (2019a), most of the works on data-driven control do not account for state and input constraints, which are one of the prime issues in many practical problems. In addition to the aforementioned papers, contributions to data-driven control in the presence of (state and input) constraints, also termed *safe control*, are found in the literature on learning-based control (Garcia and Fernández, 2015) and on safety certificates for learning-based control by convex optimization (Wabersich and Zeilinger, 2018), see also Remark 5 for a specific comparison with our approach.

In this paper, we consider data-driven safe control using notions from set invariance (Blanchini, 1999). Specifically, we consider linear time invariant systems in discrete time and study the problem of designing a control law based on a finite number of input-state data in such a way that the controlled system satisfies prescribed safety constraints given by polyhedral sets. Set invariance translates the notion of safety (i.e., if the system has initial state in a safe set, its solutions will not leave that set), so we characterize safety in terms of set invariance and λ-contractivity (recalled below in Definitions 2 and 3). Invariance of polyhedral sets for discrete-time linear systems has been thoroughly investigated in the late 80's assuming exact knowledge of the system matrices, and key results were

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given (Gutman and Cwikel, 1986; Vassilaki et al., 1988; Blanchini, 1990). These results consider, among others, the presence of disturbances on the state equation and parametric uncertainties in the system matrices. We refer the reader to the comprehensive survey Blanchini (1999) and the monograph Blanchini and Miani (2008) for an overview of these results.

Building on the notions of invariance and λ-contractivity, we show that the problem of designing safe controllers directly from data can be cast as a *linear program*, which can thus be solved efficiently. This is achieved by considering only *linear* feedback policies, although nonlinear ones are in general less conservative for linear systems with state and input constraints. Further, as in Vassilaki et al. (1988); Blanchini (1990), the solution takes the form of a state-feedback gain, which avoids to iteratively solving an online optimization problem as in receding-horizon predictive control and learning-based methods. On the other hand, in this paper we do not investigate optimality features of the safe controller.

The paper is organized as follows. Section 2 introduces the problem of interest along with some preliminaries on set invariance. The main results are given in Section 3, while Section 4 provides a preliminary result in the case of noisy data. A numerical example is discussed in Section 5.

Notation. $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ denote the sets of integers, of nonnegative integers, and of real numbers. For a positive integer n, $\mathbb{N}_n := \{1, \dots, n\}$. For column vectors $x_1 \in \mathbb{R}^{d_1}$, $\dots$, $x_m \in \mathbb{R}^{d_m}$, the notation $(x_1, \dots, x_m)$ is equivalent to $[x_1^\top \dots x_m^\top]^\top$. The $n \times n$ identity matrix is denoted by I_n. The vector $\mathbf{1}$ denotes the vector of all ones of appropriate dimension, i.e., $\mathbf{1} := (1, \dots, 1)$. Given two $n \times m$ matrices A and B, $A \geq 0$ indicates that each entry of A is nonnegative, and $A \geq B$ is equivalent to $A - B \geq 0$. For a polyhedron $\mathcal{A}$, $\operatorname{vert} \mathcal{A}$ is the set of its vertices. Given a set $\mathcal{A}$ and a scalar $\mu \geq 0$, $\mu\mathcal{A} := \{\mu x \colon x \in \mathcal{A}\}$.

2. PROBLEM STATEMENT AND PRELIMINARIES

In this section we give our problem statement and present essential preliminaries on set invariance.

2.1 Problem statement

We consider discrete-time linear time invariant systems

$$x^+ = Ax + Bu, \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. Before we introduce our sets of interest, we need the next notion.

Definition 1. (Blanchini and Miani, 2008, Def. 3.10) *A* C-set *is a convex and compact subset of* $\mathbb{R}^\nu$ *including the origin as an interior point.*

The first set of interest is the set $\mathcal{S}$ relative to the state x, which is based on a matrix $\mathrm{S} \in \mathbb{R}^{n_s \times n}$ with rows $\mathrm{S}^{(i)}$, $i = 1, \dots, n_s$. The set $\mathcal{S}$ is a polyhedral C-set represented through S as

$$\begin{aligned}\mathcal{S} :=& \{x \in \mathbb{R}^n \colon \mathrm{S}x \leq \mathbf{1}\} \\ =& \{x \in \mathbb{R}^n \colon \mathrm{S}^{(i)}x \leq 1, i = 1, \dots, n_s\}.\end{aligned} \tag{2}$$

