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Computing and optimizing the robust strong H-infinity norm of uncertain time-delay systems

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Abstract: This short paper presents a method to compute and optimize the robust strong H-infinity norm of linear time-invariant systems with discrete delays and uncertainties on the system matrices. Special attention will be paid to a fragility problem of the H-infinity norm for systems with discrete delays in the direct feed-through terms. More specifically, for such systems the H-infinity norm might be sensitive to arbitrary small delay changes. This fragility problem can be resolved by considering the strong H-infinity norm, which takes into account infinitesimal delay perturbations. The robust strong H-infinity norm is subsequently defined as the worst-case strong H-infinity norm over all instances of the uncertainties and is a measure for robust performance. It can be shown that this robust strong H-infinity norm is related to the robust distance to instability of an associated uncertain system described by delay differential-algebraic equations. Using this relation, the robust strong H-infinity norm can be computed efficiently. This efficient computation of the robust strong H-infinity norm will be used for controller design by direct minimization of the robust strong H-infinity norm as function of the controller parameters.

Keywords: Dynamical Systems - Time delay - Delay differential equations - H-infinity control - Robust Performance - Singular Systems - Distance to instability

1. INTRODUCTION

The $\mathcal{H}_\infty$-norm is an important performance measure in robust control theory, see Zhou and Doyle (1998). For an exponentially stable dynamical system with input w ($\in \mathbb{R}^m$), output z ($\in \mathbb{R}^p$), and transfer function $\mathbb{C} \ni s \mapsto T(s) \in \mathbb{C}^{p\times m}$, he $\mathcal{H}_\infty$-norm is defined as

$$\|T\|_\infty := \sup_{\omega\in\mathbb{R}^+} \sigma_1\big(T(\jmath\omega)\big)$$

with $\sigma_1(R)$ the largest singular value of the matrix R. Using Parseval's identity, the $\mathcal{H}_\infty$-norm can equivalently be characterised as the worst-case energy gain of the system with respect to energy-bounded input noise:

$$\|T\|_\infty = \sup_{w\in\mathcal{L}_2} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}},$$

with $\mathcal{L}_2$ the space of square-integratable functions equipped with the following norm: $\|z\|_{\mathcal{L}_2} = \sqrt{\int_0^{+\infty} \|z(t)\|_2^2 \, dt}$.

Here we will consider linear time-invariant (LTI) systems with discrete delays of the following form:

$$\begin{cases} \dot{x}(t) = \sum_{k=0}^{K} A_k x(t-\tau_k) + \sum_{k=0}^{K} B_k w(t-\tau_k) \\ z(t) = \sum_{k=0}^{K} C_k x(t-\tau_k) + \sum_{k=0}^{K} D_k w(t-\tau_k), \end{cases} \quad (1)$$

with $x \in \mathbb{R}^n$ the state, $w \in \mathbb{R}^m$ the performance input, $z \in \mathbb{R}^p$ the performance output, A_k, B_k, C_k and D_k real-valued matrices of appropriate dimensions, and $0 = \tau_0 < \tau_1 < \cdots < \tau_K$ discrete delays. The associated transfer function equals

$$\begin{aligned} T(s;\boldsymbol{\tau}) = \left(\textstyle\sum_{k=0}^{K} C_k e^{-s\tau_k}\right) \left(sI - \textstyle\sum_{k=0}^{K} A_k e^{-s\tau_k}\right)^{-1} \\ \times\left(\textstyle\sum_{k=0}^{K} B_k e^{-s\tau_k}\right) + \textstyle\sum_{k=0}^{K} D_k e^{-s\tau_k}, \end{aligned} \quad (2)$$

with $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)$. For the considered systems, the $\mathcal{H}_\infty$-norm can be sensitive to arbitrary small delay changes. Gumussoy and Michiels (2011) therefore introduced the strong $\mathcal{H}_\infty$-norm, which is defined as the smallest upper bound for the $\mathcal{H}_\infty$-norm which is insensitive to infinitesimal delay changes:

$$|||T|||_\infty := \lim_{\gamma\to 0+}\sup \left\{ \|T_\gamma\|_\infty : \boldsymbol{\tau}_\gamma \in \mathcal{B}(\boldsymbol{\tau},\gamma) \cap (\mathbb{R}^+)^K \right\},$$

with T_γ the transfer function of (1) where the delays are replaced by $\boldsymbol{\tau}_\gamma$ and $\mathcal{B}(\boldsymbol{\tau},\gamma)$ a ball in $\mathbb{R}^K$ centered at $\boldsymbol{\tau}$ with radius γ. Note that although infinitesimal delay changes are considered, the strong $\mathcal{H}_\infty$-norm is still a property of the nominal system.

