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Identification of a Class of Hybrid Dynamical Systems

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Abstract: This paper presents a novel identification procedure for a class of hybrid dynamical systems. In particular, we consider hybrid dynamical systems which are single flowed and single jumped and whose flow and jump maps linearly depend on two sets of unknown parameters. A systematic way to determine whether the system is flowing or jumping is introduced and used to identify the unknown parameters by employing a linear recursive estimator. Simulations have been performed to prove the validity of the proposed methodology. Results proved the efficiency and accuracy of the developed identification procedure.

Keywords: System Identification, Hydrid Systems, Linear Systems, Recursive Estimation.

1. INTRODUCTION

In recent years, there has been a growing interest in hybrid dynamical systems (HDS). These kind of systems are characterized by interacting continuous and discrete time dynamics and provide new and promising modeling perspectives for systems presenting discontinuous behaviors [Van Der Schaft and Schumacher 2000]. The presence of both discrete and continuous dynamics makes this formalism appealing also for modeling physical phenomena in many different areas, from biology and medical applications to robotics, manufacturing, traffic management and biomolecular networks [Aihara and Suzuki 2010, Bortolussi and Policriti 2008]. Overviews of this framework are given in [Van Der Schaft and Schumacher 2000, Haddad et al. 2006, Goebel et al. 2009, 2012]. As common in Automatic Controls, controller design, simulations and diagnostics of hybrid dynamical systems require an accurate knowledge of the model. However, these processes are always characterized by sets of parameters which are typically not known.

System identification techniques are the interface between real world application and mathematical world of control theory and mathematical abstraction [Ljung 2010]. These methods aim to obtain estimates of the parameters and update the model from direct measurements collected during the time evolution of the system [Åström and Eykhoff 1971].

The majority of the literature on identification of HDS is related to classes of Piece Wise Affine systems (PWA) Westra et al. [2011], i.e. systems which are defined by subdividing the space into polyhedral regions which have associated an affine state update equation. It is possible to discern four main different identification procedures for PWA systems: Bayesian, algebraic, clustering-based and bounded-error approaches. Qualitative comparison between the performance of those methodologies is reported in [Juloski et al. 2005a, Paoletti et al. 2007]. Identification of piecewise affine (PWARX), hinging hyperplanes (HHARX), and Wiener piecewise affine (WPWARX) autoregressive exogenous models of hybrid dynamical systems have been addressed in [Bemporad et al. 2001]. Although in this work global convergence is provided through a mixed-integer linear or quadratic programming, the performance of the proposed solution strictly depends on the choice of the input signal u. Picewise affine identification of submodels and the valid polyhedral partitions of the domain of hybrid systems are evaluated by combining clustering, classification and linear identification techniques in the work proposed in [Ferrari-Trecate et al. 2001]. The particular behaviour of each procedures is evaluated via experimental evaluation of the electronic components of a pick-and-place device. Even if experimental results shown the validity of the proposed solution, it requires strict assumptions on the working space and error bounds. A pick-and-place machine has also been used in [Juloski et al. 2005b] to evaluate a Bayesian scheme which model the unknown parameters as random variables described with probability distribution functions an implemented with particle filtering methodologies.

Researchers also tried to implement on-line identification of electronic componets with fuzzy clustering [Sepasi and Sadrnia 2008] and machine learning techniques. Feedforward neural networks have been used for identification of a class of hybrid systems by Messai et al. in [Messai et al. 2008, 2006]. The networks, characterised by continuous inputs, continuous outputs and binary discrete inputs, use a black-box approach to track all the mode of the system. Results are promising but highly dependent on the input sequence. Other, more recent work include [Yuan et al. 2019, Hojjatinia et al. 2019, Lauer and Bloch 2019]. Compared to previous works, in this paper we aim to solve the identification problem for a class of HDS in the form of *hybrid inclusions*, introduced in [Goebel et al. 2009, 2012]. Hybrid inclusions combine constrained differential and difference inclusions and constitute the most general representation of HDS. In particular, we consider hybrid inclusions with one *flow* and one *jump*, i.e. one constrained differential equation for the continuous–time part and one constrained difference equation for the discrete–time part. This class of systems includes ball–juggling mechanisms (see, e.g., [Tian et al. 2013]), impact pendulums and other systems from the *nonsmooth mechanics* framework [Brogliato 1999]. Furthermore, with respect to previous works, we will treat the autonomous case, in which no input–output relations are defined, with the assumption of being able to collect measurements of the state during a trajectory and linearity of the flow and jump maps with respect to two distinct sets of unknown parameters. The proposed method relies on the Lipschitz continuity assumption of the flow map to determine from observations whether the state undergoes to a discontinuity, acknowledging that a jump happened. A linear recursive estimator is then used to estimate both the flow and jump parameters while the flow and jump sets are approximated by convex hulls.

This paper is organized as follow: Section 2 gives an overview of the problem and presents the basic assumptions. The procedure used for the detection of a jump is reported in Section 3. Section 4 describes the identification methodology. The simulation results are presented in Section 5. Conclusion and future work are drawn in Section 6.

Notation The set $\mathbb{R}$ is the the set of real numbers. $\|\cdot\|$ denotes the norm induced by the inner product on $\mathbb{R}^n$. The origin of $\mathbb{R}^n$ is $\mathbb{0}_n$. Let Λ be a finite subset of $\mathbb{R}^n$, $\mathrm{conv}(\Lambda)$ is the convex hull of Λ defined by the convex combinations of its elements. The notation x^+ indicates the next value of the quantity x after a discrete–time event.