The second set of interest is the set $\mathcal{U}$ relative to the input u, which is based on a matrix $\mathrm{U} \in \mathbb{R}^{n_u \times m}$ with rows $\mathrm{U}^{(i)}$, $i = 1, \dots, n_u$. The set $\mathcal{U}$ is a polyhedral convex set (including the origin as an interior point) represented through U as

$$\begin{aligned}\mathcal{U} :=& \{u \in \mathbb{R}^m \colon \mathrm{U}u \leq \mathbf{1}\} \\ =& \{u \in \mathbb{R}^m \colon \mathrm{U}^{(i)}u \leq 1, i = 1, \dots, n_u\}.\end{aligned} \tag{3}$$

We would like to impose that the state x remains confined in the set $\mathcal{S}$, while input u is constrained in the set $\mathcal{U}$. To this end, we introduce the next notion of invariance.

Definition 2. (Blanchini and Miani, 2008, Defs. 4.1, 4.4) *A set* $\mathcal{S} \subset \mathbb{R}^n$ *is* invariant *for* $x^+ = Fx$ *if each solution to* $x^+ = Fx$ *with initial condition* $x(0) \in \mathcal{S}$ *is such that* $x(t) \in \mathcal{S}$ *for all* $t \geq 0$.

We would like to impose that $\mathcal{S}$ is invariant and u satisfies the constraints given by $\mathcal{U}$ without the knowledge of the matrices A and B, by relying only on a number of data samples collected from the system. Specifically, we make an experiment on the system by applying a sequence $u_d(0), \dots, u_d(T-1)$ of inputs and measuring the corresponding values $x_d(0), \dots, x_d(T)$ of the state response, where the subscript d emphasizes that these are data. Following the notation in De Persis and Tesi (2020), we organize these data as

$$U_{0,T} := [u_d(0) \ \dots \ u_d(T-1)] \tag{4a}$$

$$X_{0,T} := [x_d(0) \ \dots \ x_d(T-1)] \tag{4b}$$

$$X_{1,T} := [x_d(1) \ \dots \ x_d(T)] \,. \tag{4c}$$

We can now state the problem of interest.

Problem 1. *Given a polyhedral C-set* $\mathcal{S}$ *as in* (2) *and a polyhedral convex set* $\mathcal{U}$ *as in* (3)*, find a state-feedback law* $u = Kx$*, with feedback gain* K *based only on the data in* (4)*, that* guarantees *that* $\mathcal{S}$ *is invariant, the origin is asymptotically stable, and the control input* $u = Kx$ *always belongs to* $\mathcal{U}$*.*

For brevity, we say in the following that $\mathcal{S}$ is *admissible for* $\mathcal{U}$ if for each $x \in \mathcal{S}$, we have $Kx \in \mathcal{U}$ (for some matrix K that is clear from the context).

2.2 Preliminaries on (model-based) set invariance

In Problem 1, we ask that $\mathcal{S}$ is invariant and the origin is asymptotically stable. These two properties can be embedded in the notion of λ-contractivity defined next.

Definition 3. (Blanchini and Miani, 2008, Def. 4.19) *A C-set* $\mathcal{S}$ *is* λ-contractive *for* $x^+ = Fx$ *if for some* $\lambda \in [0, 1)$*, for each* $x \in \mathcal{S}$

$$\inf\{\lambda' \geq 0 \colon Fx \in \lambda'\mathcal{S}\} \leq \lambda.$$

Note that if we allow $\lambda = 1$ in Definition 3, we recover invariance of Definition 2 as a special case. We recall the next result on λ-contractivity.

Fact 1. (Blanchini and Miani, 2008, Thm. 4.43) *Given a polyhedral C-set* $\mathcal{S}$ *of the form* (2)*, the set* $\mathcal{S}$ *is* λ*-contractive for* $x^+ = Fx$ *if and only if there exists a matrix* $P \geq 0$ *such that* $P\mathbf{1} \leq \lambda\mathbf{1}$ *and* $P\mathrm{S} = \mathrm{S}F$*.*

We have the next relationship between λ-contractivity and asymptotic stability.

