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Regional Optimal Control on the Velocity Term of the Bilinear Plate Equation

Abderrahman Ait Aadi * **El Hassan Zerrik** **

* *MACS Laboratory, Department of Mathematics, Moulay Ismail University, Meknes, Morocco (e-mail: abderrahman.aitaadi@gmail.com).*
** *MACS Laboratory, Department of Mathematics, Moulay Ismail University, Meknes, Morocco (e-mail: zerrik3@yahoo.fr)*

Abstract: The aim of this paper is to study regional optimal control problem for a bilinear plate equation evolving in a spatial domain $\Omega \subset \mathbb{R}^2$. The control is bounded and acts on the velocity term. The question is to obtain a feedback control that drives such a system from an initial state to a desired one in finite time, only on a subregion ω of Ω, and minimises a quadratic functional cost. Our purpose is to prove that an optimal control exists, and characterised as solution of an optimality system. The approach is successfully illustrated by simulations.

Keywords: Bilinear systems; plate equation; feedback controls; optimal control.

1. INTRODUCTION

The controllability of distributed systems was studied in many works that led to various results. In Lions (1988), author proved exact controllability of vibrating plate model with boundary control. In Zuazua (1988), author considered exact controllability of vibrating plate equation for arbitrarily small time. In Ball et al. (1982), authors studied approximate controllability of rod and wave equations. The multiplicative controllability of parabolic and hyperbolic semilinear equations was investigated by Khapalov (2010).
The problem of optimal control for a class of distributed bilinear systems have been developed in many works : in Lenhart (1995), author proved the existence and gave characterization of an optimal control of a convective-diffusive fluid problem. In Addou and Benbrik (2002), the authors studied unbounded optimal control for a bilinear system governed by a fourth-order parabolic operator. Also, in Bradley and Lenhart (1994), the authors considered the optimal control of Kirchhoff plate equation by controls acting on the position of state.
The concept of regional controllability for a distributed linear system evolving on a spacial domain Ω concerns the study of the classical notion of controllability only on a subregion ω of Ω El Jai et al. (1995). The main reasons for considering this notion is that it is close to real applications. For example, in the problem of a tunnel furnace when one has to maintain a prescribed temperature only in a subregion of the furnace. Also, it becomes possible to control a system on a subregion of its evolution domain acting out of the subregion. Besides there exist systems which are not controllable on the whole domain Ω but controllable on some subregion. Moreover, controlling a system on a subregion is cheaper than controlling it globally El Jai et al. (1995). Regional optimal control of parabolic distributed bilinear systems with unbounded and bounded controls involving the minimization of the final state error and the energy, was considered by Zerrik and Ould Sidi (2011); Ztot et al. (2011), they established the existence and gave characterization of an optimal control.
In Ait Aadi and Zerrik (2019), authors studied regional optimal control of a bilinear plate equation by controls without constraints and acts on the position of such system.
In this work, we examine regional optimal control of a bilinear plate equation by controls with constraints and acts on the velocity term of such equation. Then, we prove the existence and we give characterization of an optimal control. Moreover, we develop a numerical approach that leads to an algorithm that we illustrate by simulations.
More precisely, we consider the following bilinear plate equation

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) = u(t) y_t(x,t), & Q = \Omega \times (0,T) \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & \Omega \\ y(x,t) = \dfrac{\partial y}{\partial \nu}(x,t) = 0, & \Sigma = \partial\Omega \times (0,T) \end{cases} \quad (1)$$

where Ω be an open bounded of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$, $Ay = \Delta^2 y$ is the unbounded bilaplacian operator with domain $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$, ν is the unit outer normal to $\partial\Omega$, $u \in \mathcal{U}_\rho := \{u \in L^\infty(0,T) : -\rho \le u \le \rho\}$ is a control function where ρ is a positive constant. Let $\mathcal{H} := H_0^2(\Omega) \times L^2(\Omega)$ be the state space and let us consider a non-empty subset $\omega \subset \Omega$, with a positive Lebesgue measure and a desired state $y_d \in L^2(\omega)$. We define $\chi_\omega : L^2(\Omega) \longrightarrow L^2(\omega)$ the restriction operator to ω, and χ_ω^* is the adjoint operator of χ_ω given by

$$(\chi_\omega^* y)(x) = \begin{cases} y(x) & \text{if } x \in \omega \\ 0 & \text{else } x \in \Omega \backslash \omega. \end{cases}$$