2. PROBLEM SETTING

Let us consider an autonomous hybrid dynamical system represented by the following equations:

$$\mathcal{H}: \quad \begin{cases} \dot{\xi} = f(\xi) & \xi \in \mathcal{C} \\ \xi^+ = g(\xi) & \xi \in \mathcal{D} \end{cases} \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state of the system, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are vector fields, $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathbb{R}^n$. Let us call $\mathcal{C}$ the *flow set*, f the *flow map*, $\mathcal{D}$ the *jump set*, g the *jump map*. The notation ξ^+ indicates the next value of the state ξ after a jump. The system can be also represented by means of the hybrid automata in Fig. 1. System (1) is the single–*flow* single–*jump* specialization of the more general *hybrid inclusions* whose framework is deeply explored in Goebel et al. [2009, 2012].

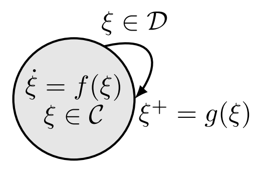

Fig. 1. Hybrid automata: Conceptual representation of the hybrid system $\mathcal{H}$ characterized by a single flow map f and one jump map g. Only one state and one reset branch are needed to picture the behaviour of the system.

Let us suppose that the flow map f and the jump map g depend on two sets of *unknown parameters* $\alpha \in \mathbb{R}^{m_f}$ and $\beta \in \mathbb{R}^{m_g}$, i.e.

$$f = f(\xi, \alpha) \qquad g = g(\xi, \beta)$$

and that no *a priori* knowledge of both, the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$, is available. Assume that the system is observable, i.e. it is possible to measure and collect samples of the state ξ.

In order to correctly simulate the system or design a controller for it, it is necessary to identify the parameters in α, β and estimate the sets $\mathcal{C}$ and $\mathcal{D}$ from measurements of the state ξ.

Hereafter, the basic assumptions required to develop the proposed identification method are presented. Firstly, as a very general hypothesis, both the flow and jump maps are assumed to be linear in the parameters. However, f and g are usually nonlinear with respect to the states. Thanks to this assumption, it is possible to employ linear identification techniques to estimate the unknown parameters.

Assumption 1. (Linearity in the parameters). The maps f and g are linear with respect to constant parameters collected in the vectors $\alpha \in \mathbb{R}^{m_f}$ and $\beta \in \mathbb{R}^{m_j}$ respectively, i.e., there exist $\varphi : \mathbb{R}^n \to \mathbb{R}^{m_f \times n}$ and $\psi : \mathbb{R}^n \to \mathbb{R}^{m_j \times n}$ such that

$$\begin{cases} \dot{\xi} = f(\xi) = \varphi(\xi)\alpha & \xi \in \mathcal{C} \\ \xi^+ = g(\xi) = \psi(\xi)\beta & \xi \in \mathcal{D} \end{cases}$$

The maps φ, ψ are assumed to be known a priori.

The second assumption deals with the regularity of the flow map.

Assumption 2. (Smoothness of the flow map). The following properties hold for the flow map f:

1. f is globally Lipschitz continuous on $\mathcal{C}$, i.e., there exists a constant $k \geq 0$ such that
$$\forall \xi_1, \xi_2 \in \mathcal{C} \quad \|f(\xi_1) - f(\xi_2)\| \leq k\|\xi_1 - \xi_2\| \ .$$
k is referred as the *Lipschitz constant*;
2. f is differentiable almost everywhere in $\mathcal{C}$;
3. f admits a fixed point in the origin which is inside $\mathcal{C}$, i.e. $f(\mathbb{0}_n) = \mathbb{0}_n$, $\{\mathbb{0}_n\} \in \mathcal{C}$.

3. JUMP DETECTION

Definition 3. (Euler derivative norm). Given the hybrid system $\mathcal{H} = (f, g, \mathcal{C}, \mathcal{D})$ and a time interval $\delta t > 0$, the norm of the Euler derivative of the state is defined as

$$D_{\delta t}\xi(t) = \frac{\|\xi(t) - \xi(t - \delta t)\|}{\delta t}$$

Definition 4. (Bounded-norm Euler derivative). The Euler derivative of a hybrid system $\mathcal{H} = (f, g, \mathcal{C}, \mathcal{D})$ has bounded-norm if there exists $\tau(t) \geq 0$ such that

$$\forall \delta t > 0, \forall t > \delta t, \forall \xi \in \mathcal{C} \quad D_{\delta t}\xi(t) \le \tau(t)$$

Proposition 5. (Norm bound of the Euler derivative). Consider the hybrid system $\mathcal{H} = (f, g, \mathcal{C}, \mathcal{D})$. If $f(\xi)$ satisfies Assumption 2 [1] and $\xi(t) \in \mathcal{C} \;\; \forall t \in [t - \delta t, t]$, then $\mathcal{H}$ has bounded-norm Euler derivative with upper bound

$$\tau(t) = \frac{1}{\delta t} \int_{t-\delta t}^{t} k \|\xi(s)\| ds \qquad (2)$$

Proof. Since $\dot{\xi} = f(\xi) \; \forall \xi \in \mathcal{C}$, integrating both sides of the equation between $t - \delta t$ and t, yields

$$\xi(t) - \xi(t - \delta t) = \int_{t-\delta t}^{t} f(\xi(s)) ds$$

Therefore,

$$D_{\delta t}\xi(t) = \frac{1}{\delta t} \|\xi(t) - \xi(t - \delta t)\| = \frac{1}{\delta t} \left\| \int_{t-\delta t}^{t} f(\xi(s)) ds \right\|$$