Fact 2. (Blanchini and Miani, 2008, Cor. 4.52) *Given a system* $x^+ = Fx$*, there exists a polyhedral C-set which is*

λ-contractive if and only if all the eigenvalues of F have modulus less or equal to λ and all the eigenvalues for which the equality holds have phases that are rational multiples of π (namely, their phase θ can be expressed as $\theta = (p/q)\pi$ for some integers p and q).

Some comments on Fact 2 are relevant for the sequel and are stated in the next remarks.

Remark 1. *As a consequence of Fact 2, if a polyhedral C-set $\mathcal{S}$ is λ-contractive, then the origin (contained in the interior of $\mathcal{S}$ by Definition 1) is asymptotically stable. Instead of imposing that $\mathcal{S}$ is invariant and the origin is asymptotically stable in Problem 1, we impose in the sequel that $\mathcal{S}$ is λ-contractive. Invariance of $\mathcal{S}$ ($\lambda = 1$) is equivalent to marginal stability of the origin along with certain conditions on the eigenvalues with unitary modulus (Blanchini and Miani, 2008, Thm. 4.50), and does not guarantee asymptotic stability of the origin as required by Problem 1. Hence, imposing $\lambda < 1$ is convenient to have asymptotic stability of the origin.*

Remark 2. *For state-feedback control laws $u = Kx$ as in Problem 1, controllability of (A, B) implies that the closed-loop eigenvalues of $A + BK$ can be assigned to satisfy the necessary and sufficient condition in Fact 2, hence there exists a polyhedral C-set which is λ-contractive for $A+BK$.*

3. DATA-BASED DESIGN AND GUARANTEES FOR λ-CONTRACTIVITY

We now present our data-based solution to Problem 1. By the foregoing considerations, we address this problem in the context of λ-contractivity.

Given system (1), $\mathcal{S}$, $\mathcal{U}$ and u as in Problem 1 and level of contractivity $\lambda \in [0, 1)$, we have that $\mathcal{S}$ is λ-contractive for $x^+ = (A + BK)x$ and admissible for $\mathcal{U}$ if and only if there exist decision variables K and $P \geq 0$ such that

$$P\mathbf{1} \leq \lambda\mathbf{1} \tag{5a}$$
$$P\mathrm{S} = \mathrm{S}(A + BK) \tag{5b}$$
$$\mathrm{U}Ks \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S}. \tag{5c}$$

Indeed, λ-contractivity of $\mathcal{S}$ is equivalent to (5a)-(5b) by Fact 1, and admissibility of $\mathcal{S}$ for $\mathcal{U}$ is equivalent to

$$Ks \in \mathcal{U} \quad \forall s \in \operatorname{vert} \mathcal{S}$$

(since $\mathcal{S}$ is a polyhedral C-set and $\mathcal{U}$ is a polyhedral convex set). As noted in Remark 1, a feedback gain K that satisfies (5) solves Problem 1. We have the next result.

Theorem 1. *Consider $\mathcal{S}$, $\mathcal{U}$ and u as in Problem 1 and level of contractivity $\lambda \in [0, 1)$. Let the data matrices $U_{0,T}$, $X_{0,T}$ and $X_{1,T}$ be as in (4). If there exist decision variables G_K and $P \geq 0$ such that*

$$P\mathbf{1} \leq \lambda\mathbf{1} \tag{6a}$$
$$P\mathrm{S} = \mathrm{S}X_{1,T}G_K \tag{6b}$$
$$\mathrm{U}U_{0,T}G_K s \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S} \tag{6c}$$
$$I_n = X_{0,T}G_K, \tag{6d}$$

then the feedback gain

$$K = U_{0,T}G_K \tag{7}$$

is such that $\mathcal{S}$ is λ-contractive for the closed-loop system $x^+ = (A + BK)x$ and admissible for $\mathcal{U}$.

Proof. The proof can be found in Bisoffi et al. (2019). ■

Remark 3. *We note that Theorem 1 corresponds to solving a* linear program *in the decision variables G_K and P, hence it is numerically appealing.*

Compared to the case where the matrices A and B are known (*cf.* (5)), the data-driven solution of Theorem 1 only provides sufficient conditions for λ-contractivity. The reason is that we made no assumptions on the data used for designing the controller. Intuitively, if the data do not carry enough information on the plant dynamics, it might be impossible to get a data-based solution.