Furthermore, by Theorems 4.3 and 4.5 in Gumussoy and Michiels (2011), the strong $\mathcal{H}_\infty$-norm of (2) is equal to

$$|||T|||_\infty = \max\{\|T\|_\infty, |||\textstyle\sum_{k=0}^{K} D_k e^{-s\tau_k}|||_\infty\}, \quad (3)$$

and we can express

$$|||\sum_{k=0}^{K} D_k e^{-s\tau_k}|||_\infty = \max_{\theta\in[0,2\pi)^K} \sigma_1\Big(D_0 + \sum_{k=1}^{K} D_k e^{\jmath\theta_k}\Big). \quad (4)$$

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The strong $\mathcal{H}_\infty$-norm of (2) thus equals the maximum of $\|T\|_\infty$, the nominal $\mathcal{H}_\infty$-norm of the system, and $|||\sum_{k=0}^{K} D_k e^{-s\tau_k}|||_\infty$, the strong $\mathcal{H}_\infty$-norm of the asymptotic part of the transfer function, i.e. the part of the transfer function that does not vanishes as $|s| \to \infty$.

Until now we considered one deterministic model. In most applications, however, there is a discrepancy between the considered model and the reality, due to modeling mismatches and/or parameter measurements with finite precision. To account for these discrepancies, uncertainties are added to the model. For such "uncertain" time-delay systems, the robust strong $\mathcal{H}_\infty$-norm is defined as the worst-case value of the strong $\mathcal{H}_\infty$-norm over all instances of the uncertainties. The robust $\mathcal{H}_\infty$-norm of uncertain time-delay systems is already examined in Kokame et al. (1998) and Ji et al. (2006). The scope of these papers is however limited to systems with one state delay, for which the robust strong $\mathcal{H}_\infty$-norm and the robust $\mathcal{H}_\infty$-norm, the worst-case value of the nominal $\mathcal{H}_\infty$-norm, coincide. This work, in contrast, considers more general systems with multiple delays in the state, input, output and direct feedthrough terms. The considered uncertain system and its robust strong $\mathcal{H}_\infty$-norm will be defined in more detail in Section 2 .

It is long established - a well known reference is the work of Hinrichsen and Pritchard (2005) - that there exists a relation between the $\mathcal{H}_\infty$-norm of an LTI system and the distance to instability of an associated autonomous system with a complex-valued uncertainty. In Appeltans and Michiels (2019) it is shown that a similar relation exists between the robust strong $\mathcal{H}_\infty$-norm of an uncertain LTI system with discrete delays and the robust distance to instability of an associated uncertain system described by delay differential-algebraic equations (DDAEs). This result will be revisited in Section 3. Based on this relation, Section 4 presents a numerical algorithm to compute the robust strong $\mathcal{H}_\infty$-norm.

Finally, Section 5 discusses and illustrates a controller design approach for uncertain LTI systems with discrete delays based on the direct optimization of the robust strong $\mathcal{H}_\infty$-norm as function of the controller parameters.

2. SYSTEM DESCRIPTION

As mentioned before, uncertainties are used to represent the mismatch between the model and reality. Here we consider L matrix uncertainties: $\delta = (\delta_1, \ldots, \delta_L)$ which are assumed to be real-valued and each bounded in Frobenius norm. We will denote the set of all feasible uncertainties as $\mathcal{D}$. These uncertainties affect the system matrices in (1) in the following way: $\tilde{M}(\delta) = M + \sum_{l=1}^{L} \sum_{s=1}^{S_l^M} G_{l,s}^M \delta_l H_{l,s}^M$, with $G_{l,s}^M$ and $H_{l,s}^M$ real-valued shape matrices of appropriate dimensions. Note that each uncertainty can affect affect multiple matrices, and can affect one matrix via multiple perturbation terms.

