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Bounds on time-optimal concatenations of arcs for two-input driftless 3D systems ⋆

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Abstract: We study a driftless system on a three-dimensional manifold driven by two scalar controls. We assume that each scalar control has an independent bound on its modulus and we prove that, locally around every point where the controlled vector fields satisfy some suitable nondegeneracy Lie bracket condition, every time-optimal trajectory has at most five bang or singular arcs. The result is obtained using first- and second-order necessary conditions for optimality.

Keywords: Optimal control, geometric control, second-order optimality conditions, time-optimal, bang arc, singular arc.

1. INTRODUCTION

The regularity of optimal trajectories is a much studied problem in control theory, because the presence of discontinuities has obvious implications on the implementability and robustness of optimal feedbacks. The regularity of the value function is also strongly related with the regularity of the optimal trajectories (see, for instance, Schättler and Ledzewicz (2012)). When the value function encodes a distance in a length-space, the regularity of geodesics carries information on the properties of balls and other relevant geometric objects (see Agrachev et al. (2019)).

We study here a class of control systems which can be written in the form

$$\dot{q}(t) = u_1(t)X_1(q(t)) + u_2(t)X_2(q(t)), \tag{1}$$

where the state q evolves on a smooth manifold M, X_1, X_2 are smooth vector fields, and the control $u = (u_1, u_2)$ takes values in the square $[-1, 1]^2$. This setup corresponds to driftless two-input systems for which the modulus of each control parameter has an independent bound. Under some natural Lie algebra rank condition, the time-optimal value function for (1) can also be seen as the length distance associated with a sub-Finsler structure (see Barilari et al. (2017)).

Little is known on the general structure of time-optimal trajectories for systems of the type (1) apart from some homogeneous cases studied in Breuillard and Le Donne (2013) (where M is the Heisenberg group and X_1, X_2 are two left-invariant generators of the horizontal distribution), Barilari et al. (2017) (where sub-Finsler versions of the Grushin plane and the Martinet distribution have been considered), and Ardentov et al. (2019) (where the Cartan group is considered). The inhomogeneous 2D case has been studied in Abdul-Latif Ali and Charlot (2019).

More is known in the case where the control $u = (u_1, u_2)$ takes values in a ball, since this leads to the sub-Riemannian framework (see Agrachev et al. (2019); Jean (2014); Rifford (2014)). It should however be mentioned that even in this case the minimal regularity of time-optimal trajectories is still an open problem (see Barilari et al. (2020); Hakavuori and Le Donne (2016); Monti et al. (2018) for recent results and a state of the art about this longstanding question). When the control takes values in a ball and a drift is added to the dynamics fewer results are known (see Agrachev and Biolo (2017, 2018); Caillau and Daoud (2012)).

Another case where more results are available is when one of the two control inputs u_1 or u_2 is constant. In this case we recover the case of a single-input control-affine system with control in a compact interval. For such kind of systems on a two-dimensional manifold, the situation has been deeply analyzed and general results covering the generic and the analytic case can be found in Sussmann (1987b,a). A monograph where 2D optimal syntheses are studied in details is Boscain and Piccoli (2004). The results in the 3D case are less complete and do not cover the generic case, but only the local behavior of time-optimal trajectories near points where the brackets of the vector fields X_1 and X_2 have degeneracies of corank 0 (i.e., near generic points) or 1 (see Agrachev and Gamkrelidze (1990); Agrachev and Sigalotti (2003); Krener and Schättler (1989); Schättler (1988); Schättler (1988); Sussmann (1986)) and some but not all cases of corank larger than 1 (Bressan (1986); Sigalotti (2005)). In Sigalotti (2005), the regularity of time-optimal trajectories near generic points in dimension 4 is also characterized.

The main type of result contained in the papers mentioned above is a guarantee that small-time optimal trajectories are the concatenation of at most a given number of bang and singular arcs (with limitations on the possible concatenation). One of the relevant consequences of this kind of result is that it allows to rule out the appearance of the Fuller phenomenon (also called *chattering*). This kind of rather radical singularity of optimal trajectory is known to be typical (i.e., not removable by small perturbations)

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in large enough dimension (see Kupka (1990); Zelikin and Borisov (1994); Zelikin et al. (2012)). Some restriction on the wildness of the Fuller behavior (in particular, on the iteration order of accumulations of switching times) can be found in Boarotto and Sigalotti (2019); Boarotto et al. (2020).