For stabilization (with no state and/or input constraints), De Persis and Tesi (2020) shows conditions on the data enabling a data-based parametrization of *all* stabilizing state-feedback gains. van Waarde et al. (2020) considers the *minimum* amount of information on the data under which *at least one* stabilizing gain can be found from data. Here, we follow the reasoning of De Persis and Tesi (2020), which lends itself to a direct extension to the case of state and/or input constraints. In fact, if the data enable a parametrization of *all* stabilizing gains, any controller guaranteeing λ-contractivity necessarily belongs to the feasibility set of (6) and is parameterized by the data since λ-contractivity is a stronger property than asymptotic stability, as shown in Fact 2. The next result holds.

Theorem 2. *Consider $\mathcal{S}$, $\mathcal{U}$ and u as in Problem 1 and level of contractivity $\lambda \in [0, 1)$. Let the data matrices $U_{0,T}$, $X_{0,T}$ and $X_{1,T}$ be as in (4). Assume further that the matrix*

$$\Theta := \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} \tag{8}$$

has full row rank. Then, there exists a feedback gain K such that $\mathcal{S}$ is λ-contractive for $x^+ = (A+BK)x$ and admissible for $\mathcal{U}$ if and only if there exist decision variables G_K and $P \geq 0$ such that (6) holds. Moreover, any such K can be expressed as in (7) for some G_K satisfying (6).

Proof. The proof can be found in Bisoffi et al. (2019). ■

Remark 4. *Under a rank condition, Theorem 2 establishes an equivalence between the model-based and the proposed data-based solution. In both cases, however, searching for a* linear *feedback policy is in general restrictive for given sets $\mathcal{S}$ and $\mathcal{U}$, and may preclude finding a solution, which could be found through a* nonlinear *feedback policy.*

An interesting result related to the matrix Θ in (8) is that if the system (1) is controllable, then one can always ensure that Θ has full row rank if the experimental data originate from persistently exciting input signals (Willems et al., 2005, Cor. 2). Moreover, controllability is important to enable the existence of a controller achieving λ-contractivity. Indeed, a controller achieving λ-contractivity of a *given* $\mathcal{S}$ may not exist. In that case, one may use the same data and search for different sets $\mathcal{S}'$ with different shapes until the constraints in (6) become feasible. Controllability is beneficial to this end because it ensures that a λ-contractive C-set $\mathcal{S}'$ exists, see Remark 2. Alternatively, if one wants to design $\mathcal{S}'$, the corresponding matrix S' becomes a decision variable and (6) becomes a *bilinear* program, as pointed out in (Blanchini, 1999, p. 1755).

Remark 5. *Compared to Wabersich and Zeilinger (2018), our approach considers unknown linear dynamics instead of known linear dynamics with unknown nonlinear term.*

*On the other hand, under a rank condition on the data, our approach always determines a solution if there is one (*cf. *Theorem 2) instead of providing ellipsoidal under-approximations of the original polyhedral set.*

3.1 λ-contractivity and decay rate

As shown in Vassilaki et al. (1988), the function $V : \mathcal{S} \to \mathbb{R}$ defined as

$$V(x) := \max_{i \in \{1,\dots,n_s\}} |\mathrm{S}^{(i)} x| \tag{9}$$

is a *polyhedral* Lyapunov function for the closed-loop dynamics $x^+ = (A+BK)x$ constrained on the set $\mathcal{S}$, and ensures that the origin is asymptotically stable. Indeed, V satisfies the following properties (which are justified in Bisoffi et al. (2019)): *(i)* $V(x) \geq 0$ for all $x \in \mathcal{S}$, and $V(x) = 0$ if and only if $x = 0$, *(ii)* it holds that

$$V(x^+) \leq \lambda V(x). \tag{10}$$

Properties *(i)* and *(ii)* imply asymptotic stability of the origin. In view of (10), the level of contractivity λ is also the decay rate of the Lyapunov function V, and it is thus of interest to minimize $\lambda \in [0, 1)$ as proposed for instance in Vassilaki et al. (1988). It is straightforward to do this based *only* on data, as shown in the next result.