Our problem is expressed by

$$\min_{u \in \mathcal{U}_\rho} J(u), \quad (2)$$

with

$$J(u) = \frac{1}{2}\int_\omega \int_0^T \big(\chi_\omega y(x,t) - y_d(x)\big)^2 dt dx + \frac{\beta}{2}\int_0^T u^2(t)dt, \tag{3}$$

where β is a positive constant.
This paper is organized as follows : in section 2, we prove the existence of an optimal control solution of problem (2). In section 3, we give characterization of an optimal control solution of (2). In section 4, we give a numerical approach that leads to an algorithm we illustrate by numerical simulations.

2. EXISTENCE OF AN OPTIMAL CONTROL

This section is devoted to the existence of an optimal control solution of problem (2).
First, we present an apriori estimate needed for the existence of an optimal control.
Lemma 1. Given $\tilde{y}_0 = (y_0, y_1) \in \mathcal{H}$ and $u \in \mathcal{U}_\rho$, then

(1) the system (1) has a unique weak solution $\tilde{y} = (y, y_t) \in \mathcal{C}([0,T], \mathcal{H})$
(2) the weak solution satisfies the estimate

$$\|\tilde{y}\|_{\mathcal{C}([0,T],\ \mathcal{H})} \le M\big(1+\rho T\big)^{\frac{1}{2}} e^{\rho C T}, \tag{4}$$

where $M = \|\tilde{y}_0\|_{\mathcal{H}}$ and C is a positive constant.

Proof. (1) The state equation (1) can be written as

$$\begin{cases} \dfrac{d}{dt}\tilde{y}(t) = \mathbb{A}\tilde{y}(t) + \mathbb{B}\tilde{y}(t) \\ \tilde{y}(0) = (y_0, y_1), \end{cases} \tag{5}$$

where $\mathbb{A} : H^4(\Omega) \times H_0^2(\Omega) \longrightarrow \mathcal{H}$

$$\mathbb{A}\tilde{y}(t) = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix} \tilde{y}(t)$$

with domain $\mathcal{D}(\mathbb{A}) = (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$ and the operator $\mathbb{B}$ is given by

$$\mathbb{B}\tilde{y}(t) = \begin{pmatrix} 0 \\ u(t)y_t(t) \end{pmatrix}.$$

The domain $\mathcal{D}(\mathbb{A})$ is dense in $\mathcal{H}$ and the operator $\mathbb{A}$ is skew-adjoint, then $\mathbb{A}$ generates a strongly continuous unitary group on $\mathcal{H}$, and $\mathbb{B}$ is bounded operator on $\mathcal{H}$. Then system (5) has a unique weak solution $\tilde{y}(t) \in \mathcal{C}([0,T], \mathcal{H})$ (see Pazy (1983)).
(2) Since $\mathcal{D}(\mathbb{A})$ is dense in $\mathcal{H}$, there exist sequences (y_0^n, y_1^n) in $\mathcal{D}(\mathbb{A})$ and $u^n \in \mathcal{U}_\rho \cap \mathcal{C}^2(0,T)$ such that

$$(y_0^n, y_1^n) \longrightarrow (y_0, y_1) \ \text{ strongly in } \ \mathcal{H},$$
$$u^n \longrightarrow u \ \text{ strongly in } \ L^2(0,T).$$