Considering the right hand side of the equation, it follows that

$$\frac{1}{\delta t} \left\| \int_{t-\delta t}^{t} f(\xi(s)) ds \right\| \le \frac{1}{\delta t} \int_{t-\delta t}^{t} \|f(\xi(s))\| ds$$

Since f is globally Lipschitz in $\mathcal{C}$ and $f(\mathbb{0}_n) = \mathbb{0}_n$,

$$\begin{aligned} \frac{1}{\delta t} \int_{t-\delta t}^{t} \|f(\xi(s))\| ds &= \frac{1}{\delta t} \int_{t-\delta t}^{t} \|f(\xi(s)) - f(\mathbb{0}_n)\| ds \\ &\le \frac{1}{\delta t} \int_{t-\delta t}^{t} k \|\xi(s)\| ds < \infty \end{aligned}$$

where k is the Lipschitz constant. Thus, there exists a $\tau \ge 0$ such that, for all t, it holds

$$D_{\delta t}\xi(t) \le \frac{1}{\delta t} \int_{t-\delta t}^{t} k \|\xi(s)\| ds \triangleq \tau(t, \delta t)$$

The above integral is always limited since, from global Lipschitz continuity of $f(\cdot)$, global existence of trajectories $\xi(t)$ is assured in $\mathcal{C}$. It follows that on any compact time interval, $[t - \delta t, t]$, where the state does not leave the flow set, the quantity $\int_{t-\delta t}^{t} \|\xi(s)\| ds$ is limited, providing the result.

■

From now on let us refer to $\tau(t)$ as the *smoothness bound.*

Example 6. Consider an hybrid system with the flow described by

$$\dot{\xi} = f(\xi) = \sin(\xi) \quad \xi \in \mathcal{C}$$

where $\mathcal{C} := [0, 2\pi]$. f clearly satisfies Assumption 2 and its Lipschitz constant k can be found as

$$k = \sup_{\xi \in \mathcal{C}} \| \frac{df}{d\xi} \| = \sup_{\xi \in \mathcal{C}} \| \cos(\xi) \| = 1$$

Furthermore, the solution of the ordinary differential equation with $\xi(0) = \xi_0$ is

$$\xi(t) = 2 \tan^{-1}(e^{t - \ln(\cot(\xi_0/2))})$$

Since $f(0) = 0$ ($\{0\} \in \mathcal{C}$) Proposition 5 holds and, for any $\delta t > 0$ yields

$$D_{\delta t}\xi(t) = \frac{1}{\delta t} \|\xi(t) - \xi(t - \delta t)\| \le \frac{1}{\delta t} \int_{t-\delta t}^{t} \|\xi(s)\| ds = \tau(t)$$

Figure 2 shows in a numerical example with $\xi_0 = 10^{-2}$ and $\delta t = 0.5$.

[1] Notice that Assumption 2.2 is not necessary for Proposition 1. However, it will become of fundamental importance later on.

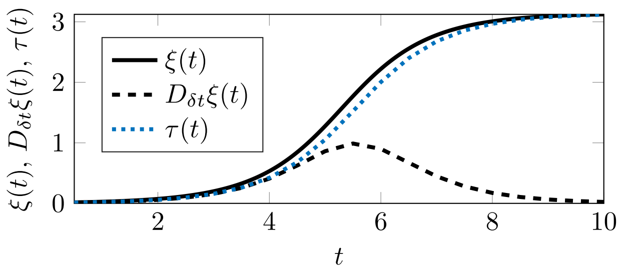

Fig. 2. Time evolution of $\xi(t)$, $D_{\delta t}\xi(t)$, $\tau(t)$ for $\xi_0 = 10^{-2}$ and $\delta t = 0.5$.

Thanks to Assumption 2 and the criteria provided by Proposition 5, any jump, i.e. state discontinuities, can be detected from a series of state measurements by inspecting the norm of the Euler derivative. In particular, the system can be considered to be jumping if $D_{\delta t}\xi(t)$ is above an empirically estimated *smoothness bound* $\hat{\tau}(t)$ and the following assumption is always satisfied.

Assumption 7. (Discontinuities and sampling time). For the chosen δt and $\hat{\tau}$, it holds:

1. $g(\xi) \in \mathcal{C} \quad \forall \xi \in \mathcal{D}$;
2. $\exists s \in [t, t - \delta t] : \xi(s) \in \mathcal{D} \Rightarrow s$ is unique.
3. Let $\xi(s) \in \mathcal{D}, \; s \in (t - \delta t, t)$. Then,
$$\mathcal{D}_{\delta t}\, \xi(t) \ge \hat{\tau}(t).$$

Therefore, a *jump detection function* can be defined as

$$\gamma(t) \triangleq \begin{cases} 0 & \|\xi(t) - \xi(t - \delta t)\|/\delta t \le \hat{\tau}(t) \\ 1 & \text{otherwise} \end{cases}$$

which is zero during the flows of the system and assumes the value 1 during the jumps.

At any instant of time $\gamma(t)$ allows to determine whether the system is *jumping* of *flowing*; $\hat{\tau}$ can be adaptively changed as function of the state and/or time. Note that $\gamma(t) = 1$ indicates that the jump just happened and thus the system's state had been inside the jump set $\mathcal{D}$ during the time interval $[t - \delta t, t]$.

Remark 8. Assumption 7 ensures two important properties. Firstly, that only one jump is possible between two samples of the state. Secondly, that the discontinuities created by jumps always make the Euler derivative norm to exceed the estimated smoothness bound for the chosen sampling time. Note that this latter assumption is quite reasonable for any smooth f and sufficiently small δt. However, presence of noise on the state's observations could indeed break this assumption.