Corollary 1. *Consider the same setting as in Theorem 1. If there exist decision variables λ, G_K and $P \geq 0$ solving*

$$\begin{aligned} &\min \lambda \\ &\text{such that } 0 \leq \lambda < 1 \text{ and (6) holds,} \end{aligned} \tag{11}$$

the feedback gain K in (7) *ensures that $\mathcal{S}$ is λ-contractive for $x^+ = (A+BK)x$ and admissible for $\mathcal{U}$.* ∎

The decision variables λ, G_K and P enter (11) in a linear fashion. Hence, (11) still corresponds to a linear program and can then be solved efficiently.

4. ROBUST DESIGN FOR NOISY DATA

In this section we present a preliminary result for the more realistic setting of noisy data. To this end, we consider a system of the form

$$x^+ = Ax + Bu + d, \tag{12}$$

where $d \in \mathcal{D} \subset \mathbb{R}^n$ and $\mathcal{D}$ is a polyhedral C-set represented through convex combinations of its n_d vertices $d^{(1)}, \dots, d^{(n_d)} \in \mathbb{R}^n$ as

$$\mathcal{D} := \left\{ \sum_{i=1}^{n_d} \alpha_i d^{(i)} : \mathbf{1}^\top \alpha = 1, \alpha \geq 0 \right\}. \tag{13}$$

The disturbance affects both the data and the invariance properties of (12). As for the data, the experiment involves the quantities in (4) and, additionally, the *unknown* sequence $d_d(0), \dots, d_d(T-1)$ of disturbances, organized as

$$D_{0,T} := [d_d(0) \ \dots \ d_d(T-1)]\,. \tag{14}$$

Because of (12), the data in (14) and (4) satisfy

$$X_{1,T} = AX_{0,T} + BU_{0,T} + D_{0,T} = [B \ \ A] \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} + D_{0,T}. \tag{15}$$

As for the invariance properties, we consider accordingly the next robust version of Definition 2.

Definition 4. (Blanchini, 1990, Def. 2.1) *A set $\mathcal{S}$ is* robustly invariant with respect to $\mathcal{D}$ *for $x^+ = Fx + d$ if for each initial condition $x(0) \in \mathcal{S}$ and each disturbance d satisfying $d(t) \in \mathcal{D}$ for all $t \geq 0$, the corresponding solution to $x^+ = Fx + d$ satisfies $x(t) \in \mathcal{S}$ for all $t \geq 0$.*

In this section we consider a slightly different setting than the rest of the paper, that is, guaranteeing that $\mathcal{S}$ is robustly invariant w.r.t. $\mathcal{D}$ for the closed-loop system and is admissible for $\mathcal{U}$, in the presence of noisy data. We recall the next instrumental result.

Fact 3. (Blanchini, 1990, Thm. 2.1) *Let $\mathcal{S}$ and $\mathcal{D}$ be C-sets. The set $\mathcal{S}$ is robustly invariant w.r.t. $\mathcal{D}$ for $x^+ = Fx + d$ if and only if for each $s \in \operatorname{vert} \mathcal{S}$ and each $w \in \operatorname{vert} \mathcal{D}$, $Fs + w \in \mathcal{S}$.*

This fact allows us to conclude that given the system in (12) and for $\mathcal{S}, \mathcal{U}$ and u as in Problem 1, and the C-set $\mathcal{D}$ in (13), $\mathcal{S}$ is

(a) robustly invariant w.r.t. $\mathcal{D}$ for $x^+ = (A+BK)x + d$,
(b) admissible for $\mathcal{U}$

if and only if

$$\mathrm{S}((A+BK)s + w) \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S}, \forall w \in \operatorname{vert} \mathcal{D} \tag{16a}$$

$$\mathrm{U}Ks \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S}. \tag{16b}$$