Denote by $\tilde{y}^n(t)$ the weak solution of system (1) corresponding to the initial data (y_0^n, y_1^n) with control u^n.
Multiplying the equation (1) by $y_t(t)$ and integrating over $\Omega \times (0,\tau)$, we obtient

$$\begin{aligned} 0 &= \int_0^\tau \int_\Omega \big(y_{tt}^n\, y_t^n + \Delta^2 y^n\, y_t^n - u^n (y_t^n)^2\big) dx dt \\ &= \int_0^\tau \int_\Omega \big(\frac{1}{2}\frac{d}{dt}\big[\big(y_t^n\big)^2 + (\Delta y^n)^2\big] - u^n (y_t^n)^2\big) dx dt \\ &= \int_0^\tau \int_\Omega \frac{1}{2}\frac{d}{dt}\big(y_t^n\big)^2 dx dt + \frac{1}{2}\int_0^\tau \frac{d}{dt} q(y^n, y^n) dt \\ &- \int_0^\tau \int_\Omega u^n (y_t^n)^2 dx dt, \end{aligned}$$

where $q(v,w) = \displaystyle\int_\Omega \Delta v \Delta w dx$, for all $v, w \in H_0^2(\Omega)$.
Thus, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big(y_t^n\big)^2(x,\tau)dx + q(y^n, y^n)(\tau) \\ &= \frac{1}{2}\|y_1^n\|^2_{L^2(\Omega)} + \frac{1}{2} q(y_0^n, y_0^n) + \int_0^\tau \int_\Omega u^n (y_t^n)^2 dx dt \\ &\le \frac{1}{2}\|\tilde{y}(0)^n\|^2_{\mathcal{H}} + \rho \int_0^\tau \|\tilde{y}(t)^n\|^2_{\mathcal{H}} dt. \end{aligned}$$

Gronwall's inequality gives

$$\sup_{0\le t\le T}\big(\int_\Omega \big(y_t^n\big)^2(x,\tau)dx + q(y^n, y^n)(\tau)\big) \le \|\tilde{y}(0)^n\|^2_{\mathcal{H}}\big(1 + 2\rho T\big)e^{\rho C T}. \tag{6}$$

We pass to the limit and obtain (4) for $\tilde{y}(t)$.

Now, we obtain the existence of an optimal control.
Theorem 1. There exists an optimal control $u^* \in \mathcal{U}_\rho$, solution of problem (2).

Proof. Let u^n be a minimizing sequence in $\mathcal{U}_\rho$, such that

$$\lim_{n\to+\infty} J(u^n) = \inf_{u\in\mathcal{U}_\rho} J(u). \tag{7}$$

By Lemma 1, we have the estimate

$$\|y^n\|^2_{H_0^2(\Omega)} + \|y_t^n\|^2_{L^2(\Omega)} \le M e^{\rho C T}, \tag{8}$$

where M is a positive constant.
From system (1) and (8), we conclude that

$$\|y_{tt}^n\|^2_{H^{-2}(\Omega)} \le M' e^{\rho C T}, \tag{9}$$

where M' is a positive constant.
Using (8) and (9), we deduce the following convergence properties

$$y^n \rightharpoonup y^* \quad \text{weakly}^* \text{ in } L^\infty([0,T], H_0^2(\Omega)) \tag{10}$$
$$y_t^n \rightarrow y_t^* \quad \text{weakly}^* \text{ in } L^\infty([0,T], L^2(\Omega)) \tag{11}$$
$$y_{tt}^n \rightharpoonup y_{tt}^* \quad \text{weakly}^* \text{ in } L^\infty([0,T], H^{-2}(\Omega))$$
$$u^n \rightharpoonup u^* \quad \text{weakly in } L^2(0,T). \tag{12}$$