In this paper we provide a bound on the worst-regularity behavior of time-optimal trajectories of (1) near generic points in the case where M has dimension 3. Our main result, Theorem 3, states that such trajectories are necessarily concatenations of bang and singular arcs, and that the number of such arcs is not larger than 5. It is interesting to observe that in the corresponding homogenous 3D case, the sub-Finsler structure on the Heisenberg group, small-time optimal trajectories with 5 arcs do exist (Breuillard and Le Donne (2013); Barilari et al. (2017)). This means that the bound that we provide in this paper is sharp. The main technical step in the proof of our main result used the second-order necessary conditions for optimality proposed in Agrachev and Gamkrelidze (1990).

2. STATEMENT OF THE TIME-OPTIMAL PROBLEM AND NECESSARY CONDITIONS FOR OPTIMALITY

Throughout the paper M denotes a smooth (i.e., $\mathcal{C}^\infty$) complete manifold and X_1, X_2 are two smooth vector fields on M. We associate with M, X_1, and X_2 the dynamics

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad q \in M, \quad u_1, u_2 \in [-1, 1]. \quad (2)$$

We also introduce the notations $u = (u_1, u_2)$ and

$$X(u) = u_1 X_1 + u_2 X_2,$$
$$X_+ = X_1 + X_2,$$
$$X_- = X_1 - X_2.$$

It is well known that the time-optimal trajectories of a control system of the form (2) heavily depend on the commutativity properties of the vector fields X_1, X_2, which are infinitesimally characterized by the iterated Lie brackets between them.

Given two smooth vector fields X and Y we write $[X, Y]$ to denote the Lie bracket between X and Y, and ad_X for the operator $[X, \cdot]$ acting on the space of smooth vector fields by

$$\mathrm{ad}_X(Y) = [X, Y].$$

To reduce the notational burden we also set

$$X_{12} = [X_1, X_2], \quad X_{+12} = [X_+, X_{12}], \quad X_{-12} = [X_-, X_{12}].$$

The Lie algebra generated by $\{X_1, X_2\}$, denoted by $\mathrm{Lie}\{X_1, X_2\}$, is the minimal linear subspace of the space of all smooth vector fields on M containing $\{X_1, X_2\}$ and invariant both for ad_{X_1} and ad_{X_2}.

The time-optimal control problem for (2) consists in finding the trajectories of (2) connecting two given points q_0 and q_1 of M in minimal time. The existence of at least one trajectory connecting any pair of points and minimizing the time is a consequence of Chow's theorem and Filippov's theorem, under the assumption that

$$\{V(q) \mid V \in \mathrm{Lie}\{X_1, X_2\}\} = T_q M$$

for every $q \in M$ (see, for instance, Agrachev and Sachkov (2004)).

2.1 *First-order necessary conditions for optimality: the Pontryagin maximum principle*

A well-known optimality condition satisfied by all time-optimal trajectories is the Pontryagin maximum principle (PMP, for short). In order to fix some notations, let us recall here its statement.

Let $\pi : T^*M \to M$ be the cotangent bundle, and $s \in \Lambda^1(T^*M)$ be the tautological Liouville one-form on T^*M. The non-degenerate skew-symmetric form $\sigma = ds \in \Lambda^2(T^*M)$ endows T^*M with a canonical symplectic structure. With any smooth function $p : T^*M \to \mathbf{R}$ let us associate its smooth Hamiltonian lift $\vec{p} \in \mathcal{C}(T^*M, TT^*M)$ by the condition

$$\sigma_\lambda(\cdot, \vec{p}) = d_\lambda p. \quad (3)$$

Introducing the control-dependent Hamiltonian function $\mathcal{H} : T^*M \times \mathbf{R}^2 \to \mathbf{R}$ by

$$\mathcal{H}(\lambda, v) = \langle \lambda, v_1 X_1(q) + v_2 X_2(q) \rangle, \quad q = \pi(\lambda), \quad (4)$$

the statement of the PMP for the time-optimal problem associated with system (2) is the following (see, for instance, Agrachev and Sachkov (2004)).

