Minimum phase properties of systems with a new signal reconstruction method ⋆

Minghui Ou, **Shan Liang** *, **Hao Zhang**, **Tong Liu** and **Jing Liang**

College of Automation, Chongqing University, Chongqing 400044, China (e-mail: lightsun@cqu.edu.cn).

Abstract: The Minimum Phase (MP) properties of linear control systems can be reflected by its zero stability. The stability of zeros affects the system control performance. When a continuous-time system is discretized to a discrete-time system, the discretization process may render continuous-time system models have nonminimum phase. This paper analyses the MP properties of system and deduces a new stable condition of the zeros when continuous-time system is discretized by Forward Triangle Sample and Hold (FTSH) for sufficiently small sampling periods. Finally, two numerical examples have verified our results.

Keywords: Minimum Phase; discrete-time system; stability; discretized; forward triangle sample and hold.

1. INTRODUCTION

A linear continuous-time control system with a rational function is non-minimum phase(NMP) if it has at least one zero in the right-half plane. Analogously, a discrete-time system is NMP if there exists zero outside the unit circle. The control of NMP systems is more difficult than that of MP systems. It is essential to check the minimum phase (MP) properties of system before we attempt to design control procedures (Hsu and Lu (2008)). Unfortunately the MP properties of the system can not be always preserved when a continuous-time model is transformed to a discrete-time model by a sample and hold device (Åström et al. (1984)). The MP characteristics of linear system can be reflected by the corresponding zero stability. The location of zeros in transfer function has a significant impact on the behaviour of linear systems, and the MP property plays a key role in systems analysis and control design of discrete-time system (Hoagg and Bernstein (2007); Isidori (2013)). The unstable zeros arising from the sampling process make it difficult to construct some control strategies, such as inverse systems, model matching systems, and model reference adaptive controllers(Ishitobi (1992)). Therefore, extensive researchers have poured efforts to avoid the appearance of unstable zeros in the process of sampling, see Åström et al. (1984); Ishitobi (1992); Weller et al. (2001); Liang et al. (2003); Yucra et al. (2013); Carrasco et al. (2017); Ou et al. (2019).

During the discretization process of continuous-time system, it is well known that the connection between continuous-time poles $p_i, i = 1, 2, \cdots, n.$ are transformed as : $p_i \leftrightarrow e^{p_i T}$, where T is the sampling period (i.e. the system stability can be preserved). However, the relation of zeros are much more complicated, and the simple transcendental relation cannot be preserved. The zeros of linear discrete-time systems are classified into two categories: intrinsic zeros and sampling (or limiting) zeros (Åström et al. (1984); Yuz and Goodwin (2005)). The location of sampling zeros mainly depends on the methods of signal reconstruction, sampling period and relative degree of system (Schrader and Sain (1989)). Åström et al. (1984) were the first to study this problem for the case of system with zero-order hold (ZOH), and they revealed the limiting zeros are unstable if the relative degree of a continuous-time transfer function is greater than or equal to two. Hagiwara et al. (1992, 1993) studied another signal reconstruction method–first order hold (FOH) which provides no improvement over the ZOH.

Moreover, fractional-order hold (FROH), as an alternative signal reconstruction method, has attracted considerable attentions. Extensive studies have found that FROH can provide a more general feasible stability condition than the ZOH, see Passino and Antsaklis (1988); Ishitobi (1996); Bárcena and Etxebarria (2010); Ishitobi and Zhu (1997). This is because FROH can provide an adjustable parameter in the process of obtaining discrete-time model. However, FROH schemes have some limitations, such as it requires the continuous-time system has to be low pass (Passino and Antsaklis (1988); Ishitobi (1996)). Another side, the initiate research about generalized sample hold function (GSHF) can be traced to the works of (Chammas and Leondes (1979)). Extensive studies about the control of linear-invariant systems and the MP properties of discrete-time systems in the case of GSHF, can be seen in Kabamba (1987); Ortega and Kreisselmeier (1988) and Zeng et al. (2018). The non-MP properties of discretized models can be avoided by selecting appropriate values of piecewise constant impulse response of GSHF (Yuz et al. (2004)). However, GSHF leads to the intersample ripples(Ugalde et al. (2012)). Recently, Wang et al. (2016) investigated a new signal reconstruction method,

⋆ This work is partially supported by the National Natural Science Foundation of China (61771077), the Science and Technology Major Special Project of Guangxi (GKAA17129002).

namely forward triangle sample and hold (FTSH). As an alternative to the ZOH and FROH, FTSH enables the discrete-time system has MP characteristics by selecting the proper sampling period and parameters of the FTSH, while the ZOH fails to do so. The method of reconstruction input signal of system for GSHF and FTSH is different. However, Wang et al. (2016) only numerically studied the properties of the FTSH. The theoretical research about how FTSH affects the MP properties of linear system is still under studied.