This leads to the following "uncertain" model:

$$\begin{cases} \dot{x}(t) = \sum_{k=0}^{K} \tilde{A}_k(\delta) x(t-\tau_k) + \sum_{k=0}^{K} \tilde{B}_k(\delta) w(t-\tau_k) \\ z(t) = \sum_{k=0}^{K} \tilde{C}_k(\delta) x(t-\tau_k) + \sum_{k=0}^{K} \tilde{D}_k(\delta) w(t-\tau_k), \end{cases} \tag{5}$$

with $\tilde{A}_k(\delta)$, $\tilde{B}_k(\delta)$, $\tilde{C}_k(\delta)$, and $\tilde{D}_k(\delta)$ uncertain matrices as defined above and $0 = \tau_0 < \cdots < \tau_K$ discrete delays. The associated "uncertain" transfer function equals

$$\tilde{T}(s;\boldsymbol{\tau},\delta) = (\sum_{k=0}^{K} \tilde{C}_k(\delta) e^{-s\tau_k}) \left(sI - \sum_{k=0}^{K} \tilde{A}_k(\delta) e^{-s\tau_k}\right)^{-1} \times (\sum_{k=0}^{K} \tilde{B}_k(\delta) e^{-s\tau_k}) + \sum_{k=0}^{K} \tilde{D}_k(\delta) e^{-s\tau_k},$$

and its robust strong $\mathcal{H}_\infty$-norm is defined as:

$$\begin{aligned} |||\tilde{T}|||_\infty^{\mathcal{D}} :&= \max_{\delta\in\mathcal{D}} |||\tilde{T}(\cdot;\cdot,\delta)|||_\infty \\ &= \max\{ \max_{\delta\in\mathcal{D}} \|\tilde{T}(\cdot;\cdot,\delta)\|_\infty, \\ &\qquad \max_{\delta\in\mathcal{D}} |||\sum_{k=0}^{K} \tilde{D}_k(\delta) e^{-s\tau_k}|||_\infty\}. \end{aligned} \tag{6}$$

3. ASSOCIATED UNCERTAIN SYSTEM OF DELAY-DIFFERENTIAL ALGEBRAIC EQUATIONS

Consider the "uncertain" autonomous system described by the following delay differential-algebraic equations:

$$Q\,\dot{\xi}(t) = P_0(\delta,\Delta)\,\xi(t) + \sum_{k=1}^{K} P_k(\delta)\,\xi(t-\tau_k), \tag{7}$$

in which

$$Q = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ P_0(\delta,\Delta) = \begin{bmatrix} \tilde{A}_0(\delta) & \tilde{B}_0(\delta) & 0 \\ 0 & -I & \Delta \\ \tilde{C}_0(\delta) & \tilde{D}_0(\delta) & -I \end{bmatrix} \text{ and}$$

$$P_k(\delta) = \begin{bmatrix} \tilde{A}_k(\delta) & \tilde{B}_k(\delta) & 0 \\ 0 & 0 & 0 \\ \tilde{C}_k(\delta) & \tilde{D}_k(\delta) & 0 \end{bmatrix},$$

with $\tilde{A}_k(\delta)$, $\tilde{B}_k(\delta)$, $\tilde{C}_k(\delta)$, $\tilde{D}_k(\delta)$, and τ_k as previously defined, $\delta \in \mathcal{D}$, I_n the n dimensional identity matrix, and $\Delta \in \mathbb{C}^{m\times p}$. Notice that (7) has two types of uncertainties: the real-valued uncertainties on the system matrices, δ, and an additional complex-valued uncertainty, Δ.

Using the results from Fridman (2002), one can show that systems of form (7) may be of advanced type or admit impulsive solutions if $I_p - \tilde{D}_0(\delta)\Delta$ is not invertible. Therefore, we define the robust distance to non well-posedness as:

$$\begin{aligned} \mathrm{d_{NWP}} := \min\{\epsilon \geq 0 : \exists\delta \in \mathcal{D} \text{ and } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 \leq \epsilon \\ \text{such that } I_p - \tilde{D}_0(\delta)\Delta \text{ is not invertible}\}, \end{aligned}$$

and for completeness we define $\min\emptyset = +\infty$.