Let us apply to (16) the same approach as in Section 3 in light of the new dynamics in (15). If there exists a decision variable G_K such that

$$\mathrm{S}((X_{1,T} - D_{0,T}) G_K s + w) \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S}, \forall w \in \operatorname{vert} \mathcal{D} \tag{17a}$$

$$\mathrm{U} U_{0,T} G_K s \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S} \tag{17b}$$

$$I_n = X_{0,T} G_K, \tag{17c}$$

then the feedback gain $K = U_{0,T} G_K$ would ensure for $\mathcal{S}$ its desired properties *(a)–(b)* above. In particular, (17a) follows from

$$A + BK = [B \ A] \begin{bmatrix} K \\ I_n \end{bmatrix} = [B \ A] \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} G_K = (X_{1,T} - D_{0,T}) G_K$$

where the last equality uses the new dynamics in (15). However, the disturbance sequence leading to $D_{0,T}$ in (17a) is unknown. A possible way of overcoming this issue is to ask conservatively that (17a) be satisfied for all the possible sequences of the disturbance $d_d(0), \dots, d_d(T-1)$ as long as each $d_d(0), \dots, d_d(T-1)$ belongs to $\mathcal{D}$. To this end, define for $j \in \mathbb{N}_T$ and $i \in \mathbb{N}_{n_d}$ the matrix $\delta_{ji} \in \mathbb{R}^{n \times T}$ being zero except for its j-th column equal to $Td^{(i)}$, i.e.,

$$\delta_{ji} := \big[\underbrace{0}_{\text{1-st,}} \,|\, \dots \,|\, \underbrace{Td^{(i)}}_{j\text{-th,}} \,|\, \dots \,|\, \underbrace{0}_{T\text{-th column}} \big].$$

The reason for the dependence on T in the j-th column of δ_{ji} becomes clear from the proof of our next result.

Proposition 1. *Consider $\mathcal{S}$, $\mathcal{U}$ and u as in Problem 1, the disturbance d belonging to the C-set $\mathcal{D}$ in* (13)*, and let the data matrices $U_{0,T}$, $X_{0,T}$, $X_{1,T}$ be as in* (4)*. If there exists a decision variable G_K such that*

$$\mathrm{S}((X_{1,T} - \delta_{ji}) G_K s + w) \leq \mathbf{1}$$
$$\forall s \in \operatorname{vert} \mathcal{S}, \forall w \in \operatorname{vert} \mathcal{D}, \forall j \in \mathbb{N}_T, \forall i \in \mathbb{N}_{n_d} \tag{18a}$$

$$\mathrm{U} U_{0,T} G_K s \leq \mathbf{1} \quad \forall s \in \operatorname{vert} \mathcal{S} \tag{18b}$$

$$I_n = X_{0,T} G_K, \tag{18c}$$

then the feedback gain $K = U_{0,T} G_K$ is such that $\mathcal{S}$ is robustly invariant w.r.t. $\mathcal{D}$ for $x^+ = (A+BK)x + d$ and admissible for $\mathcal{U}$.

Proof. The proof can be found in Bisoffi et al. (2019). ∎

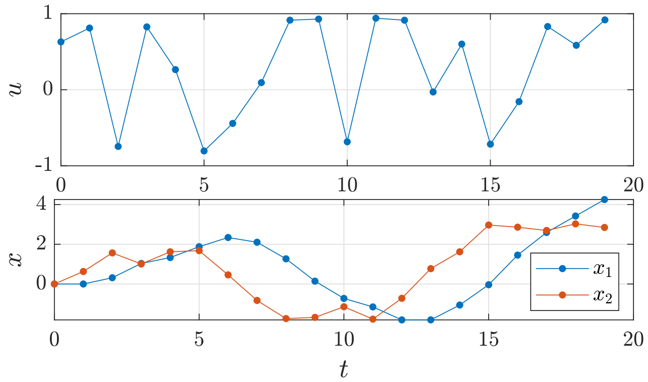

Fig. 1. Input and state data as in (4), with $T = 20$.

Proposition 1 is a preliminary result due to the conservatism of replacing the constraints in (17a) (where $D_{0,T}$ is unknown) with $n_d T$ as many such constraints in (18a). On the other hand, Proposition 1 still corresponds to solving a linear program in the decision variable G_K.