The purpose of this paper is therefore to consummate the theoretical framework of the FTSH, and to investigate the MP properties of discrete-time system with FTSH. In this paper, we address both issues, as a result, deduce the corresponding exact discrete-time models of linear continuous-time system. Secondly, we provide the expression of the sampling zeros and analyze the asymptotic behavior of them. Furthermore, we derive the stability conditions of the sampling zeros for sufficiently small T. Finally, two examples are provided to verify the results of this paper.

Notations: Throughout this paper, N and R^+ denote the set of natural and non-negative real numbers, respectively. R^n and $R^{n\times n}$ denote, respectively, $n-$dimensional real valued vectors and $n \times n$ real valued matrices. A zero or a pole of continuous-time systems $G(s)$ is said to be stable (respectively unstable) if it lies inside the open left (right) half-plane. Similarly, a zero or pole of discrete-time systems $G(z)$ is said to be stable (respectively unstable) if it lies inside the open unite disc (outside the closed unite disc).

2. PRELIMINARIES

Consider a single input single output (SISO) n-th order linear continuous-time system, which is time-invariant, controllable, observable.

$$S_C : \begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = cx(t) \end{cases}, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^{n\times 1}$ and $c \in R^{1\times n}$ are n-th order state matrix and column and row vectors, respectively. Obviously, the system (1) with an input vector $u(t) \in R$, an output vector $y(t) \in R$ and a state vector $x(t) \in R^{n\times 1}$. On the other hand, the relation between transfer function $G(s)$ and the state space of the linear system can be expressed as follows.

$$G(s) = c(sI_n - A)^{-1}b. \tag{2}$$

Under the Assumption of system S_C is invertible, the definitions of system zeros, invariant zeros and transmission zeros for continuous-time system are coincide. The zeros of S_C can be computed from the roots of the numerator polynomial in (2).

Lemma 1. System S_C relative degree is v, if and only if

$$cA^{l-1}b = \begin{cases} 0; & l < v \\ b_m \neq 0; & l = v \end{cases}. \tag{3}$$

Remark 1. The proof details of Lemma 1 is omitted (see Yuz and Goodwin (2014)). The parameter b_m is the coefficient of term s^m in the numerator polynomial of (2).

We are here interested in the MP properties of systems with the FTSH signal reconstruction method. In other word, it is an interesting work to reveal the relations of the zeros of discrete-time system to those of continuous-time system in this situation.

The mentioned discrete-time system in this paper contains hold circuit, the continuous-time system and sampling device, where the signal reconstruction method forward triangle sample and hold (FTSH) (Wang et al. (2016)) was considered to yield the system input signal as shown in Fig. 1 and the corresponding expression is

$$u_F(t) = \begin{cases} \frac{u_Z(t)(kT-t)}{fT} + u_Z(t), t \in [kT, kT + fT), \\ 0, \quad t \in [kT + fT, kT + T), \end{cases} \tag{4}$$

where $k \in N$, $f \in (0, 1]$ is the adjustable parameter of the switched input, and $u_Z(t)$ represents the ZOH values over interval $[kT, kT + T)$. Thus, for each sampling interval, the following notation of $u_Z(t)$ is equivalent.