4. IDENTIFICATION PROCEDURE

4.1 Approximation of the Smoothness Bound

In order to estimate on-line the smoothness bound (2), it is necessary to approximate the Lipschitz constant k and then numerically integrate the norm of the state between two sampling instants. At any time instant t, the estimated flow map $\hat{f}_t$ is defined as

$$\hat{f}_t(\xi) = \varphi(\xi)\hat{\alpha}(t)$$

where $\hat{\alpha}(t)$ is the estimated vector of flow parameters at time t. Thanks to Assumption 2, f is differentiable (and so is $\hat{f}_t$) and globally Lipschitz on $\mathcal{C}$, the Lipschitz constant will bound from above the supremum of the norm of the Jacobian of f in the flow set $\mathcal{C}$:

$$k \geq \sup_{\xi \in \mathcal{C}} \|\frac{\partial f}{\partial \xi}\|$$

Since f is not known a priori (α is unknown), the estimation of k must be carried out employing the estimated flow map $\hat{f}_t$. In this paper, as an estimate of the Lipschitz constant, the following quantity is considered:

$$\hat{k} = r \sup_{\xi \in \mathcal{C}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \quad r > 1$$

Therefore, the estimated smoothness bound $\hat{\tau}(t)$ is given by

$$\hat{\tau}(t) = \frac{\hat{k}}{\delta t} \int_{t-\delta t}^{t} \|\xi(s)\| ds = \frac{r}{\delta t} \sup_{\xi \in \mathcal{C}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \int_{t-\delta t}^{t} \|\xi(s)\| ds$$

Notice that the integral below must be computed numerically. A possible simple choice is to use the first order Newton-Cotes formula (trapezoidal rule). This would yield to the following approximation of $\hat{\tau}(t)$

$$\begin{aligned} \hat{\tau}(t) &= \frac{r}{\delta t} \sup_{\xi \in \mathcal{C}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \int_{t-\delta t}^{t} \|\xi(s)\| ds \\ &\approx \frac{r}{2} \sup_{\xi \in \mathcal{C}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \left[\|\xi(t)\| + \|\xi(t-\delta t)\|\right] \end{aligned} \tag{3}$$

It worths to be observed that with this numerical integration algorithm the estimated smoothness threshold does not explicitly depends on δt. Other empirical methods for computing the Lipschitz constant of multivariable functions are presented in Mladineo [1986], Wood and Zhang [1996]. Notice that we do not have any *a priori* knowledge of $\mathcal{C}$ and thus, in equation (3) an approximated flow set $\hat{\mathcal{C}}$ should be employed.

Remark 9. The tuning of the multiplicative constant r should be addressed empirically. In fact, it has to be chosen considering a trade off between robustness (high r ensures that $\hat{\tau}(t) \geq \tau(t)\ \forall t$) and accuracy (low r let $\hat{\tau}$ stay below the peaks of $D_{\delta t}\xi(t)$ during jumps, i.e. Assumption 3 would be violated).

4.2 Parameters Estimation

Let us assume to observe the system and measure the state ξ with a sampling time δt. Suppose to collect N_f samples during the flows of the system. It holds:

$$\begin{bmatrix} \dot{\xi}(t_1) \\ \dot{\xi}(t_2) \\ \vdots \\ \dot{\xi}(t_{N_f}) \end{bmatrix} = \begin{bmatrix} \varphi(\xi(t_1)) \\ \varphi(\xi(t_2)) \\ \vdots \\ \varphi(\xi(t_{N_f})) \end{bmatrix} \alpha \tag{4}$$

Similarly, if N_j samples of the states are collected during jumps, i.e., if $\gamma(t_i) = 1$ for all $i = 1, \ldots, N_j$, yields

$$\begin{bmatrix} \xi(t_1) \\ \xi(t_2) \\ \vdots \\ \xi(t_{N_j}) \end{bmatrix} = \begin{bmatrix} \psi(\xi(t_1 - \delta t)) \\ \psi(\xi(t_2 - \delta t)) \\ \vdots \\ \psi(\xi(t_{N_j} - \delta t)) \end{bmatrix} \beta \tag{5}$$

However, in practice, measurements are always affected by noise and, thus, relations (4) and (5) do not hold. In particular, let's assume that an additive noise, $\tilde{\xi}(t_i)$, with zero-mean affects the system, i.e., $\xi(t_i) = \bar{\xi}(t_i) + \tilde{\xi}(t_i)$, where $\bar{\xi}(t)$ is the true value of the state variable.

Thanks to Assumption 1, the parameters in α and β can be estimated by linear identification techniques. In fact, both equations, (4) and (5) belong to the error-in-variables (EIV) models:

$$y \approx Xa \tag{6}$$

where $y = \bar{y} + \tilde{y}$ is an output's measurements vector and $X = \bar{X} + \tilde{X}$ is an observation matrix, both comprise of a noiseless part ($\bar{y}$, $\bar{X}$), and a noisy part ($\tilde{y}$, $\tilde{X}$). In this linear model, the vector of parameters a could be estimated by mean of a standard linear estimator Ljung [1987]. The choice of the proper estimation scheme should be done considering how the measurement noise is distributed on the possible state–nonlinearities of φ and ψ. The overall identification experiment is carried out as follows: at each sampling time, the jump detection function is computed. If the system is considered to be flowing, the estimated flow parameter $\hat{\alpha}$ is updated and the jump parameter $\hat{\beta}$ remains unchanged. On the contrary, if a jump state is detected $\gamma(t) = 1$, the jump parameter $\hat{\beta}$ will be updated and the flow parameters $\hat{\alpha}$ will be unchanged.