Since $\mathcal{U}_\rho$ is a closed and convex subset of $L^\infty(0,T) \subset L^2(0,T)$, $\mathcal{U}_\rho$ is weakly closed in $L^2(0,T)$.
Then $u^* \in \mathcal{U}_\rho \subset L^2(0,T)$. On the other hand, since $-\rho \le u^n(t) \le \rho$ for all n, $u^n \rightharpoonup u^{**}$ weakly* in $L^\infty(0,T)$, and hence $u^n \rightharpoonup u^{**}$ weakly in $L^2(0,T)$.
By the uniqueness of the weak limit, we obtain $u^* = u^{**}$ and $u^* \in \mathcal{U}_\rho \subset L^\infty(0,T)$.
In other hand, we have y^n satisfies the weak form

$$\int_0^T \langle y_{tt}^n, \phi\rangle dt + \int_0^T q(y^n, \phi) dt = \int_Q u^n y_t^n \phi dx dt, \quad \forall \phi \in H_0^2(\Omega). \tag{13}$$

We define the sequence of function $v^n(t)$ by

$$v^n(t) = \int_\Omega y_t^n(x,t)\phi(x,t)dx.$$

So that $\int_Q u^n y_t^n \phi dxdt$ becomes

$$\int_0^T u^n(t)v^n(t)dt.$$

By the estimate of Lemma (1), v^n is uniformly bounded independent of n. Using the continuity of y_t^n in time into $L^2(\Omega)$, for each fixed t,

$v^n(t) \to v(t) = \int_\Omega y_t^*(x,t)\phi(x,t)dx,$ pointwise as $n \to +\infty$, using the weak convergences above. By Egorof's Theorem, for any $\varepsilon > 0$, there exists a set $L \subset [0,T]$ such that $\mu(L) < \varepsilon$ and

$$v^n(t) \to v(t), \text{ uniformly on } [0,T]\backslash L.$$

Then

$$\int_0^T |u^n v^n - u^* v|dt \le \int_0^T |(u^n v^n - u^* v)1_L|dt + \int_0^T |(u^n v^n - u^* v)1_{[0,T]\backslash L}|dt.$$

The integral term on $[0,T]\backslash L$ approaches 0 as $n \to +\infty$ by the uniform convergence of $v^n \to v$ on $[0,T]\backslash L$. The integral term on L can be estimated

$$\begin{aligned}\int_0^T |(u^n v^n - u^* v)1_L|dt &\le \rho \int_0^T (|v^n| - |v|)1_L dt \\ &\le \rho(\|v^n\|_{L^2(0,T)} + \|v\|_{L^2(0,T)})\mu(L) \\ &\le C\mu(L),\end{aligned}$$

where C does not depend on n and $\mu(L) < \varepsilon$.
Hence

$$\lim_{n\to+\infty} \int_Q u^n y_t^n \phi dxdtdt = \int_Q u^* y_t^* \phi dxdt.$$

Taking the limit as $n \to +\infty$ in (13), we conclude

$$\int_0^T \langle y_{tt}^*(t), \phi(t)\rangle dt + \int_0^T q(y^*(t), \phi(t))dt = \int_0^T \langle u^* y_t^*(t), \phi(t)\rangle dt, \text{for all } \phi \in H_0^2(\Omega).$$

Thus $y^* = y(u^*)$ is the solution of state equation (1) with control u^*. Since

$$J(u^*) = \frac{1}{2}\int_\omega \int_0^T \left(\chi_\omega y^*(x,t) - y_d(x)\right)^2 dtdx + \frac{\beta}{2}\int_0^T (u^*)^2(t)dt,$$

using lower-semicontinuity of L^2 norm with respect to weak convergence, we have

$$\begin{aligned}J(u^*) &\le \frac{1}{2}\lim_{n\to+\infty}\int_\omega \int_0^T \left(\chi_\omega y^n(x,t) - y_d(x)\right)^2 dtdx \\ &+ \frac{\beta}{2}\liminf_{n\to+\infty}\int_0^T (u^n)^2(t)dt \\ &\le \liminf_{n\to+\infty} J(u^n) \\ &= \inf_{u\in\mathcal{U}_\rho} J(u).\end{aligned}$$

Finally, we conclude that u^* is an optimal control.