$$u_Z(t) = u_Z(kT) = u(kT); \ \ kT \le t < kT + T. \tag{5}$$

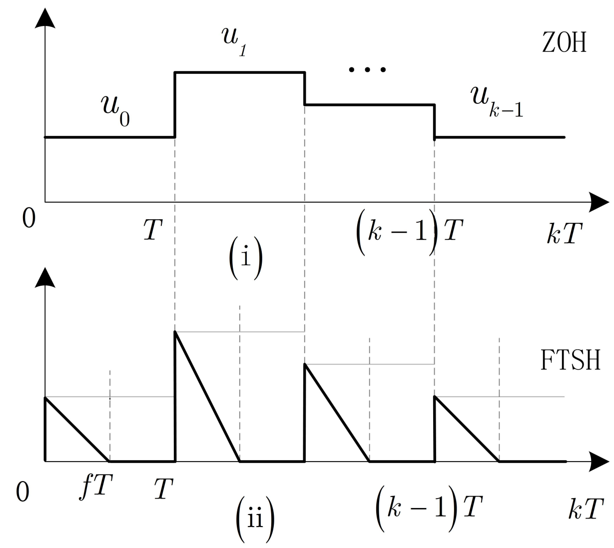

Fig. 1. The signal reconstruction (i) ZOH (ii) FTSH

If we use the FTSH as signal reconstruction to generate the input $u(t)$, then, the corresponding discrete-time system of original continuous-time system (1) or (2) is given by:

$$S_D : \begin{cases} x((k+1)T) = \Phi x(kT) + \Psi u(kT) \\ y(kT) = cx(kT) \end{cases}, \tag{6}$$

where

$$\Phi = e^{AT}, \quad \Psi = \int_0^{fT} e^{A(T-\tau)} \left(1 - \frac{\tau}{fT}\right) bd_t. \tag{7}$$

Thus, the corresponding transfer function $G_F(z)$ can be obtained from the above state space form.

$$\begin{aligned} G_F(z) &= c(zI - \Phi)^{-1}\Psi \\ &= \frac{B_F(z)}{A_F(z)}, \end{aligned} \tag{8}$$

where $A_F(z)$ and $B_F(z)$ represent the denominator and numerator polynomials, respectively, and each of them can be computed using the following determinant.

$$A_F(z) = \det(zI - \Phi), \tag{9}$$

$$B_F(z) = \det \begin{bmatrix} zI - \Phi & -\Psi \\ c & 0 \end{bmatrix}. \tag{10}$$

Here, the roots of B_F represent the zeros (including intrinsic zeros and limiting zeros) of sampled-data system $G_F(z)$, while the roots of A_F denote the poles of sampled-data system $G_F(z)$.

Before we show the properties of limiting zeros of system (8) with the sampling period tends to zero. A new polynomial will be first introduced here, i.e. the authors are not find any description about the polynomial defined as follows.

Definition 1. When the input signals of continuous-time system are reconstructed by FTSH, a new polynomial is defined by

$$B_{F,v}(z,f) = v! \cdot \det Z_{F,v}, \tag{11}$$

where $f \in (0,1]$, $v \in N \geq 0$, $p = 1-f$ and

$$Z_{F,v} = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(v-1)!} & \frac{-1+(v+1)f+p^{v+1}}{(v+1)!f} \\ 1-z & 1 & \cdots & \frac{1}{(v-2)!} & \frac{-1+vf+p^v}{v!f} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1-z & 1 & \frac{-1+3f+p^3}{3!f} \\ 0 & \cdots & 0 & 1-z & \frac{-1+2f+p^2}{2!f} \end{bmatrix}.$$

Remark 2. The new polynomial will play an important role in the asymptotic properties of limiting zeros. This polynomial is very similar to other famous polynomials, such as the standard and modified Euler-Fröbenius polynomials– see Weller et al. (2001); Foata (2010); Carrasco et al. (2017). Moreover, more details of the matrix in Definition 1 can refer to the proof process of Theorem 2.

3. MAIN RESULTS

In this section, firstly, the exact discrete-time model and the asymptotic properties of limiting zeros of the continuous-time system with FTSH for sufficiently small sampling periods are researched. Secondly, according to the results obtained, the stability condition of the limiting zeros is derived. The corresponding results are described in the following theorems.

Theorem 2. Let a FTSH as signal reconstruction to generate the input value to a linear continuous-time system. Then, the discrete-time model is obtained as follows.

Case (a) Consider a r-th order integrator with continuous-time transfer function $G(s) = 1/s^r$. Then the exact discrete-time system satisfies the following discrete time state space equation:

$$x_{k+1} = A_d x_k + B_{FTSH} u_k, \tag{12}$$

where

$$x_k = [\, x_{1,k} \; x_{2,k} \; \cdots \; x_{r,k} \,]^{\mathrm{T}},$$

$$A_d = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{T^{r-2}}{(r-2)!} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$B_{FTSH} = \begin{bmatrix} \frac{(-1+(r+1)f+p^{r+1})T^{r+1}}{(r+1)!fT} \\ \vdots \\ \frac{(-1+3f+p^3)T^3}{3!fT} \\ \frac{(-1+2f+p^2)T^2}{2!fT} \end{bmatrix},$$

with the output $y_k = x_{1,k}$. The transfer function $G_F(z)$ of the exact discrete-time system is given by

$$G_F(z) = \frac{T^r \cdot B_{F,r}(z,f)}{r! \cdot (z-1)^r}, \tag{13}$$

where $B_{F,r}(z,f)$ is defined in (11).