Notice that, in order to update the flow parameters $\hat{\alpha}$ according to the chosen linear estimation scheme, the knowledge of the derivative of the state, $\dot{\xi}(t)$, is needed. Since we assume to be able of collecting only samples of the state, this quantity has to be computed numerically (Euler derivative, sliding mode, etc.). This cumbersome calculation can be avoided by considering that the linear relation holds even if it is integrated in an interval of time. In fact,

$$\begin{aligned} & \dot{\xi}(t) = \varphi(\xi(t))\alpha \\ \Leftrightarrow \quad & \underbrace{\xi(t) - \xi(t-\delta t)}_{\Delta\xi(t)} = \underbrace{\left[\int_{t-\delta t}^{t} \varphi(\xi(s)) ds\right]}_{\Phi(t)} \alpha \\ \Leftrightarrow \quad & \Delta\xi(t) = \Phi(t)\alpha \end{aligned} \tag{7}$$

This propriety has been exploit in the identification experiments by employing the model (7). Note that, since the value of ξ is available only at t and $t-\delta t$, the above integral must be computed by some explicit numerical scheme An overview of the procedure used to update the parameters at time t is presented in Algorithm 1.

Algorithm 1 Identification of the Hybrid System

1: **Input:** $\hat{\alpha}(t-\delta t)$, $\hat{\beta}(t-\delta t)$, $\xi(t)$, $\xi(t-\delta t)$,
2: Compute $\hat{\tau}(t) = \frac{r}{2} \sup_{\xi \in \hat{\mathcal{C}}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \left[\|\xi(t)\| + \|\xi(t-\delta t)\|\right]$
3: Compute $\gamma(t)$
4: **if** $\gamma(t) = 0$ **then**
5: $\quad \Delta\xi(t) = \xi(t) - \xi(t-\delta t)$, $\Phi(t) = \int_{t-\delta t}^{t} \varphi(\xi(s)) ds$
6: $\quad$ Update $\hat{\alpha}(t)$ (Linear Estimator)
7: $\quad \hat{\beta}(t) \leftarrow \hat{\beta}(t-\delta t)$
8: **else**
9: $\quad$ Update $\hat{\beta}(t)$ (Linear Estimator)
10: $\quad \hat{\alpha}(t) \leftarrow \hat{\alpha}(t-\delta t)$
11: **Output:** $\hat{\alpha}(t)$, $\hat{\beta}(t)$,

4.3 Reduced Order Identification

Let $\dot{\xi} = f(\xi) = \varphi(\xi)\alpha$. Suppose that some parameters in α are known before the identification experiment. In particular, let us assume that all the m_l parameters of the first l components of f are known

$$\alpha = [\alpha_1, \alpha_2]^\top$$

where $\alpha_1 \in \mathbb{R}^{m_l}$ are the known parameters and $\alpha_2 \in \mathbb{R}^{m_f - m_l}$. In this case, $\varphi(\xi)$ can be partitioned as

$$\varphi(\xi) = \begin{bmatrix} \varphi_1(\xi) & 0 \\ 0 & \varphi_2(\xi) \end{bmatrix}$$

with $\varphi_1(\xi) \in \mathbb{R}^{l \times m_l}$ and $\varphi_2(\xi) \in \mathbb{R}^{(n-l)\times(m_f - m_l)}$. Therefore, the identification experiment should be applied only to the subsystem

$$\dot{\xi}_2 = \varphi_2(\xi)\alpha_2 \tag{8}$$

This is justified in many physical systems described by a set of second order differential equations (see Example 10).

Notice that the same situation might happen for a jump map. In this case,

$$\psi(\xi) = \begin{bmatrix} \psi_1(\xi) & 0 \\ 0 & \psi_2(\xi) \end{bmatrix}, \quad \beta = [\beta_1, \beta_2]^\top$$

and the model considered for the identification would be

$$\xi_2^+ = \psi_2(\xi)\beta_2 \tag{9}$$

Example 10. Let the flows of the hybrid system be described by the following differential equation

$$\ddot{x} = a\dot{x} + b\sin(x)$$

Define $\xi = [\xi_1, \xi_2]^\top = [x, \dot{x}]^\top$. The canonical first order state space equation becomes

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ a\xi_2 + b\sin(\xi_1) \end{bmatrix} = \begin{bmatrix} \xi_2 & 0 & 0 \\ 0 & \xi_2 & \sin(\xi_1) \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$

It is clear that we are interested only in the parameters a and b. Therefore it is convenient to consider only the subsystem

$$\dot{\xi}_2 = [\xi_2 \;\; \sin(\xi_1)] \begin{bmatrix} a \\ b \end{bmatrix}$$

in the identification experiment.

4.4 Flow and Jump Sets Approximation

The method to approximate the flow and jump sets proposed in this paper is developed from the following idea. Since it is possible to determine the “state” of the system, i.e. whether the system is flowing or it has jumped, it is possible to subdivide the samples of the state in two different sets, a set Λ containing the state's samples during flows and a set Γ containing the state's samples corresponding to jumps. Then, the approximated flow set $\hat{\mathcal{C}}$ and jump set $\hat{\mathcal{D}}$ are obtained by computing the convex hull of Λ and Γ respectively. The detailed procedure is reported in Algorithm 2.

