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A pedagogical path from the internal model principle to Youla-Kučera parametrization

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Abstract: We propose a sequence of pedagogical steps for introducing the Youla-Kučera parametrization, starting from the *internal model principle*, and introducing the control structures of *disturbance observer* and *internal model control* along the way. We provide some background on the concepts and a brief survey of their treatment in textbooks on control.

Keywords: Youla-Kučera parametrization, IMC, DOB, Internal model principle

1. INTRODUCTION

The Youla-Kučera parametrization (YKP) of all stabilizing controllers for a given linear time-invariant system is an important result in control theory and often appears in courses on the subject. For stable plants, the YKP takes on a particularly simple form, and it is therefore often first introduced to students in that setting. For unstable plants, the formulations become more involved and the transition from the stable case can be perceived as difficult to follow by some students. Especially the multivariable case, where the concept of coprime matrix factorization is an important tool, can pose difficulties.

The aim of this paper is to a) propose a line of reasoning that we hope can provide inspiration for teaching the YKP, and b) to draw attention to some useful perspectives on the related control structures.

The proposed line of reasoning, illustrated in Fig. 1, takes as point of departure the *internal model principle* (IMP), which can be a useful conceptual tool for students when reasoning about control. Then two alternative controller structures, both using a model of the plant, are introduced; *internal model control* (IMC) and *disturbance observer* (DOB). These two structures are used to introduce a more general structure, based on factorization, which provides a natural bridge to the general formulation of the YKP.

The paper is structured as follows: In Section 2 we summarize the concepts to be introduced, and briefly survey how they are presented in control textbooks. In Section 3 we present the procedure, in Section 4 we discuss the pedagogical merits of the proposed line of reasoning and in Section 5 we present some possible extensions.

2. PRELIMINARIES

2.1 The internal model principle

The internal model principle (IMP) is the idea that in order to control a system and compensate for disturbances, the controller needs to have an understanding, i.e. an internal model, of how the system works and the nature of the disturbances. Intuitively this seems like a reasonable proposition, and it agrees with everyday experience.

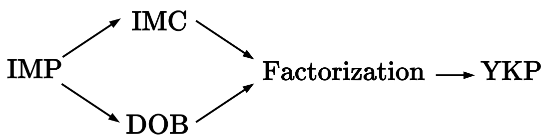

Fig. 1. The internal model principle (IMP) is used to introduce internal model control (IMC) and disturbance observer (DOB). Comparing these control structures naturally leads to the concept of factorization, which is a bridge to the general Youla-Kučera parametrization (YKP) of all stabilizing controllers.

The concept has been formalized for multivariable linear systems by Francis and Wonham (1975), and generalized to a setting of abstract automata by Wonham (1976).

An interesting paper by Conant and Ashby (1970) derives a general principle using a broad concept of mappings and an argument based on entropy. The title of that paper, *Every good regulator of a system must be a model of that system*, succinctly summarizes the internal model principle even though the paper does not refer to it by that name.

The internal model principle does not seem to figure prominently in textbooks on automatic control. Åström and Murray (2008) for example mentions it in passing when discussing observer-based state feedback, and Corriou (2004) calls it *"a recommendation of general interest"*, briefly giving a frequency description and referencing the work of Francis and Wonham (1975). Of the surveyed textbooks, Goodwin et al. (2001) most prominently features the IMP. There, the internal model principle is discussed in relation to disturbance rejection and reference tracking, as those require the controller to incorporate models describing how disturbances and references are generated. It is also mentioned in relation to achieving integral action in LQ control.

In some literature, e.g. Åström and Wittenmark (1984), Maciejowski (1989), Chen (1999), Franklin et al. (2002), Skogestad and Postlethwaite (2005), Dorf and Bishop (2008), the IMP is only used to refer to the principle that in order to compensate for a persistent disturbance, the controller needs to contain a model of the the system generating the disturbance. Doyle et al. (1992) does not refer to an IMP by name, but talks of "an elementary principle" required for asymptotic tracking.