Case (b) When the system is a strictly proper n-th order general linear continuous-time system with transfer function $G(s)$. Then, the corresponding discrete-time model has the following limit expression

$$\frac{K \cdot (z-1)^m B_{F,r}(z,f)}{(n-m)! \cdot (z-1)^n} \tag{14}$$

as $T \to 0$, where $r = n - m$ is the relative degree.

Proof. Case (a) of Theorem 2 will be proved in Appendix A.

Remark 3. The Proof process case (b) of Theorem 2 is omitted. If one has interest in the proof process, combining the details in (Åström et al. (1984); Carrasco et al. (2017); Ou et al. (2019)) can obtain the results in Theorem 2.

Remark 4. As the results shown in Theorem 2, the numerator part of the limiting expression contains the new polynomial (11). This implies the new polynomial can reflect the properties of the limiting zeros of the discrete-time system to some extent.

Remark 5. Notice that for rational linear system (2) with FTSH as the input signal reconstruction method, the poles of discrete-time system S_D converge to 1 as $e^{p_i T}$ with sampling period T converge to zero, m zeros close to 1 as the same transcendental relation and other remaining zeros converge to the roots of $B_{F,r}(z,f) = 0$, where the sampling zeros are influenced by the relative degree of original continuous-time system and the parameter f of FTSH.

From Theorem 2, the following result is immediately obtain. Thereafter, because of the relative degree of many linear or nonlinear mechanical systems in practical field is two (Ishitobi (2000); Zeng et al. (2014)), we mainly analysis the stability condition, sufficiently small sampling period, of limiting zeros for continuous-time system with relative degree two.

Theorem 3. Suppose that the original continuous-time system $G(s)$ has no zeros on the imaginary axis. Its corresponding discrete-time system is obtained by using FTSH as the signal reconstruction. Therefore, when the relative degree $r = 2$, if the original system $G(s)$ is MP and the adjustable variable parameter f satisfies $0 < f \leq 1$, then the discrete-time system is MP.

Proof. From Theorem 2, when a FTSH was used to generate the input signal of a general linear system, the corresponding discrete-time model will be obtained and its limiting zeros will go to the roots of polynomial $B_{F,r}(z,f) = 0$ as the sampling period $T \to 0$. When the original continuous-time system relative degree $r = 2$, from the limit expression of the sampled-data system, let numerator polynomial $B_{F,2}(z, f) = 0$ can obtain the limiting zeros. Through simple computing the equation can obtain the stability fulfill the condition that variable parameter of FTSH is $0 < f \leq 1$.

4. NUMERICAL SIMULATION

In this section, two numerical examples will be given to verify the results of this paper. Firstly, a relative degree two continuous-time system is considered and its' transfer function is shown as follows

$$G(s) = \frac{s+7}{(s+1)(s+2)(s+3)}. \tag{15}$$

Obviously, system is stable and minimum phase. From the results of Åström et al. (1984), the corresponding discrete-time system is NMP. However, based on the results of Theorem 2 and 3, one can select a proper duty cycle value of FTSH to stabilize the limiting zeros. As shown in the system transfer function (15), the original continuous-time system has $m = 1$ zero and $n = 3$ poles. Thus, the corresponding discrete-time system has one intrinsic zero and one sampling zero, where the intrinsic zero converge to 1 as e^{-7T} and the sampling zero goes to the root of $B_{F,2}(z, f) = 0$ with the sampling period goes to zero. For example, if we select the parameter $f = 0.5$, the sampling zero will equal to -0.2 , which is coincide with the simulation result in Fig. 2.

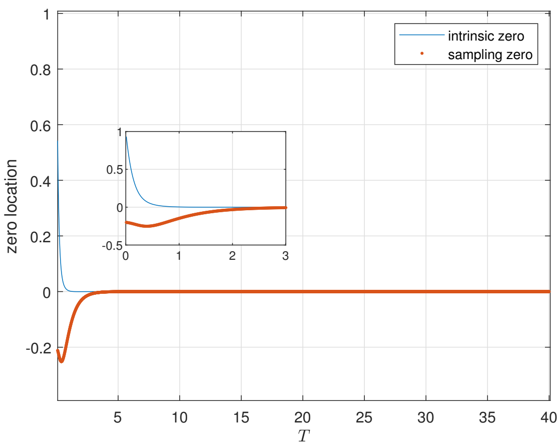

Fig. 2. The location of limiting zeros for system (15) with respect to sampling period T and $f = 0.5$ of FTSH.