Remark 11. In case $\mathcal{C}$ and $\mathcal{D}$ are assumed convex sets, the approximation results offer an accurate representations of them, consistent with the regions explored by the state of the system. Otherwise, $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ will be a redundant representation of the flow and jump sets. In this case, other techniques might be implemented for a better approximation, e.g. machine–learning–related methods. However, these further investigations fall out of the scope of this paper and will be treated in future work.

Algorithm 2 Approximation of the flow and jump sets

Input: $\hat{\alpha}(t-\delta t)$, $\xi(t)$, $\xi(t-\delta t)$, $\Lambda(t-\delta t)$, $\Gamma(t-\delta t)$
Compute $\hat{\tau}(t) = \frac{r}{2}\sup_{\xi\in\hat{\mathcal{C}}} \|\frac{\partial \hat{f}_t}{\partial \xi}\| \left[\|\xi(t)\| + \|\xi(t-\delta t)\|\right]$
Compute $\gamma(t)$
if $\gamma(t) = 0$ **then**
    $\Lambda(t) \leftarrow \Lambda(t-\delta t) \cup \{\xi(t)\}$
    $\Gamma(t) \leftarrow \Gamma(t-\delta t)$
else
    $\Lambda(t) \leftarrow \Lambda(t-\delta t) \cup \{\xi(t)\} \backslash \{\xi(t-\delta t)\}$
    $\Gamma(t) \leftarrow \Gamma(t-\delta t) \cup \{\xi(t-\delta t)\}$
$\hat{\mathcal{C}}(t) = \mathrm{conv}(\Lambda(t))$
$\hat{\mathcal{D}}(t) = \mathrm{conv}(\Gamma(t))$
Output: $\hat{\mathcal{C}}(t)$, $\hat{\mathcal{D}}(t)$, $\Lambda(t)$, $\Gamma(t)$

5. SIMULATION EXPERIMENTS

5.1 Case of Study: Impact Mass-Spring-Damper System

Let us consider an impact mass-spring-damper system represented in Fig. 3. Assume the mass to be unitary and the rest position of the spring to be at the origin ($x = 0$).

Model of the Flows When the ball is not touching the floor, the system behaves as a simple damped linear oscillator. Let $\xi = [\xi_1, \xi_2]^\top = [x, \dot{x}]^\top$. The flows of the system are described by

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ -a\xi_1 - b\xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 & 0 & 0 \\ 0 & -\xi_1 & -\xi_2 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$

where a and b are the spring stiffness and the damping coefficient, respectively. The corresponding flow set flow set is

$$\mathcal{C} = \{\xi : \xi_1 \geq 0\} \setminus \{\xi : \xi_1 = 0, \xi_2 < 0\}$$

Model of the Jumps It is clear that during the collision between ball and the floor ($x = 0$) there is a discontinuity in the velocity. In this example, collisions are considered partially inelastic while the ball is modeled as a rigid body. The selected jump map to model the collisions is the following:

$$\xi^+ = \begin{bmatrix} \xi_1 \\ -\lambda\xi_2 + \mu \end{bmatrix} = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & -\xi_2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \mu \end{bmatrix}$$

where λ is the restitution coefficient and μ is introduced to consider possible unmodeled impact dynamics. The jump set is therefore,

$$\mathcal{D} = \{\xi : \xi_1 = 0, \xi_2 \leq 0\}$$

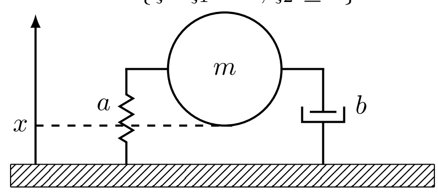

Fig. 3. Impact mass-spring-damper system

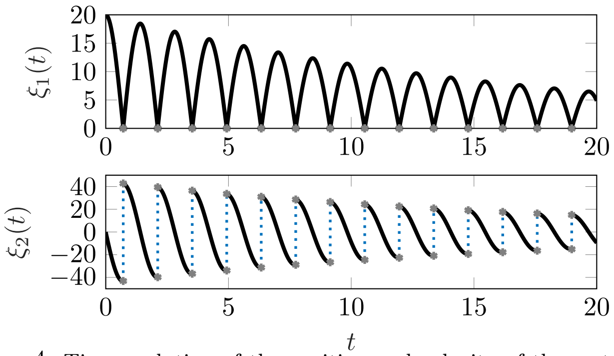

Fig. 4. Time evolution of the position and velocity of the system. The gray dots correspond to system's jumps while dotted blue lines highlight the change of state which creates discontinuities.

Model Identification In order to estimate the system's parameters, a reduced order identification has to be performed for both the flow and the jump as in (8) and (9). Let $\alpha = [\alpha_1, \alpha_2]^\top = [a, b]^\top$ and $\beta = [\beta_1, \beta_2]^\top = [\lambda, \mu]^\top$ The reduced model for the flow is

$$\dot{\xi}_2 = [-\xi_1 \ -\xi_2] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

and the one for the jump is

$$\xi_2^+ = [-\xi_2 \ 1] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

The approximation of the smoothness bound can be derived by computing the supremum of the norm of the estimated flow map Jacobian. Hence, the estimated flow map will be

$$\hat{f}_t(\xi) = -\hat{\alpha}_1(t)\xi_1(t) - \hat{\alpha}_2(t)\xi_2(t)$$

Therefore,

$$\frac{\partial \hat{f}_t}{\partial \xi} = \begin{bmatrix} -\hat{\alpha}_1(t) \\ -\hat{\alpha}_2(t) \end{bmatrix} \Rightarrow \sup_{\xi \in \hat{\mathcal{C}}} \| \frac{\partial \hat{f}_t}{\partial \xi} \| = \|\hat{\alpha}(t)\|$$