2.2 Internal model control

Internal model control (IMC) as introduced in García and Morari (1982) refers to a particular control structure that uses an *"internal model to predict the effect of the manipulated variables on the output"*. The book by Morari and Zafiriou (1989) is a widely cited reference in which IMC is the basis for a robust design procedure.

Brosilow and Tong (1978) introduces the same concept under a different name, *inferential control*, and focuses on the estimation aspect, making the connection between the IMC and DOB ideas clear.

The main idea is that only the deviation from the predicted output, which can be interpreteted as an output disturbance estimate $\hat{d}_y$, is fed back to the controller.

IMC is commonly presented with the block-diagram in Fig. 2, although García et al. (1989) [1] observes that the two-degree-of-freedom (2-DOF) structure presented in Fig. 3 is preferable. They also give a brief historical summary of how the concepts leading up to IMC control developed.

Horowitz (1963) discusses how 2-DOF structures are all equivalent, and among the examples one can find the IMC structure under the name of *model feedback* (Ch. 6, figure 6.1-1f). The book also offers the following illuminating quote relating to the equivalence of 2-DOF controllers:

"*... it destroys the mystique of structure which seems to some to be of great importance in feedback theory. The designer need not fear that, if he were only clever enough, he could find some exotic structure with new and wonderful properties.* "

Frank (1974) describes how the model feedback idea evolved, and provides an experimental IMC design procedure (section 2.7.1) that nicely illustrates the relation between control and modeling.

There is a one-to-one correspondence between an error feedback controller C, as shown in Fig. 4, and an IMC controller Q such as in Fig. 2, given by

$$C = \frac{Q}{1 - \hat{P}Q}, \; Q = \frac{C}{1 + \hat{P}C}. \tag{1}$$

In case of a perfect model $\hat{P} = P$, the transfer functions characterizing the closed loop system (the so-called *Gang of Four*, see e.g. Åström and Murray (2008)) takes on a particularly simple form

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} PQ & P(1-PQ) \\ Q & -PQ \end{bmatrix} \begin{bmatrix} r \\ d_u \end{bmatrix} \tag{2}$$

making them easy to analyze.

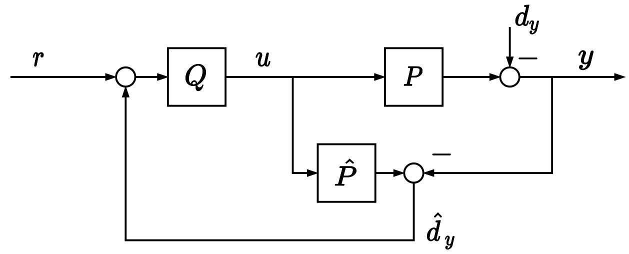

Fig. 2. IMC is often introduced using a 1-degree-of-freedom (1-DOF) structure.

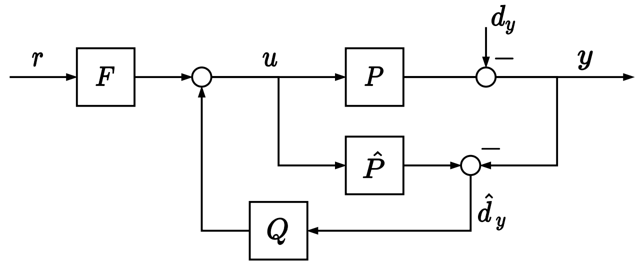

Fig. 3. García et al. (1989) advocates using a 2-DOF IMC structure. This structure illustrates the interpretation of error feedback as feedforward combined with disturbance estimation and rejection.

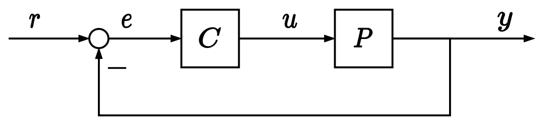

Fig. 4. A common 1-DOF feedback controller structure.