Through the numerical analyzes about discrete-time zeros of system (15), the asymptotic behavior of the intrinsic zero and limiting zero are stable, and they approach the unite circle form inside of the circle as T converges to zero. The MP properties of the system are preserved with the FTSH as the signal reconstruction method.

Furthermore, we consider the inverted-pendulum system in Fig. 3 as the second example. Assume that the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. Note that angle θ indicates the rotation of the pendulum rod about point p, and x is the location of the cart. Then, the transfer function between input force u and rotation angle θ obtained as follows (more details–see Ogata (2009))

$$G(s) = \frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M+m)g}. \tag{16}$$

We select the parameter value of system (16) follows the value in Table 1 and obtain the transfer function as follow

$$G(s) = \frac{1}{3(s-4)(s+4)}. \tag{17}$$

Table 1. Parameter value of the inverted-pendulum

Variable	Value	Units SI
M	3	kg
m	1.8	kg
l	1	m
g	10	N/kg

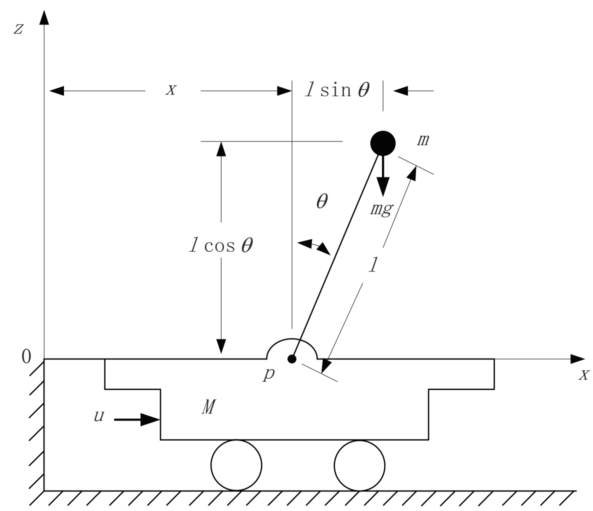

Fig. 3. Inverted-Pendulum system

Note that the relative degree of the inverted-pendulum system is two. Here we need to analyze the properties of limiting zero for inverted-pendulum system with FTSH as the discrete signal reconstruction method. From the results in Theorem 2, the limiting zero should be stable and it should be converge to a fixed value for each f as the sampling period tends to zero. Based on the result in (13), the limiting zero is equal -0.3953488 if select the FTSH parameter $f = 0.85$. Another side, the asymptotic behavior of the limiting zeros of discrete-time inverted-pendulum system with respect to the sampling period in the case of FTSH with $f = 0.85$ is showed in Fig. 4.

Remark 6. When the inverted-pendulum system is discretized by the FTSH, the discrete-time system is MP. However, the inverted-pendulum system (16) always has an unstable pole $s = \left(\sqrt{M+m}\big/\sqrt{Ml}\right)\sqrt{g}$, and Theorem 2 does not state the original continuous-time system whether has unstable part. The result in this paper reveals the relationship of limiting zeros and the relative degree of stable system. We shall investigate the properties of limiting zeros for the system with unstable part in the near future.

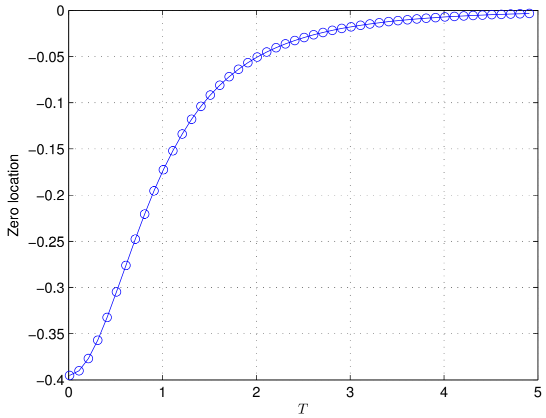

Fig. 4. The location of limiting zeros for inverted-pendulum system with respect to sampling period T and $f = 0.85$ of FTSH.