5.2 Numerical Simulation

To validate the proposed identification method, numerical simulations have been performed. The simulation experiments have been carried out using Hybrid Equations (HyEQ) Toolbox Sanfelice et al. [2013] for the MATLAB environment. The parameters of the system have been chosen as $a = 5$, $b = 0.1$, $\lambda = 0.99$, $\mu = 0$. The initial condition and the time span of the simulation have been set to $\xi(t_0) = [20, 0]^\top$ and 20 seconds, respectively. Furthermore, the sampling time δt has been chosen to be $0.5 \cdot 10^{-3}$ s, the multiplicative constant r in the smoothness bound has been set to 220 and the estimated smoothness bound has been initialized to 10^5.

Here, as both the flow and jump maps result linear in the state, the chosen estimation scheme is the *recursive least squares* (RLS) Ljung [1987]. However, due to how noise would influence the system, i.e. it affects the observation matrix, alternative results might be achieved with a *recursive generalized total least squares* Rhode et al. [2014] or a *recursive Frisch scheme* Massaroli et al. [2018, 2019]. The resulting time evolution of the system is represented in Fig. 4. It can be noticed that the behavior of the system is similar to the one of a bouncing ball under the effect of constant gravitational acceleration. However, in this case, the ball is pulled to the ground by the force of the spring. Along the trajectory, the norm of the Euler time derivative $D_{\delta t}\xi(t)$ and the estimated smoothness bound $\hat{\tau}(t)$, have been computed. These quantities have been employed to evaluate the function $\gamma(t)$ and thus, to identify the parameters α and β through the RLS. Initially, to test the overall procedure in nominal conditions, no additive noise has been added to the measurements of the states. The trend of $D_{\delta t}\xi(t)$ during the simulation, is shown in the upper part of Fig. 5. The figure also shows the comparison between the approximated $\hat{\tau}$ and the true smoothness bound τ (scaled by an arbitrary constant r). The approximation error of the smoothness bound, defined as $e_\tau(t) = \|\hat{\tau}(t) - r \cdot \tau(t)\|$, it is plotted in the lower part of Fig. 5. It can be noticed that after just few iterations the estimated smoothness bound $\hat{\tau}$ can accurately track the scaled nominal bound τ. At the same time, as the estimation of the parameters becomes more accurate, the tracking error decreases. The performance of the system in the identification of the parameters has been evaluated by defining the absolute estimation errors $e_\alpha(t) = \|\alpha - \hat{\alpha}(t)\|$ and $e_\beta(t) = \|\beta - \hat{\beta}(t)\|$. The time evolution of the components of $\hat{\alpha}$ and $\hat{\beta}$, and the one of the estimation errors $e_\alpha(t)$ and $e_\beta(t)$, are plotted in Fig. 6. As expected, in the nominal case, the estimation error decreases with the number of iterations. A zero-mean Gaussian noise with standard deviation $\tilde{\sigma}_\xi = 0.25$ has successively been added to the state observations and the experiment repeated The results are shown in Figs. 7 and 8. The tracking error of the smoothness quickly converges to a values oscillating around 10^3 which correspond to a mean relative error of about 2.5%, Fig. 7. This proves a certain level of robustness to noise in the approximation of $\tau(t)$. This result is also pictured in Fig. 8 where good estimates are achieved for both, the flow and the jump parameters. Besides, it must be mentioned that, in case of excessive noise, the approximation of the smoothness becomes less accurate and the difference of $D_{\delta t}\xi(t)$ during flows and jumps less prominent, making the detection of the jumps difficult. In fact, the system is sensitive to the choice of the multiplicative constant r and δt, which should be empirically made. To ensure a clear separation between the Euler derivatives and thus, increase the robustness to noise, a small value should be assigned to δt. Indeed, this would also help to enforce the validity of Assumption 3. A major drawback of the proposed approach is that the erroneous classification of some of the states as *flows* or *jumps* inevitably introduce errors in the estimation process. This might be due to a poor choice of the scalar r or, in the firsts iterations of the identification algorithm, to rough estimates of the flow parameters. However, the problem might be partially solved introducing a *forgetting* term in the estimation scheme Söderström [2018] allowing a quick recover of the estimates together with a method to retroactively re–classify the wrong sample. These further investigations are left for future work.

6. CONCLUSIONS

In this paper a new methodology for the identification of a class of hybrid dynamical systems, which evaluates the unknown parameters by employing a linear recursive estimator, has been proposed. The developed procedure is able to identify the state of the system and explicitly

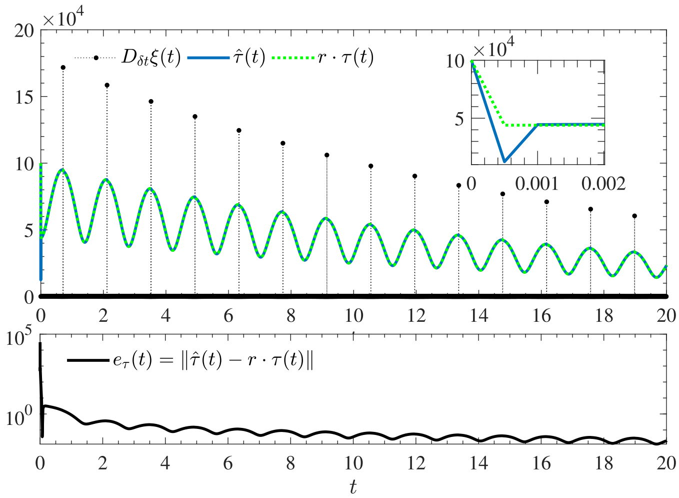

Fig. 5. Nominal experiment (no measurement noise). [Above] Norm of the Euler derivative $D_{\delta t}\xi(t)$, (dotted black line and circe marker), estimated smoothness bound $\hat{\tau}(t)$ (blue solid line) and nominal smoothness bound $\tau(t)$ scaled by the arbitrary constant r. [Below] smoothness bound approximation error $e_\tau(t)$ (log scale).