Next, we restrict our attention to instances of (7) for which $\|\Delta\|_2$ is bounded to the interval $[0, \mathrm{d_{NWP}})$ and examine the exponential stability of their null solution for particular δ and Δ. The notion of exponential stability is however not trivial for such systems as stability can be sensitive to arbitrary small delay perturbations. Therefore we will consider strong exponential stability, as defined in Michiels (2011), which takes into account infinitesimal delay changes. Furthermore, in the same work it was shown that a necessary and sufficient condition for strong exponential stability of the null solution of (7) for particular δ and Δ is:

$$\gamma_0(\delta,\Delta) < 1 \text{ and } \alpha(\delta,\Delta) < 0, \tag{8}$$

with

$$\gamma_0(\delta,\Delta) := \max_{\theta\in[0,2\pi)^K} \rho\left(\left(-I + \Delta\tilde{D}_0(\delta)\right)^{-1} \Delta \sum_{k=1}^{K} \tilde{D}_k(\delta) e^{\jmath\theta_k}\right)$$

where $\rho(R)$ gives the spectral radius of the matrix R and

$$\alpha(\delta,\Delta) := \sup\Big\{\Re(\lambda) : \det\Big(Q\lambda - P_0(\delta,\Delta) - \sum_{k=1}^{K} P_k(\delta)e^{-\lambda\tau_k}\Big) = 0\Big\}.$$

To define the robust distance to instability, we are interested in the smallest ϵ for which there exist $\delta \in \mathcal{D}$ and $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \le \epsilon$ such that the null solution of the corresponding instance of (7) is not strongly stable. Condition (8) leads to the definition of the following two distances

$$\mathrm{d}_\infty := \min\{\epsilon \in [0, \mathrm{d}_{\mathrm{NWP}}) : \exists \delta \in \mathcal{D} \text{ and } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 \le \epsilon \text{ such that } \gamma_0(\delta,\Delta) \ge 1\}$$

and

$$\mathrm{d}_\mathrm{f} := \min\{\epsilon \in [0, \min\{\mathrm{d}_{\mathrm{NWP}}, \mathrm{d}_\infty\}) : \exists \delta \in \mathcal{D} \text{ and } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 \le \epsilon \text{ such that } \alpha(\delta,\Delta) \ge 0\}.$$

The robust distance to instability is then defined as

$$d = \min\{\mathrm{d}_{\mathrm{NWP}}, \mathrm{d}_\infty, \mathrm{d}_\mathrm{f}\}.$$

For more details on these distances, we refer to Appeltans and Michiels (2019).

The following theorem formalizes the relation between this robust distance to instability and the robust strong $\mathcal{H}_\infty$-norm of (5).

Theorem 1. If system (5) is internally exponentially stable for all $\delta \in \mathcal{D}$, then its robust strong $\mathcal{H}_\infty$-norm is equal to the reciprocal of the robust distance to instability of (7).

For the proof, see Appeltans and Michiels (2019). The most important ingredients of this proof are

$$\min\{\mathrm{d}_{\mathrm{NWP}}, \mathrm{d}_\infty\} = \left(\max_{\delta\in\mathcal{D}} |||\textstyle\sum_{k=0}^{K} \tilde{D}_k(\delta)e^{-s\tau_k}|||_\infty\right)^{-1}$$

and

$$|||\tilde{T}|||_\infty^{\mathcal{D}} = \max\{\max_{\delta\in\mathcal{D}} |||\textstyle\sum_{k=0}^{K} \tilde{D}_k(\delta)e^{-s\tau_k}|||_\infty, \mathrm{d}_\mathrm{f}^{-1}\} \quad (9)$$

where we define $(+\infty)^{-1} = 0$.