Theorem 1. (PMP). Let $q : [0, T] \to M$ be a time-optimal trajectory of (2), associated with a control $u(\cdot)$. Then there exists an absolutely continuous curve $\lambda : [0, T] \to T^*M$ such that $(q(\cdot), \lambda(\cdot), u(\cdot))$ is an *extremal triple*, i.e., in terms of the control-dependent Hamiltonian $\mathcal{H}$ introduced in (4), one has

$$\lambda(t) \in T^*_{q(t)} M \setminus \{0\}, \quad \forall t \in [0, T], \quad (5)$$

$$\mathcal{H}(\lambda(t), u(t)) = \max\{\mathcal{H}(\lambda(t), v) \mid v \in [-1, 1]^2\} \quad (6)$$
$$\text{for a.e. } t \in [0, T],$$

$$\dot{\lambda}(t) = \vec{\mathcal{H}}(\lambda(t), u(t)), \quad \text{for a.e. } t \in [0, T]. \quad (7)$$

For any extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$, we call the corresponding trajectory $t \mapsto q(t)$ an *extremal trajectory* and the curve $t \mapsto \lambda(t)$ its We associate with an extremal triple the absolutely continuous functions

$$t \mapsto \phi_i(t) = \langle \lambda(t), X_i(q(t)) \rangle, \quad i = 1, 2.$$

A maximal open interval of $[0, T]$ on which ϕ_1 and ϕ_2 are both different from zero is said to be a *bang arc*. A u_1-*singular* (respectively, u_2-*singular*) arc is a maximal open interval of $[0, T]$ on which $\phi_1 \equiv 0$ while ϕ_2 is different from zero (respectively, $\phi_2 \equiv 0$ while ϕ_1 is different from zero). An arc which is either u_1-singular or u_2-singular is said to be *singular*. A point separating two bang arcs is said to be a *switching time*. We say that the control u_i *switches* at the switching time τ if u_i (which is constant and of modulus 1 on sufficiently small left- and right-neighborhoods of τ) changes sign at τ. In particular, if u_i switches at τ then it follows from the maximality condition of the PMP that $\phi_i(\tau) = 0$. A trajectory is said to be *bang-bang* if it is the concatenation of finitely many bang arcs.

A useful consequence of the PMP is that for every smooth vector field Y on M, for every extremal triple associated with (2) the identity

$$\frac{d}{dt} \langle \lambda(t), Y(q(t)) \rangle = \langle \lambda(t), [X(u(t)), Y](q(t)) \rangle \quad (8)$$

holds true for a.e. t.

2.2 Second-order optimality conditions

We recall in this section a useful second-order necessary condition for a trajectory with piecewise constant control to be time-optimal, obtained in Agrachev and Gamkrelidze (1990) using time-reparameterizations as variations of the reference control signal.

In order to state the theorem, let us introduce the following notation. Given a control $u : [0,T] \to [-1,1]^2$ and two times $s,t \in [0,T]$, denote by $P^u(s,t)$ the flow of (2) from time s to time t. Since we are interested in local properties, we can always assume that solutions of (2) exist globally, which ensures that $P^u(s,t)$ is defined on the entire manifold M. As a consequence, by standard properties of solutions of ODEs, $P^u(s,t) : M \to M$ is a diffeomorphism.

For every diffeomorphism $P : M \to M$, every point $q \in M$, and every $v \in T_qM$, $P_{*,q}v \in T_{P(q)}M$ denotes the push-forward of v obtained by applying the differential of P at q. For every diffeomorphism $P : M \to M$ and every vector field Y, the *push-forward of* Y *by* P is the vector field

$$P_*Y : q \mapsto P_{*,P^{-1}(q)}Y(P^{-1}(q)).$$

Using these notations, we can deduce from the PMP that, if $(q(\cdot),\lambda(\cdot),u(\cdot))$ is an extremal triple on $[0,T]$, then, for $i = 1,2$ and for $t,\overline{\tau} \in [0,T]$,

$$\phi_i(t) = \langle \lambda(\overline{\tau}), (P^u(\overline{\tau},t)_*X_i)(q(\overline{\tau}))\rangle. \tag{9}$$

Theorem 2. Let $q : [0,T] \to M$ be a time-optimal trajectory for (2) and let $u(\cdot) = (u_1(\cdot),u_2(\cdot))$ be the corresponding control function. Assume that $u(\cdot)$ is piecewise constant on $[0,T]$, with K non-removable discontinuities $\tau_1 < \tau_2 < \cdots < \tau_K$ in $(0,T)$. Denote by $u_0,\dots,u_k$ the successive values of $u(\cdot)$ on the $K+1$ bang arcs. Assume that $q(\cdot)$ admits a unique extremal lift $\lambda(\cdot)$ up to multiplication by a positive scalar. Fix $\overline{\tau}$ in $[0,T]$ and let

$$h_i = P^u(\overline{\tau},\tau_i)_*X(u_i), \tag{10}$$

for $i = 0,\dots,K$. Let Q be the quadratic form

$$Q(\alpha) = \sum_{0\le i<j\le K} \alpha_i\alpha_j \left\langle \lambda(\overline{\tau}), [h_i,h_j](q(\overline{\tau}))\right\rangle , \tag{11}$$

defined on the space

$$H = \Big\{\alpha = (\alpha_0,\dots,\alpha_K) \in \mathbf{R}^{K+1} \mid \sum_{i=0}^{K}\alpha_i = 0,\ \sum_{i=0}^{K}\alpha_i h_i(q(\overline{\tau})) = 0\Big\}. \tag{12}$$

Then $Q \le 0$.