5. NUMERICAL EXAMPLE

In this section we illustrate the previous results through an example taken from Vassilaki et al. (1988). The sets $\mathcal{S}$ in (2) and $\mathcal{U}$ in (3) are determined by the matrices

$$\mathrm{S} := \begin{bmatrix} 1/5 & 2/5 \\ -1/5 & -2/5 \\ -3/20 & 1/5 \\ 3/20 & -1/5 \end{bmatrix}, \quad \mathrm{U} := \begin{bmatrix} 1/7 \\ -1/7 \end{bmatrix}, \qquad (19)$$

so that the set $\mathcal{S}$ corresponds to the quadrilateral in a green, solid line in Figure 2, while the set $\mathcal{U}$ corresponds to the condition $-7 \leq u \leq 7$. The level of contractivity is selected as $\lambda = 0.84$. The data are collected from an open-loop experiment as in Figure 1, where u is the realization of a random variable uniformly distributed on $[-1, 1]$, and show that the underlying linear system is unstable. The matrices A and B generating these data are

$$A := \begin{bmatrix} 4/5 & 1/2 \\ -2/5 & 6/5 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and are reported *only* for illustrative purposes, because our solution relies *only* on the collected data.

Remark 6. *Full row rank of* Θ *in* (8) *can be checked from data. However, this condition holds by (Willems et al., 2005, Cor. 2) if* (A, B) *is controllable and the input sequence is persistently exciting of order* $n + 1$ *(see, e.g., (De Persis and Tesi, 2020, Def. 1)). As noted in (De Persis and Tesi, 2020, Sect. II.A), persistence of excitation poses a mild necessary condition on the number of samples, i.e.,* $T \geq (m + 1)n + m = 5$ *in the considered case.*

The linear optimization problem in Theorem 1 is solved in the variables G_K and P, and the resulting K in (7) is

$$K = [0.420 \ \ -0.610]. \qquad (20)$$

Only for illustrative purposes, we also solve the problem in (5) and obtain a gain matrix

$$K_{A,B} = [0.313 \ \ -0.671]. \qquad (21)$$

The solutions resulting from simulating the system with state feedback law $u = Kx$ (our data-based solution) and $u = K_{A,B}\, x$ (the model-based solution) are in Figure 2 and show that Problem 1 is solved. As an alternative to solving the feasibility problem in Theorem 1, we solve

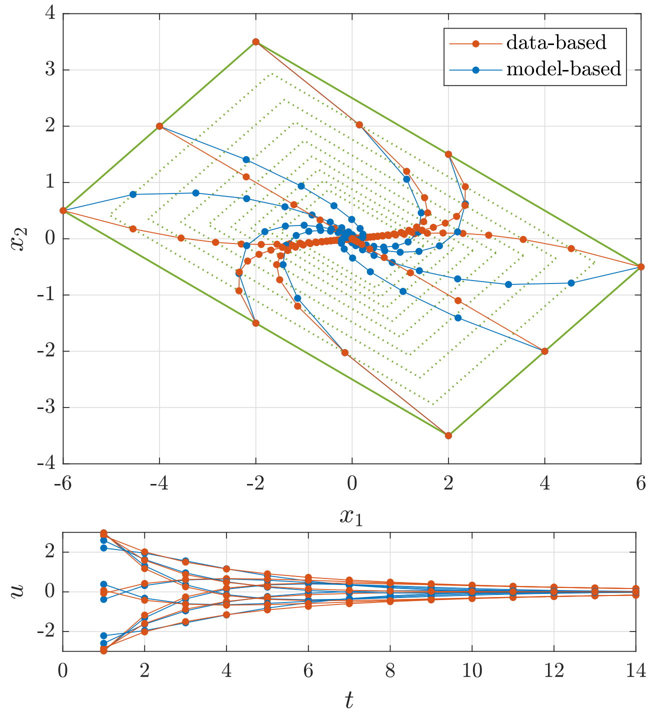

Fig. 2. Sets $\mathcal{S}$ and $\mathcal{U}$ with parameters in (19) and $\lambda = 0.84$. (Top) Solutions arising from the state feedback law $u = Kx$ (see (20)) designed based on data (orange), and from $u = K_{A,B}\, x$ (see (21)) based on the classical model-based approach (blue), set $\mathcal{S}$ (green, solid) and the sets $\lambda\mathcal{S}$, $\lambda^2\mathcal{S}$, $\lambda^3\mathcal{S}$, ... (green, dotted). (Bottom) Control signal u corresponding to the solutions in orange and blue depicted on top. The control signal satisfies the constraints given by $\mathcal{U}$.

the minimization problem in Corollary 1 using the same data. In this case we obtain $\lambda = 0.758$ and $K = K_{A,B} = [0.379 \ \ -0.692]$ and the resulting solutions are in Figure 3.