4. NUMERICAL ALGORITHM

Based on relation (9), Appeltans and Michiels (2019) presented a two step algorithm to compute the robust strong $\mathcal{H}_\infty$-norm of (5). In the first step, the algorithm computes $\max_{\delta\in\mathcal{D}} |||\sum_{k=0}^{K} \tilde{D}_k(\delta)e^{-s\tau_k}|||_\infty$. By (4), this quantity is found by solving the following optimization problem

$$\max_{\delta\in\mathcal{D}} \max_{\theta\in[0,2\pi)^K} \sigma_1\Big(\tilde{D}_0(\delta) + \sum_{k=1}^{K} \tilde{D}_k(\delta)e^{\jmath\theta_k}\Big).$$

To solve this optimization problem, the projected gradient flow method is used. This method looks for a flow through the space of permissible θ and δ along which the objective function monotonically increases and whose attractive stationary points are (local) optimizers of the optimization problem. These (local) optimizers are found by discretizing the flow until it converges to a stationary point. To improve the chance of converging to the global optimum, the algorithm is restarted with different initial points.

In a second step, one has to compute d_f. This quantity can be found by interpreting d_f as the zero-crossing of the pseudo-spectral abscissa,

$$\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon) = \max_{\delta\in\mathcal{D}, \|\Delta\|_2\le\epsilon} \alpha(\delta,\Delta),$$

in function of ϵ for $\epsilon \in \big[0, \min\{\mathrm{d}_{\mathrm{NWP}}, \mathrm{d}_\infty\}\big)$. This zero-crossing can be found using the Newton-bisection method. To compute $\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon)$ for given ϵ and $\mathcal{D}$ one has to solve the following optimization problem

$$\begin{array}{ll} \max_{\delta,\Delta} \sup_\lambda & \Re(\lambda) \\ \text{subjected to} & \det\big(M(\lambda;\delta,\Delta)\big) = 0 \\ & \delta \in \mathcal{D}, \Delta \in \mathbb{C}^{m\times p}, \|\Delta\|_2 \le \epsilon. \end{array}$$

This optimization problem can also be solved using the projected gradient flow method.

5. CONTROLLER DESIGN

In this section we present a controller design methodology based on the direct optimization framework. The idea is to find a suitable controller by directly minimizing the robust strong $\mathcal{H}_\infty$-norm as function of the controller parameters. Solving this minimization problem is however not trivial as the robust strong $\mathcal{H}_\infty$-norm may be a non-smooth and non-convex function of the controller parameters. We will therefore use an optimization method for non-smooth functions such as HANSO, see Overton (2009). This solver requires the evaluation of the robust strong $\mathcal{H}_\infty$-norm for given values of the control parameters and its derivative with respect to the controller parameters at values for which this derivative exists. The former can be obtained using the numerical algorithm described above, the latter can be obtained as a by-product at almost no additional cost.

We will illustrate this methodology by designing a decentralized controller for a networked system that consists of N identical carts that each balance an inverted pendulum and that are connected using identical springs [1]. A schematic representation of a single cart is given in Figure 1. After linearization we obtain the following state-space description for this system:

$$\begin{cases} \dot{x}(t) = (I_N \otimes A)x(t) + (I_N \otimes B_u)u(t-\tau_u) \\ \qquad\quad +(I_N \otimes B_w)w(t) + (P_N \otimes B_n C_n)x(t) \\ z(t) = (I_N \otimes C_z)x(t) \end{cases} \quad (10)$$

with $x(t) = [x_1(t)^T \cdots x_N(t)^T]^T$, $x_i(t)$ the internal state of cart i, $u(t) = [u_1(t)^T \cdots u_N(t)^T]^T$ the control inputs, $w(t) = [w_1(t)^T \cdots w_N(t)^T]^T$ the performance inputs, $z(t) = [z_1(t)^T \cdots z_N(t)^T]^T$ the performance outputs,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{M} & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2k}{Ml} & 0 & \frac{g}{l} + \frac{mg}{Ml} & 0 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix}, \; B_n = \begin{bmatrix} 0 \\ \frac{k}{M} \\ 0 \\ \frac{-k}{Ml} \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 & 0 \\ \frac{1}{M} & \frac{-m}{M} \\ 0 & 0 \\ \frac{-1}{Ml} & \frac{1}{l} + \frac{m}{Ml} \end{bmatrix}, \; C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; C_n = [2 \; 0 \; 0 \; 0],$$

$M = 1$ kg, $m = 0.05$ kg, $k = 1$ N/m, $l = 1$ m, $g = 9.8$ m/s^2 and $\tau_u = 0.1$ s, I_N the N dimensional identity matrix, $\otimes$ the Kronecker product and

[1] The first and last cart are at one side connected to a wall.