Remark 1. The theorem above can be extended from time-minimal to (locally) time-maximal trajectories. This can be done following the lines of Theorem 2 in Agrachev and Sigalotti (2003) where the notion of quasi-optimal trajectory is introduced to cover both kind of properties.

3. BOUND ON THE NUMBER OF ARCS

Let us assume from now on that M has dimension 3. This section contain the main results on the admissible concatenation of arcs, locally near a point where suitable bracket independence conditions are satisfied. We collect the main results in a single statement, Theorem 3 below.

Theorem 3. Let $q_0 \in M$ and Ω be a neighborhood of q_0, compactly contained in M, such that (X_1,X_2,X_{12}), (X_1,X_{12},X_{+12}), (X_1,X_{12},X_{-12}), (X_2,X_{12},X_{+12}), and (X_2,X_{12},X_{-12}) are moving bases on the closure $\overline{\Omega}$ of Ω. Then there exists $T > 0$ such that every time-optimal trajectory $q : [0,T'] \to M$ of (2) contained in Ω and such that $T' \le T$ is the concatenation of at most 5 bang or singular arcs. Moreover, if $q(\cdot)$ contains a singular arc, then it is the concatenation of at most a bang, a singular, and a bang arc.

The main technical step in the proof of the theorem is contained in the next lemma, which focuses on the situation in which both u_1 and u_2 switch along the time-optimal trajectory. Notice that in this case we can relax the assumptions on the triples of vector fields which should be linearly independent on the considered neighborhood Ω. Actually, we just need to assume that X_1, X_2, and X_{12} are linearly independent on $\overline{\Omega}$.

Lemma 1. Let $q_0 \in M$ and Ω be a neighborhood of q_0, compactly contained in M, such that X_1, X_2, and X_{12} are linearly independent on the closure $\overline{\Omega}$ of Ω. Then there exists $T > 0$ such that every bang-bang time-optimal trajectory $q : [0,T'] \to M$ of (2) contained in Ω, undergoing switchings both in u_1 and in u_2, and such that $T' \le T$ is the concatenation of at most 5 arcs.

Proof. Consider a bang-bang extremal trajectory $q : [0,T'] \to M$ of (2) contained in Ω. Assume that $q(\cdot)$ is the concatenation of 6 bang arcs and that both u_1 and u_2 switch. We are going to show that, for T' small enough, $q(\cdot)$ is not optimal.

Let $\lambda(\cdot)$ be an extremal lift of $q(\cdot)$ and define

$$\phi_\star(t) = \langle \lambda(t), X_\star(q(t))\rangle , \qquad t \in [0,T'], \quad \star \in \{1,2,12\}. \tag{13}$$

It follows from (8) that

$$\dot{\phi}_\star(t) = \langle \lambda(t), [u_1(t)X_1 + u_2(t)X_2, X_\star](q(t))\rangle \tag{14}$$

for $\star \in \{1,2,12\}$ and for almost every $t \in [0,T']$. In particular,

$$\begin{aligned}\dot{\phi}_1(t) &= -u_2(t)\phi_{12}(t),\\ \dot{\phi}_2(t) &= u_1(t)\phi_{12}(t),\end{aligned} \tag{15}$$

for almost every $t \in [0,T']$.