5. CONCLUSION

This paper analyzes the MP properties of the system with the FTSH. We have investigated the asymptotic behavior of the limiting zeros for a discrete-time system with FTSH and given the stability condition of the zeros for the discrete-time system with sufficiently small sampling periods. Results find that MP properties of the system can be preserved when the continuous-time system is discretized by the FTSH and fulfill the condition of this paper. In simulation, our result is verified in a inverted-pendulum system. Future work will focus on the problem stated in *Remark 6*.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge reviews and editors for their helpful comments and suggestions, which have greatly improved the quality of this paper.

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Appendix A. PROOF OF CASE (A) IN THEOREM 2

Consider the mentioned system $G(s) = 1/s^r$ with output variable $y(t) = x_1(t)$. Then, the time domain state space of this original system is

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A.1}$$

where

$$x(t) = [\, x_1(t)\ x_2(t)\ \cdots\ x_r(t)\,]^{\mathrm{T}}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{r\times r}$$

$$B = [\,0\ 0\ \cdots\ 0\ 1\,]_{1\times r}^{\mathrm{T}}$$

The state variables of (A.1) at time $kT + T$ is given in the following:

$$\begin{aligned} x(kT + fT) &= e^{AfT}x_k + \int_0^{fT} e^{A(fT-\tau)}Bu_F(kT+\tau)d\tau \\ x(kT + T) &= e^{A(T-fT)}x(kT + fT) \\ &= e^{AT}x_k + \int_0^{fT} e^{A(T-\tau)}Bu_F(kT+\tau)d\tau \end{aligned} \tag{A.2}$$

Based on the definition of FTSH, which is used to generate the input value of the continuous-time system, we can get the discrete time state space matrices of the exact sampled-data model.

$$A_d = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{T^{r-2}}{(r-2)!} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$B_{FTSH} = \begin{bmatrix} \frac{(-1+(r+1)f+p^{r+1})T^{r+1}}{(r+1)!fT} \\ \vdots \\ \frac{(-1+3f+p^3)T^3}{3!fT} \\ \frac{(-1+2f+p^2)T^2}{2!fT} \end{bmatrix}$$

Therefore, the discrete-time system with FTSH can be rewritten using the forward shift operator (q-operator) as

$$\begin{aligned} &[\,(q-1)x_{1,k}\ 0\ \cdots\ 0\ 0\,]^{\mathrm{T}} \\ =&D_{F,r}[\,x_{2,k}\ x_{3,k}\ \cdots\ x_{r,k}\ u_k\,]^{\mathrm{T}} \end{aligned} \tag{A.3}$$

where

$$D_{F,r} = \left[\begin{array}{cccc|c} T & \frac{T^2}{2} & \cdots & \frac{T^{r-1}}{(r-1)!} & \\ -(q-1) & T & \cdots & \frac{T^{r-2}}{(r-2)!} & \\ \vdots & \vdots & \ddots & \vdots & B_{FTSH} \\ 0 & 0 & \ddots & T & \\ 0 & 0 & \cdots & -(q-1) & \end{array}\right]$$

Then, using the Cramers rule to solve (A.3) and obtaining the result of u_k as

$$u_k = \frac{\det N}{\det D_{F,r}}$$

where

$$N = \begin{bmatrix} T & \frac{T^2}{2} & \cdots & \frac{T^{r-1}}{(r-1)!} & (q-1)x_{1,k} \\ -(q-1) & T & \cdots & \frac{T^{r-2}}{(r-2)!} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T & 0 \\ 0 & 0 & \cdots & -(q-1) & 0 \end{bmatrix} \tag{A.4}$$

Now, let us compute the determinant of N along with the last column, we finally obtain

$$\begin{aligned} \det N &= (-1)^{1+r}(q-1)x_{1,k}\cdot(-(q-1))^{r-1} \\ &= (q-1)^r x_{1,k} \end{aligned} \tag{A.5}$$

Another side, based on the results in Definition 1, $\det D_{F,r} = T^r \det Z_{F,r}$, further

$$\det D_{F,r} = \frac{T^r}{r!}\cdot r!\det Z_{F,r} = \frac{T^r}{r!}\cdot B_{F,r}(q,f)$$

From the relation between input and output of system. Note the truth that $G_{F,r}(z) = y_k/u_k$, thus, the result (13) can be obtained.

As a result, the proof is complete.

Q.E.D