2.3 Disturbance observer

Another, less widespread, controller structure that also uses an internal model is the *disturbance observer* structure (DOB), where an inverse model of the system is used to estimate an input disturbance d_u and try to cancel that disturbance. The concept is illustrated in Fig. 5, where the filter $H(s)$ should ideally be equal to 1, but has to have sufficient relative degree to ensure realizablility.

The term disturbance observer is introduced in Nakao et al. (1987), and Oboe (2018) gives a contemporary as well as enthusiastic introduction.

2.4 The Youla-Kučera parametrization

The idea behind the Youla-Kučera parametrization (YKP) was presented independently in papers by Youla et al. (1976) and Kučera (1975).

The YKP describes the set of all controllers that stabilize a given linear system, and parametrizes that set by a stable transfer function, often denoted Q. It is therefore sometimes referred to as *Q-parametrization*. The fact that

[1] The title uses *Model predictive control*, a term that has taken on a slightly different meaning in modern control terminology. The process control oriented text Marlin (2000) uses it in the same sense as García et al. (1989) for IMC-like control.

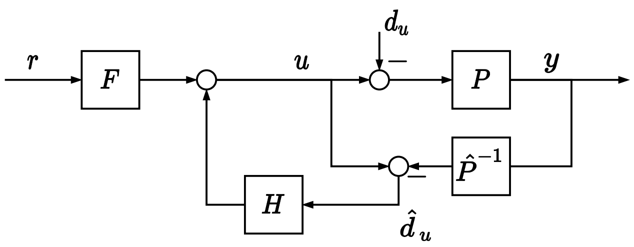

Fig. 5. The disturbance observer controller structure uses the model to estimate an input disturbance d_u, instead of an output disturbance d_y like IMC does.

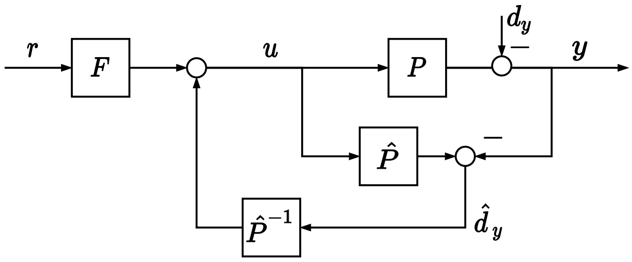

Fig. 6. Ideal IMC, where the estimated output disturbance is fed through an inverse of the model.

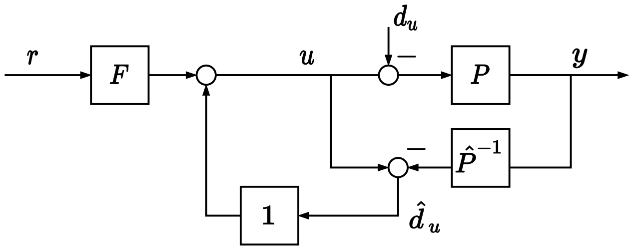

Fig. 7. Ideal DOB, where the estimated input disturbance is cancelled directly.

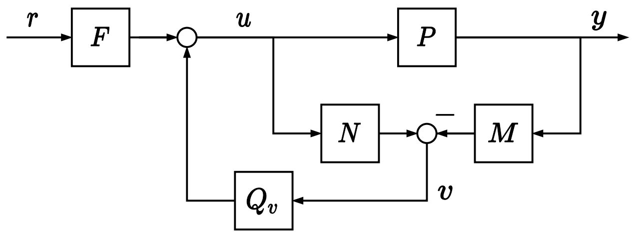

Fig. 8. A general 2-DOF IMC-like controller structure, similar to what Chen (1999) calls the *controller-estimator* or *plant-input-output-feedback* structure.

the closed loop transfer functions become linear in Q makes the YKP a powerful tool for optimization-based synthesis of controllers. The textbooks Doyle et al. (1992) and Skogestad and Postlethwaite (2005) both use the this parametrization to discuss stabilizing controllers.