Assume for now that $\lambda(\cdot)$ is normalized in such a way that

$$\max\{|\phi_1(0)|,|\phi_2(0)|,|\phi_{12}(0)|\} = 1. \tag{16}$$

We first claim that, up to taking T small enough, ϕ_{12} does not change sign on $[0,T']$. In order to prove the claim, notice that, since $\lambda(\cdot)$ solves the time-dependent Hamiltonian system (7) described by the Pontryagin maximum principle, its growth admits a uniform bound among extremal trajectories in $T^*\Omega$. It then follows from (14) that there exist $T,C > 0$ (independent of $q(\cdot)$) such that $|\dot{\phi}_\star(t)| \le C$ for $\star \in \{1,2,12\}$ and $t \in [0,T']$, provided that $T' < T$. Because of the assumption that both u_1 and u_2 switch along the trajectory $q(\cdot)$, we know that both ϕ_1 and ϕ_2 have a zero on $[0,T']$. This implies, in particular, that $|\phi_1|,|\phi_2| \le CT'$ on $[0,T']$. Let T' be smaller that $1/C$. It follows that $|\phi_1|,|\phi_2| < 1$ on $[0,T']$, and hence, from (16), one has $|\phi_{12}(0)| = 1$. As a consequence, up to modifying C uniformly with respect to $q(\cdot)$, $||\phi_{12}| - 1| \le CT'$ on $[0,T']$, and, in particular, ϕ_{12} has constant sign. This concludes the proof of the claim that ϕ_{12} can be assumed not to change sign on $[0, T']$.

Let us focus now on the admissible concatenations of switches. Assume for a moment that u_2 switches at the same time $\tau \in (0, T')$ at which u_1 does. Then $\dot{\phi}_1$ changes sign at τ. This means that ϕ_1 does not change sign at τ, contradicting the assumption that u_1 switches at τ. We can then assume that the switches of u_1 and u_2 are distinct.

Denote by $\tau_0 \in (0, T')$ the first switching time for $q(\cdot)$. Up to exchanging the roles of X_1 and X_2, we can assume that τ_0 is a switching time for u_1. Up to exchanging X_2 and $-X_2$, we can assume that $u_2 = -1$ in a neighborhood of τ_0. Denote by τ_1 the smaller switching time for $q(\cdot)$ with $\tau_1 > \tau_0$. Assume for now that the (constant) sign of ϕ_{12} on $[0, T']$ is -1. Hence, ϕ_2 is increasing and ϕ_1 decreasing on (τ_0, τ_1). This implies that τ_1 is a switching time for u_2. The same reasoning allows to continue the argument and deduce that the sequence of constant controls (u_1, u_2) for $q(\cdot)$ on its 6 bang arcs follows the (periodic) pattern

$$\begin{pmatrix}1\\-1\end{pmatrix} \vec{\tau_0} \begin{pmatrix}-1\\-1\end{pmatrix} \vec{\tau_1} \begin{pmatrix}-1\\1\end{pmatrix} \vec{\tau_2} \begin{pmatrix}1\\1\end{pmatrix} \vec{\tau_3} \begin{pmatrix}1\\-1\end{pmatrix} \vec{\tau_4} \begin{pmatrix}-1\\-1\end{pmatrix}, \tag{17}$$

where we have denoted by $\tau_0, \dots, \tau_4$ the 5 switching times of $q(\cdot)$. The 6 bang arcs correspond then to the intervals $I_1 = (0, \tau_0)$, $I_2 = (\tau_0, \tau_1)$, and so on, up to $I_6 = (\tau_4, T')$.

In the case where $\phi_{12} > 0$ on $[0, T']$, one analogously shows that the pattern is

$$\begin{pmatrix}-1\\-1\end{pmatrix} \vec{\tau_0} \begin{pmatrix}1\\-1\end{pmatrix} \vec{\tau_1} \begin{pmatrix}1\\1\end{pmatrix} \vec{\tau_2} \begin{pmatrix}-1\\1\end{pmatrix} \vec{\tau_3} \begin{pmatrix}-1\\-1\end{pmatrix} \vec{\tau_4} \begin{pmatrix}1\\-1\end{pmatrix}$$

and the argument below can be adapted easily.

Let us denote by $t_i = \tau_{i+1} - \tau_i$, $i = 0, 1, 2, 3$, the length of the four bang arcs connecting two switching times.

We are going to apply Theorem 2 to the trajectory $q(\cdot)$ taking $\overline{\tau} = \tau_2$. Notice that $\phi_1(\overline{\tau}) = 0$, since u_1 is switching at $\overline{\tau}$.

It is convenient to consider a new normalization of the covector $\lambda(\cdot)$ by imposing that

$$\phi_{12}(\overline{\tau}) = -1. \tag{18}$$

One of the assumptions of Theorem 2 is the uniqueness of the extremal lift for $q(\cdot)$. In order to prove uniqueness, notice that $\lambda(\cdot)$ depends only on the value $\lambda(\overline{\tau})$ and that

$$\langle \lambda(\overline{\tau}), X_1(q(\overline{\tau})) \rangle = 0,$$

because $\overline{\tau}$ is a switching time for u_1, and

$$\langle \lambda(\overline{\tau}), X_{12}(q(\overline{\tau})) \rangle = -1,$$

because of the normalization of $\lambda(\cdot)$. We are left to prove the uniqueness of the component $\phi_2(\overline{\tau})$ of $\lambda(\overline{\tau})$ along $X_2(q(\overline{\tau}))$.