3. CHARACTERIZATION OF AN OPTIMAL CONTROL

In this section, we give characterization of an optimal control solution of problem (2).
Let now examine the differentiability of the mapping $u \to \tilde{y}(u)$.

Lemma 2. The mapping $u \in \mathcal{U}_\rho \to \tilde{y}(u) \in \mathcal{C}([0,T],\mathcal{H})$ is differentiable in the following sense $\dfrac{\tilde{y}(u+\varepsilon h) - \tilde{y}(u)}{\varepsilon} \rightharpoonup \tilde{\lambda}$ weakly in $L^\infty([0,T],\mathcal{H})$ as $\varepsilon \to 0$, for any u, $u + \varepsilon h \in \mathcal{U}_\rho$. Moreover, $\tilde{\lambda} = (\lambda, \lambda_t)$ is a weak solution of the following system

$$\begin{cases}\lambda_{tt}(x,t) + \Delta^2\lambda(x,t) = u(t)\lambda_t(x,t) + h(t)y_t(x,t) \\ \lambda(x,0) = \lambda_t(x,0) = 0 \\ \lambda(x,t) = \dfrac{\partial\lambda}{\partial\nu}(x,t) = 0\end{cases} \tag{14}$$

Proof. Denote $\tilde{y}^\varepsilon = \tilde{y}(u+\varepsilon h) = (y^\varepsilon, y_t^\varepsilon)$ and $\tilde{y} = \tilde{y}(u)$. Then $\left(\frac{\tilde{y}^\varepsilon - \tilde{y}}{\varepsilon}\right)$ is a weak solution of

$$\begin{cases}\left(\dfrac{y^\varepsilon - y}{\varepsilon}\right)_{tt} + \Delta^2\left(\dfrac{y^\varepsilon - y}{\varepsilon}\right) = u\left(\dfrac{y^\varepsilon - y}{\varepsilon}\right)_t + hy_t^\varepsilon \\ \left(\dfrac{y^\varepsilon - y}{\varepsilon}\right)(x,0) = \left(\dfrac{y^\varepsilon - y}{\varepsilon}\right)_t(x,0) = 0 \\ \left(\dfrac{y^\varepsilon - y}{\varepsilon}\right) = \dfrac{\partial}{\partial\nu}\left(\dfrac{y^\varepsilon - y}{\varepsilon}\right) = 0\end{cases}$$

Using Lemma 1 with source term hy_t^ε, we obtain

$$\|\frac{\tilde{y}^\varepsilon - \tilde{y}}{\varepsilon}\|_{\mathcal{C}([0,T],\mathcal{H})} \le \|hy_t^\varepsilon\|_{L^2(Q)}e^{\rho CT}.$$

y_t^ε satisfies the estimate

$$\begin{aligned}\|hy_t^\varepsilon\|_{L^2(Q)} &\le T\|h\|_\infty\|\tilde{y}^\varepsilon\|_{\mathcal{C}([0,T],\mathcal{H})} \\ &\le (1+\rho T)^{1/2}e^{\rho CT}\|\tilde{y}(0)\|_{\mathcal{H}}.\end{aligned}$$

Hence, we have

$$\frac{\tilde{y}^\varepsilon - \tilde{y}}{\varepsilon} \rightharpoonup \tilde{\lambda} \text{ weakly in } L^\infty([0,T],\mathcal{H}) \ \ as \ \ \varepsilon \to 0.$$

We conclude that λ is a weak solution of system (14).

Now, we give characterization of an optimal control.