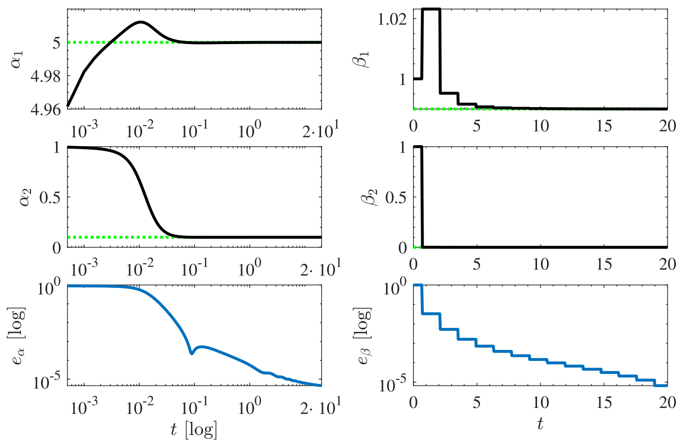

Fig. 6. Nominal experiment (no measurement noise): parameters estimation results. [Left] Estimates of the parameters in α and absolute estimation error $e_\alpha(t) = \|\alpha - \hat{\alpha}(t)\|$ (below). [Right] Estimates of the parameters in β and absolute estimation error $e_\beta(t) = \|\beta - \hat{\beta}(t)\|$ (below).

determine the flowing and jumping states. Here we have derived a systematic approach for the identification of hybrid dynamical systems, that to the best of authors' knowledge, is the first application of system identification methodologies to hybrid dynamical systems which can be analytically represented by Equation 1. Further development will include a more rigorous approach for the estimation of the smoothness bound without the need of any empirical coefficient, the exploration of alternative estimation schemes and a possible extension to the identification of *hybrid inclusions*. Problems related to the non–convex approximation of the the flow and jump sets will also be regarded investigating new machine learning strategies.

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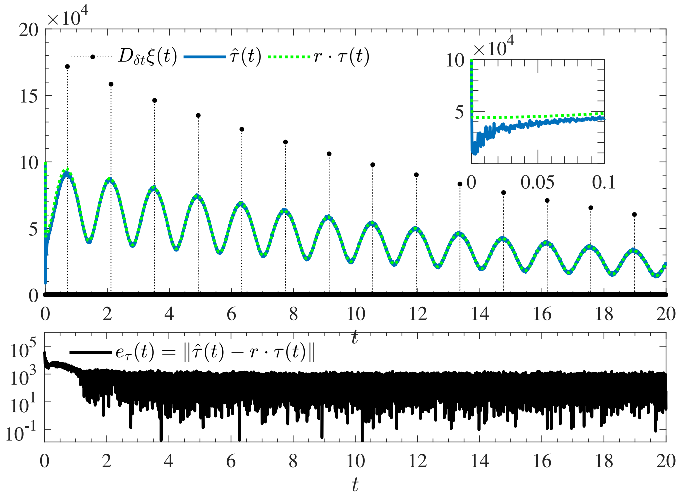

Fig. 7. Noisy experiment (measurement noise standard deviation $\tilde{\sigma}_\xi$=0.25). [Above] Norm of the Euler derivative $D_{\delta t}\xi(t)$, (dotted black line and circle marker), estimated smoothness bound $\hat{\tau}(t)$ (blue solid line) and nominal smoothness bound $\tau(t)$ scaled by the arbitrary constant r. [Below] smoothness bound approximation error $e_\tau(t)$ (log scale).

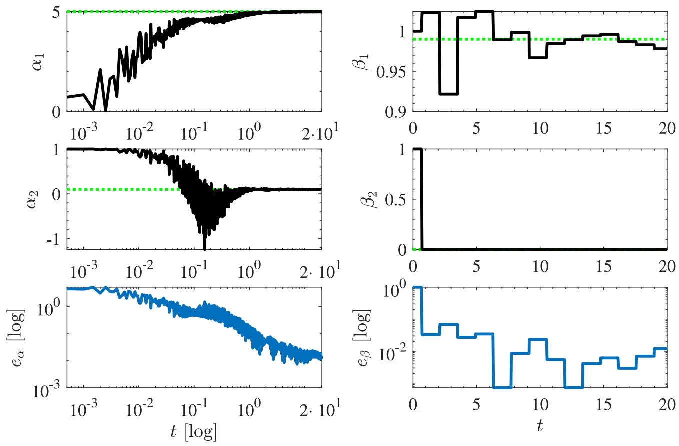

Fig. 8. Noisy experiment (measurement noise standard deviation $\tilde{\sigma}_\xi$=0.25): parameters estimation results. [Left] Estimates of the parameters in α and absolute estimation error $e_\alpha(t) = \|\alpha - \hat{\alpha}(t)\|$ (below). [Right] Estimates of the parameters in β and absolute estimation error $e_\beta(t) = \|\beta - \hat{\beta}(t)\|$ (below).

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