Using (9), let us consider, for $t \in (0, t_2)$, the development

$$\begin{aligned}\phi_2(\overline{\tau} + t) &= \langle \lambda(\overline{\tau}), (e^{t(X_1+X_2)}{}_* X_2)(q(\overline{\tau})) \rangle \\ &= \phi_2(\overline{\tau}) a(q(\cdot), t) + b(q(\cdot), t),\end{aligned} \tag{19}$$

where the functions a and b depend on t and on the bang-bang trajectory $q(\cdot)$, but not on the extremal lift $\lambda(\cdot)$.

We write below $O(t)$ to denote a quantity which can be bounded from above by a term of the form $C|t|$, for t small enough, with C uniform with respect to $q(\cdot)$.

With this notation,

$$a(q(\cdot), t) = 1 + O(t), \quad b(q(\cdot), t) = O(t).$$

Since $\phi_2(\overline{\tau} + t_2) = \phi_2(\tau_3) = 0$, we deduce from (19) that $\phi_2(\overline{\tau})$ is a function of $q(\cdot)$ only (including the dependence on t_2), i.e., that $\phi_2(\overline{\tau})$ is uniquely identified by $q(\cdot)$. This completes the proof of the uniqueness of the extremal lift.

Pushing the computations a step further and noticing that

$$e^{t_2(X_1+X_2)}{}_* X_2 = X_2 + t_2[X_1, X_2] + O(t_2^2)$$

on Ω, we have that

$$a(q(\cdot), t_2) = 1 + O(t_2^2), \quad b(q(\cdot), t_2) = -t_2 + O(t_2^2).$$

Together with (19) evaluated at $t = t_2$, this yields

$$\phi_2(\overline{\tau}) = t_2 + O(t_2^2).$$

Repeating the argument on the interval $[\tau_1, \overline{\tau}]$, we have that

$$\begin{aligned}\phi_2(\overline{\tau} + t) &= \langle \lambda(\overline{\tau}), (e^{-t(-X_1+X_2)}{}_* X_2)(q(\overline{\tau})) \rangle \\ &= \phi_2(\overline{\tau})(1 + O(t^2)) - t + O(t^2),\end{aligned}$$

for $t \in [0, t_1]$, yielding

$$t_2 + O(t_2^2) = \phi_2(\overline{\tau}) = t_1 + O(t_1^2).$$

Henceforth,

$$t_2 = t_1 + O(t_1^2).$$

Similarly,

$$t_0, t_3 = t_1 + O(t_1^2),$$

and we then deduce that $T' = O(t_1)$. In particular,

$$\max_{t \in [0, T'],\, i=1,2} |\phi_i(t)| = O(t_1).$$

According to the relation $T' = O(t_1)$, we are left to prove that $q(\cdot)$ is not optimal if t_1 is small enough.

Using the notation from Theorem 2, we have

$$\begin{aligned}
h_0 &= e^{-t_1(-X_-)}{}_* e^{-t_0(-X_+)}{}_*(X_-) = X_- + O(t_1),\\
h_1 &= e^{-t_1(-X_-)}{}_*(-X_+) = -X_+ + O(t_1),\\
h_2 &= -X_-,\\
h_3 &= X_+,\\
h_4 &= e^{t_2(X_+)}{}_*(X_-) = X_- + O(t_1),\\
h_5 &= e^{t_2(X_+)}{}_* e^{t_3(X_-)}{}_*(-X_+) = -X_+ + O(t_1).
\end{aligned}$$

We can then evaluate

$$\sigma_{ij} = \langle \lambda(\overline{\tau}), [h_i, h_j](q(\overline{\tau})) \rangle$$

for $0 \le i < j \le 5$, obtaining

$$\begin{aligned}
&\sigma_{01} = 2 + O(t_1), && \sigma_{02} = O(t_1),\\
&\sigma_{03} = -2 + O(t_1), && \sigma_{04} = O(t_1),\\
&\sigma_{05} = 2 + O(t_1), && \sigma_{12} = 2 + O(t_1),\\
&\sigma_{13} = O(t_1), && \sigma_{14} = -2 + O(t_1),\\
&\sigma_{15} = O(t_1), && \sigma_{23} = 2,\\
&\sigma_{24} = O(t_1), && \sigma_{25} = -2 + O(t_1),\\
&\sigma_{34} = 2 + O(t_1), && \sigma_{35} = O(t_1),\\
&\sigma_{45} = 2 + O(t_1).
\end{aligned}$$