Theorem 2. An optimal control solution of problem (2) is given by the formula

$$u^*(t) = \max(-\rho, \min(-\frac{1}{\beta}\int_\Omega \chi_\omega^*\chi_\omega y_t^*(x,t)p(x,t)dx, \rho)), \tag{15}$$

where $(p, p_t) \in \mathcal{C}([0,T],\mathcal{H})$ is the weak solution of the adjoint system

$$\begin{cases}p_{tt}(x,t) + \Delta^2 p(x,t) = u^*(t)p_t(x,t) + y^*(x,t) - \chi_\omega^* y_d(x) \\ p(x,T) = p_t(x,T) = 0 \\ p(x,t) = \dfrac{\partial p}{\partial\nu}(x,t) = 0\end{cases} \tag{16}$$

Proof. The proof of existence of the solution to the adjoint system is similar to the proof of existence of

solution of the state equation (1) since the source term $(y^* - \chi_\omega^* y_d) \in \mathcal{C}([0,T], L^2(\Omega))$.
We now proceed to characterize the optimal control in terms of the state (y, y_t) of system (1) and the one (p, p_t) of the adjoint system (16). Let $u^* \in \mathcal{U}_\rho$ be an optimal control and $\tilde{y} = \tilde{y}(u^*)$ be the corresponding optimal solution. Let $u^* + \varepsilon h \in \mathcal{U}_\rho$ for $\varepsilon > 0$ and $\tilde{y}^\varepsilon = \tilde{y}(u^* + \varepsilon h)$ be the corresponding weak solution of system (1). We compute the directional derivative of the cost functional J with respect to u^* in the direction of h.
Since J reaches its minimum at u^*, we have

$$\begin{aligned}
0 \leq & \lim_{\varepsilon\to 0^+} \frac{J(u^*+\varepsilon h) - J(u^*)}{\varepsilon} \\
= & \lim_{\varepsilon\to 0^+} \frac{1}{2}\int_\omega \int_0^T \frac{(\chi_\omega y^\varepsilon - y_d)^2 - (\chi_\omega y^* - y_d)^2}{\varepsilon} dt dx \\
+ & \lim_{\varepsilon\to 0^+} \frac{\beta}{2}\int_0^T \frac{(u^*+\varepsilon h)^2 - {u^*}^2}{\varepsilon} dt \\
= & \lim_{\varepsilon\to 0^+} \frac{1}{2}\int_\omega \int_0^T \chi_\omega \frac{(y^\varepsilon - y^*)}{\varepsilon}(\chi_\omega y^\varepsilon + \chi_\omega y^* - 2y_d) dt dx \\
+ & \lim_{\varepsilon\to 0^+} \frac{\beta}{2}\int_0^T (2hu^* + \varepsilon h^2) dt.
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{J(u^*+\varepsilon h) - J(u^*)}{\varepsilon} = & \int_Q \chi_\omega^* \chi_\omega \lambda\, (y^* - \chi_\omega^* y_d) dx dt \\
& + \beta \int_0^T h u^* dt,
\end{aligned}$$

where λ is solution of system (14).
Using the adjoint system (16), and the solution of system (14), we obtain

$$\begin{aligned}
0 \leq & \int_\Omega \chi_\omega^* \chi_\omega \int_0^T \lambda(x,t)\big[p_{tt}(x,t) + \Delta^2 p(x,t) \\
& - u^*(t) p_t(x,t)\big] dt dx + \beta \int_0^T h(t) u^*(t) dt \\
= & \int_\Omega \chi_\omega^* \chi_\omega \int_0^T (\lambda_{tt}(x,t) \\
& + \Delta^2 \lambda(x,t) - u^*(t)\lambda_t(x,t)) p(x,t)\, dt dx \\
& + \beta \int_0^T h(t) u^*(t)\, dt \\
= & \int_\Omega \chi_\omega^* \chi_\omega \int_0^T h(t) y_t^*(x,t)\, p(x,t) dt dx \\
& + \beta \int_0^T h(t) u^*(t) dt \\
= & \int_Q h(t)\,(\beta u^*(t) + \chi_\omega^* \chi_\omega y_t^*(x,t)\; p(x,t)\;)\, dx dt.
\end{aligned}$$

Using a standard control argument based on the choices for the variation $h(t)$, an optimal control is given by

$$u^*(t) = \max(-\rho, \min(-\frac{1}{\beta}\int_\Omega \chi_\omega^* \chi_\omega y_t^*(x,t) p(x,t) dx, \rho)).$$