Some comments on the results corresponding to Figures 2 and 3 can be made. Because Θ in (8) has full row rank, feasibility of conditions (5) in the variables K and P is equivalent to feasibility of conditions (6) in the variables G_K and P by Theorem 2. In general, the two feasibility problems yield different solutions as in Figure 2, e.g., due to different initializations of the decision variables. However, since feasible linear programs have a global minimum, minimizing λ under (5) or (6) yields the same value for λ. Moreover, minimizing λ reduces the size of the feasibility set (due to the constraints $P \geq 0$ and $P\mathbf{1} \leq \lambda\mathbf{1}$), which leads in this case to the fact that the minimizers G_K and P under (6) yield the same feedback gain as the minimizers K and P under the conditions in (5).

Finally, we tested the design with noisy data given by Proposition 1. The data are generated according to (12) with matrices A, B and input signal u and sets $\mathcal{S}$, $\mathcal{U}$ as before. The set $\mathcal{D}$ in (13) is taken as $e\mathcal{E}$ where $e > 0$ and $\mathcal{E} := \{(e_1, e_2) \colon |e_1| \leq 1, |e_2| \leq 1\}$, so that larger values of e dilate $\mathcal{E}$ and yield a larger $\mathcal{D}$, which determines in turn the size of disturbance d in the data (see (14)) and for robust invariance (see (12)). The feasibility problem in Proposition 1 could be solved for e up to $8 \cdot 10^{-2}$ (against an input signal u in $[-1, 1]$). The development of methods tailored for highly noisy data is currently under study.

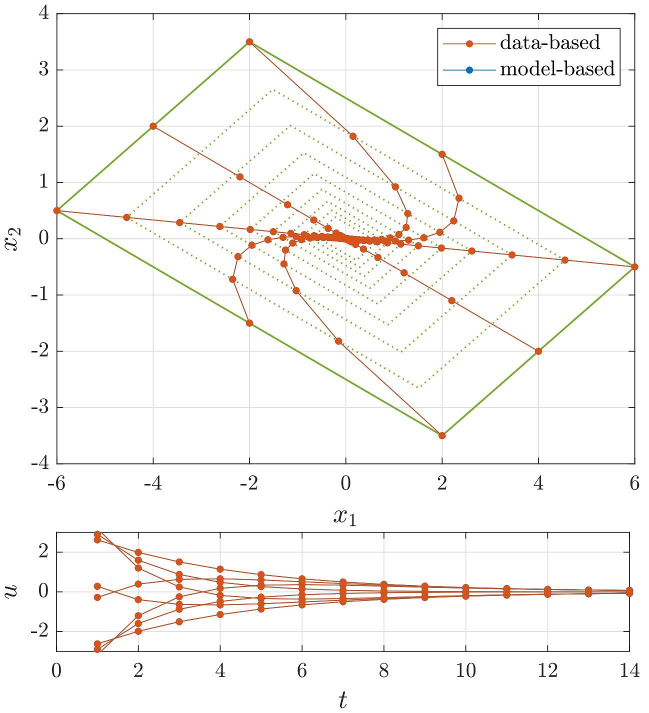

Fig. 3. See the caption of Figure 2 for the illustration convention of the quantities in this figure, which correspond to $\lambda = 0.758$ minimized as in Corollary 1.

6. CONCLUSIONS

This paper proposes a data-based solution for designing a linear feedback controller enforcing that a given polyhedral C-set for the state is λ-contractive (hence, invariant) and given polyhedral convex constraints on the control are satisfied. With respect to classical approaches from set-invariance, we show that the data-based solution still arises from a numerically-efficient linear program, and that, under a rank condition on the collected data, the data-based solution is feasible if and only if the model-based solution is feasible. The level of λ-contractivity is guaranteed based on the data. Our main results are given for the nominal case of input and state data not affected by noise, and a preliminary result is given for noisy data.

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