$$P_N = \begin{bmatrix} 0 & 0.5 & & & \\ 0.5 & 0 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 0 & 0.5 \\ & & & 0.5 & 0 \end{bmatrix}.$$

Each individual subsystem is controlled by a local state feedback controller, that is identical for each subsystem:

$$u_i(t) = Kx_i(t) \text{ or equivalently } u(t) = (I_N \otimes K)\, x(t). \quad (11)$$

The goal is to find a matrix K such that the strong $\mathcal{H}_\infty$-norm of the closed loop is minimized. However, for large N this is computationally costly. Therefore, we will use the result from Section 4 in Dileep et al. (2018) which showed that the strong $\mathcal{H}_\infty$-norm of (10) is equal to the maximal strong $\mathcal{H}_\infty$-norm from $\hat{w}$ to $\hat{z}$ of

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + B_u K\hat{x}(t-\tau_u) + B_w\hat{w}(t) + \lambda B_n C_n x(t) \\ \hat{z}(t) = C_z\hat{x}(t) \end{cases} \quad (12)$$

with λ a parameter whose allowable values correspond to the eigenvalues of P_N. Furthermore, as the eigenvalues of P_N lie in the interval $[-1,\ 1]$ for all N, the robust strong $\mathcal{H}_\infty$-norm of (12) with λ an uncertain parameter confined to $[-1,\ 1]$ gives an upper bound for the strong $\mathcal{H}_\infty$-norm of the overall network. By minimizing this upper bound instead of the strong $\mathcal{H}_\infty$-norm of the complete network, the computational cost significantly decreases and this cost now only depends on the dimension of a single subsystem. Furthermore, by minimizing this upper bound one can guarantee a level of disturbance rejection even if the exact number of subsystems is not known or changes. Using the approach outlined in the beginning of this section to minimize the robust strong $\mathcal{H}_\infty$-norm of (12), we find the following controller

$$K = [17.6417\ 14.5064\ 69.1021\ 24.8179] \quad (13)$$

The robust strong $\mathcal{H}_\infty$-norm of the corresponding instance of (12) equals 0.531752.

Now we consider again the original networked system (10). By applying controller (11) with K given by (13), the strong $\mathcal{H}_\infty$-norm of the closed loop is equal to 0.517543 for $N = 3$, to 0.528727 for $N = 10$ and to 0.530313 for $N = 15$. The obtained controller thus guarantees good disturbance suppression over a wide range of N. Finally, Figure 2 shows the input $w_{10}(t)$ and the output $z_{10}(t)$ of the closed loop networked system with $N = 20$ for $t \in [0, 10]$ and each disturbance input signal low pass filtered ($\omega_{cutoff} = 3$ Hz) white Gaussian noise scaled to have 0.1 root mean square energy after filtering. We again observe that the disturbances are well attenuated by the system.

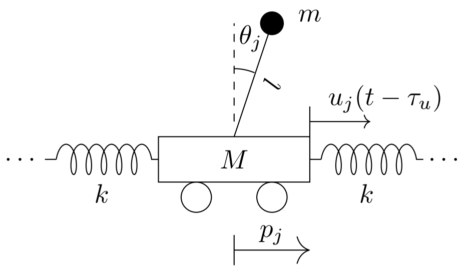

Fig. 1: Schematic representation of a single cart.

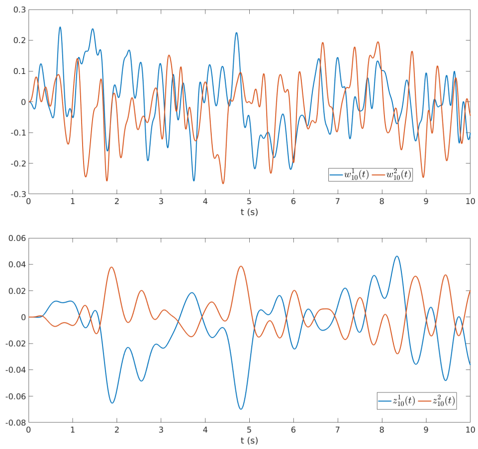

Fig. 2. Input $w_{10}(t)$ and the output $z_{10}(t)$ of the closed loop for $N = 20$ and K given by (13).

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