Decomposing the relation $\sum_{i=0}^{5} \alpha_i h_i(q(\overline{\tau})) = 0$ on the basis $X_+(q(\overline{\tau})), X_-(q(\overline{\tau})), X_{12}(q(\overline{\tau}))$ and collecting the components along $X_+(q(\overline{\tau}))$ and $X_-(q(\overline{\tau}))$, we get

$$\begin{aligned}0 =&\alpha_0 O(t_1) - \alpha_1(1 + O(t_1)) + \alpha_3\\ &+ \alpha_4 O(t_1) - \alpha_5(1 + O(t_1)),\\ 0 =&\alpha_0(1 + O(t_1)) + \alpha_1 O(t_1) - \alpha_2\\ &+ \alpha_4(1 + O(t_1)) + \alpha_5 O(t_1).\end{aligned}$$

Considering, in addition, the relation $\sum_{i=0}^{5} \alpha_i = 0$ and solving with respect to $\alpha_0, \alpha_1, \alpha_2$, we get

$$\begin{aligned}\alpha_0 &= -\alpha_3 - \alpha_4 + O(t_1; \alpha_3, \alpha_4, \alpha_5),\\ \alpha_1 &= \alpha_3 - \alpha_5 + O(t_1; \alpha_3, \alpha_4, \alpha_5),\\ \alpha_2 &= -\alpha_3 + O(t_1; \alpha_3, \alpha_4, \alpha_5),\end{aligned}$$

where $(\alpha_3, \alpha_4, \alpha_5) \mapsto O(t_1; \alpha_3, \alpha_4, \alpha_5)$ denotes a linear function whose coefficients are $O(t_1)$ in the sense introduced above.

We claim now that the space H defined in (12), taken here with $K = 5$, is of dimension 3 in $\mathbf{R}^6$. Indeed, if a vector p is orthogonal to

$$W = \left\{ \sum_{i=0}^{5} \alpha_i h_i(q(\overline{\tau})) \mid \sum_{i=0}^{5} \alpha_i = 0 \right\},$$

then p annihilates, in particular, $h_{i+1}(q(\overline{\tau})) - h_i(q(\overline{\tau}))$ for $i = 0, \ldots, 4$. The same computations as those used to prove the uniqueness of the extremal lift of $q(\cdot)$ show that the orthogonal to W is one-dimensional, that is, W is 2-dimensional. Hence, the kernel H of the map

$$(\alpha_0, \ldots, \alpha_5) \mapsto \sum_{i=0}^{5} \alpha_i h_i(q(\overline{\tau}))$$

on the 5-dimensional space

$$\left\{ (\alpha_0, \ldots, \alpha_5) \mid \sum_{i=0}^{5} \alpha_i = 0 \right\}$$

is of dimension $5 - \dim W = 3$.

The quadratic form Q from Theorem 2 is then

$$Q(\alpha_3, \alpha_4, \alpha_5) = (\alpha_3, \alpha_4, \alpha_5)(M + O(t_1))(\alpha_3, \alpha_4, \alpha_5)^T,$$

where

$$M = \begin{pmatrix} -4 & 0 & 2\\ 0 & 0 & 2\\ 2 & 2 & 0 \end{pmatrix}.$$

Since M has one positive and two negative eigenvalues, we conclude from Theorem 2 that the trajectory $q(\cdot)$ is not optimal for t_1 small enough. This concludes the proof of Lemma 1.

We are now ready to conclude the proof of Theorem 3.

Proof of Theorem 3. The proof works by considering separately several types of time-optimal trajectories $q : [0, T'] \to \Omega$ of (2).

First consider the case where either ϕ_1 or ϕ_2 never vanish on $[0, T']$. Up to exchanging the roles of X_1 and X_2, let us assume that ϕ_2 does not change sign and, consequently, u_2 is constantly equal to $+1$ or -1 on $[0, T']$.

This means that the trajectory $q(\cdot)$ is time-optimal also for the single-input control-affine system

$$\dot{q} = f(q) + ug(q), \quad u \in [-1, 1], \tag{20}$$

with $g = X_1$ and f equal either to X_2 or to $-X_2$. In particular $g, [f, g]$, and $[f + g, [f, g]]$ are linearly independent on $\bar{\Omega}$ and the same is true for $g, [f, g]$, and $[f - g, [f, g]]$. We can deduce from Theorem 3 in Agrachev and Sigalotti (2003) that $q(\cdot)$ is the concatenation of at most 3 bang arcs or a bang, a singular, and a bang arc.