4. NUMERICAL APPROACH AND SIMULATIONS

We have seen that the solution of problem (2) is given by

$$u^*(t) = \max(-\rho, \min(-\frac{1}{\beta}\int_\Omega \chi_\omega^* \chi_\omega y_t^*(x,t) p(x,t) dx, \rho)),$$

where y^* is the solution of system (1) associated with the control u^* and p is the solution of the adjoint system (16). The computation of such control can be realised by the following formula

$$\begin{cases} u_{n+1}^*(t) = \max(-\rho, \min(-\frac{1}{\beta}\int_\Omega \chi_\omega^* \chi_\omega (y_t^*)_n p_n dx, \rho)) \\ u_0^* = 0, \end{cases} \tag{17}$$

where y_n^* is the solution of system (1) associated to the control u_n^* and p_n is the solution of the adjoint system (16). This allows to consider the following algorithm :

Step 1: Initials system data.

- ⊙ Initial state y_0, y_1 and u_0^*.
- ⊙ Desired state y_d.
- ⊙ Threshold accuracy ϵ, subregion ω and time T.

Step 2 :

- ⊙ Solving equation (1) gives y_n^*.
- ⊙ Solving equation (16) gives p_n.
- ⊙ Calculate u_{n+1}^* by (17).

Until $\|u_{n+1}^* - u_n^*\|_{L^\infty(0,T)} \leq \epsilon$ stop, else $n = n+1$ go to step 2.
Step 3 : The control u_n^* is optimal.

Simulations
On $\Omega =]0,1[\times]0,1[$, consider a bilinear plate equation

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) = u(t) y_t(x,t) \\ y(x,0) = y_0(x), \;\; y_t(x,0) = y_1(x) \\ y(x,t) = \dfrac{\partial y}{\partial \nu}(x,t) = 0 \end{cases} \tag{18}$$

where $x = (x_1, x_2)$ and consider problem (2) with the control set $\mathcal{U}_\rho = \{u \in L^\infty(0,T) \;:\; -\rho \leq u(t) \leq \rho\}$.
An optimal control solution of problem (2) is given by the following formula

$$u^*(t) = \max\big(-\rho, \min\big(-\frac{1}{\beta}\int_\Omega \chi_\omega^* \chi_\omega y_t^*(x,t) p_n(x,t) dx, \rho\big)\big),$$

where y^* is solution of system (18) associated to the control u^* and p is the solution of the following adjoint system

$$\begin{cases} p_{tt}(x,t) + \Delta^2 p(x,t) = u^*(t) p_t(x,t) + y^*(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad -\chi_\omega^* y_d(x,t) \\ p(x,T) = p_t(x,T) = 0 \\ p(x,t) = \dfrac{\partial p}{\partial \nu}(x,t) = 0 \end{cases}$$

We take $T = 1$, $\rho = 1$, $\beta = 0.1$, $y_0(x_1,x_2) = x_1 x_2 (1-x_1)(1-x_2)$, $y_1(x_1,x_2) = 0$, and the desired state $y_d(x_1,x_2,T) = 0$ on $\omega \subset \Omega$.
Applying the previous algorithm, with $\epsilon = 10^{-4}$ we obtain.

- For $\omega =]0.7, 1[\times]0, 1[$

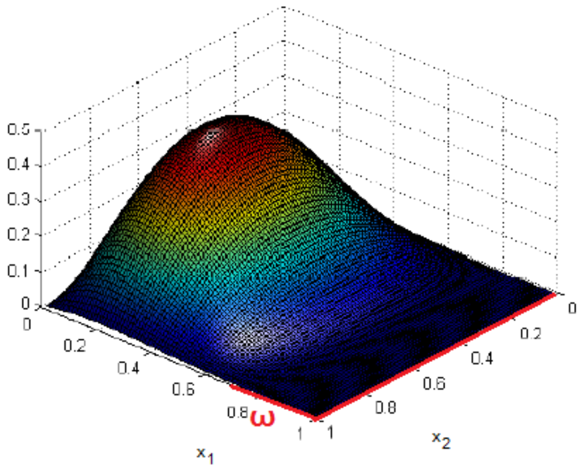

Fig. 1. Desired and final state on Ω.

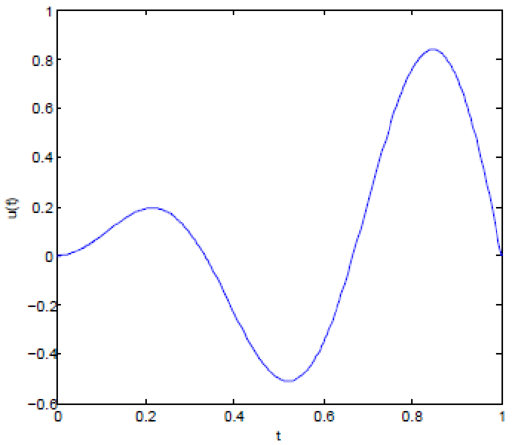

Fig. 2. Desired and final state on Ω.

Figure 1 shows that the reached state is very close to the desired state on ω and the evolution of control function is given by the figure 2. The desired state is obtained with error $\|\chi_\omega y^*(.,T)\|^2_{L^2(\omega)} = 4.2 \times 10^{-4}$, and cost $J(u^*) = 3.8 \times 10^{-3}$.

- For $\omega =]0.5, 1[\times]0, 1[$

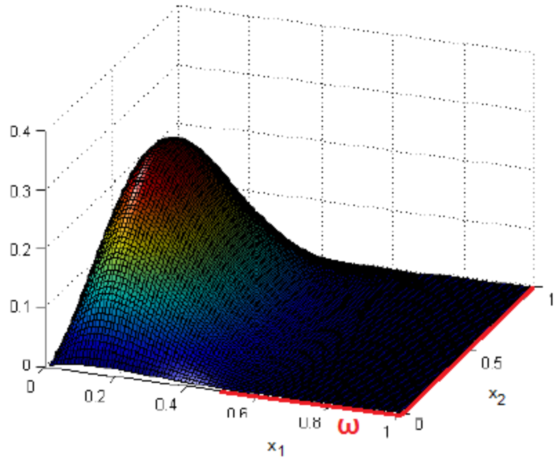

Fig. 3. Desired and final state on Ω.

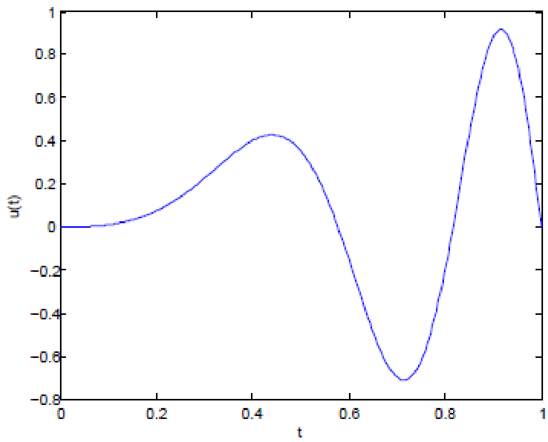

Fig. 4. Desired and final state on Ω.

Figure 3 shows that the reached state is very close to the desired state on ω and the evolution of control function is given by the figure 4. The desired state is obtained with error $\|\chi_\omega y^*(.,T)\|^2_{L^2(\omega)} = 7.5 \times 10^{-4}$, and cost $J(u^*) = 6.4 \times 10^{-3}$.

5. CONCLUSION

Regional optimal control problem of a bilinear plate equation was considered using bounded control. The existence of an optimal control is proved and characterised as a solution of an optimality system. The approach allows us to assess the fitness of the final state to a prescribed target restricted in a subregion of the system domain. The obtained results are successfully tested through numerical examples. Questions are still open, as is the case for regional optimal control of nonlinear plate equation.

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