Consider now the case where both ϕ_1 and ϕ_{12} have at least one zero on $[0, T']$. We claim that in this case ϕ_2 is never zero on $[0, T']$, and the conclusion then follows by the case just considered. In order to prove the claim, normalize $\lambda(\cdot)$ in such a way that

$$\max(|\phi_1(0)|, |\phi_2(0)|, |\phi_{12}(0)|) = 1.$$

Since the growth of λ can be uniformly bounded on Ω, we have that $|\phi_1|$ and $|\phi_{12}|$ can be bounded by CT' for some positive constant C independent on the trajectory $q(\cdot)$. By taking T' small enough, it follows that $|\phi_2(0)| = 1$. Since, moreover, according to (15),

$$|\dot{\phi}_2| \leq |\phi_{12}| \leq CT',$$

we can conclude that ϕ_2 never vanishes, as claimed.

The case where both ϕ_2 and ϕ_{12} vanish on $[0, T']$ being completely analogous, we can assume from now on that ϕ_{12} never vanishes on $[0, T']$.

We claim that in this case $q(\cdot)$ is bang-bang. Indeed, consider first the case where $q(\cdot)$ has a u_1-singular arc (τ_0, τ_1). We deduce from the expression of $\dot{\phi}_1$ (cf. (15)) and the nonvanishing of ϕ_{12} that u_2 must vanish on (τ_0, τ_1). This, in turn, implies that also ϕ_2 vanishes on (τ_0, τ_1). Notice now that, by (15), $\dot{\phi}_2 = u_1 \phi_{12}$ which would also imply that $u_1 \equiv 0$ on (τ_0, τ_1). But, clearly, the control corresponding to a time-optimal trajectory cannot vanish on a nontrivial interval, invalidating the assumption that $q(\cdot)$ has a u_1-singular arc. The case of a u_2-singular arc is completely symmetric.

In order to prove that $q(\cdot)$ is bang-bang we can then assume that there exists an open interval (τ_0, τ_1) contained in $[0, T']$ on which ϕ_1 and ϕ_2 are different from zero. Assume that (τ_0, τ_1) is maximal with this property and that $\tau_1 < T'$. Then either ϕ_1 or ϕ_2 vanish at τ_1. If they both vanish at τ_1, then we deduce from (15) that if one of the two functions ϕ_i changes sign, then also the derivative of the other function ϕ_{3-i} changes sign, which means that $\text{sign}(\phi_{3-i})$ and hence u_{3-i} stay constant. Moreover, since the sign of $\dot{\phi}_1$ and $\dot{\phi}_2$ do not change on a bang arc, the interval (τ_1, T') will be a bang arc. If they do not both vanish at the same point, then an analogous reasoning shows that $q|_{(\tau_1, T')}$ is either a single bang arc or that $q|_{(\tau_0, T')}$ is the concatenation of bang arcs undergoing switches both in u_1 and u_2. Reasoning similarly for the interval $[0, \tau_0]$, we conclude the proof of the claim that $q(\cdot)$ is bang-bang.

Moreover, the argument above shows that, under the condition that ϕ_{12} never vanishes, the bang-bang trajectory $q(\cdot)$ is either the concatenation of at most 2 bang arcs, or it undergoes switches both in u_1 and u_2. The conclusion of the proof of Theorem 3 then follows from Lemma 1.

4. CONCLUSION

We considered in this paper the class of time-optimal driftless two-input control problems in which each scalar control has an independent bound on its modulus. When the system is defined on a three-dimensional manifold and the controlled vector fields satisfy some generic Lie bracket independence condition at a given point, we prove that all small-time optimal trajectories near such a point are the concatenation of at most five bang and singular arcs. The result ensures, in particular, that the time-optimal synthesis does not exhibit the Fuller phenomenon. The proposed bound is sharp, as it has been illustrated by previous works in the literature considering the homogeneous case of the Heisenberg group, also known as Brockett integrator in the control literature.

The proof of this fact extensively uses the Hamiltonian formalism provided by the Pontryagin maximum principle and its related second-order sufficient conditions for optimality. Extensions of the technique to more degenerate bracket conditions or to four-dimensional manifolds could be possible, but would involve more intricate computations.

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