3. PROPOSED PROCEDURE

3.1 Introduce the internal model principle

Introduce the idea that to control something, the controller needs knowledge about the process to be controlled. Appeal to everyday experience to show that this proposition is reasonable.

Mention that this intuitive principle can be formalized in different theoretical settings, one of which is linear time-invariant systems.

Note that in our field, system knowledge often takes the form of a mathematical model. And that the power of feedback control is that often a simple approximation can suffice as model, in some cases as simple as knowing the sign of the steady state gain.

3.2 Introduce IMC and DOB

Explain that one way of using a model in a controller, is to compare the actual system behaviour with the behaviour predicted by the model, and then using only the difference, i.e. the unexpected behaviour, to make an adjustment.

Introduce the IMC structure in Fig. 3 as an example of the above, and mention that here all unexpected behaviour is in a sense interpreted as an output disturbance d_y. Note that ideally, we would like to use a perfect model inverse, as in Fig. 6, but that this is rarely possible.

Now, explain that it is also possible to regard all unexpected behaviour as an input disturbance, and that this perspective is beheind the DOB structure in Fig. 5. Note that we would ideally like to achieve the structure in Fig. 7, but that this, again, is generally not possible.

3.3 Demonstrate equivalence between IMC and DOB

Note that the structure in Fig. 5 can easily be turned into Fig. 3 by "pulling" the block with the inverse model $\hat{P}^{-1}$ down through the summation junction and choosing $H = Q\hat{P}$.

Table 1. Parameter choices to achieve equivalent controllers in different formulations.

Structure	Q_v	N	M
IMC	Q	$\hat{P}$	1
DOB	$Q\hat{P}$	1	$\hat{P}^{-1}$
PF	$Q\hat{A}^{-1}$	$\hat{B}$	$\hat{A}$
IPF	$QC\hat{A}^{-1}$	$\hat{B}C^{-1}$	$\hat{A}C^{-1}$

3.4 Interpret in terms of factorization

Introduce Fig. 8 as a way of describing both structures, and remark that between IMC and DOB, one invariant is that $NM^{-1} = \hat{P}$, i.e. that N and M^{-1} is a factorization of $\hat{P}$. Remark that this might lead a curious mind to ask if other interesting controller structures might be obtained by different factorizations.

3.5 Introduce the polynomial factorization

Note that since we are dealing with a rational transfer function $\hat{P} = \frac{\hat{B}}{\hat{A}}$, one natural choice to investigate would be to simply factorize $\hat{P}$ into its numerator and denominator polynomials, choosing $N = \hat{B}$ and $M = \hat{A}$. Remark that this gives us two stable transfer functions in the structure in Fig. 9, as by definition polynomials have no poles.

Admit that of course, polynomials are not proper transfer functions, and if we would like to use the structure in Fig. 8 for implementation we can instead choose a factorization

$$N = \hat{B}C^{-1},\ M = \hat{A}C^{-1} \tag{3}$$

where C is a polynomial without roots in the right half plane and of sufficient degree to make both N and M proper and thus realizable. Illustrate this by Fig. 10.

If suitable, show Table 1 to illustrate how a given IMC controller translates to the DOB, the polynomial factorization (PF) and the implementable PF (IPF).

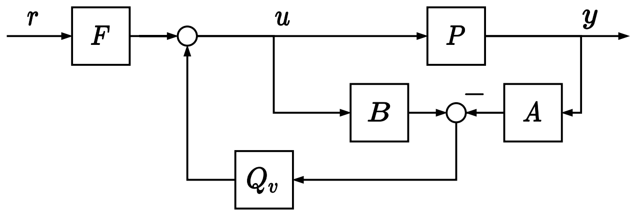

Fig. 9. For a rational SISO system, factorization by the numerator and denominator polynomials yields stable transfer functions.

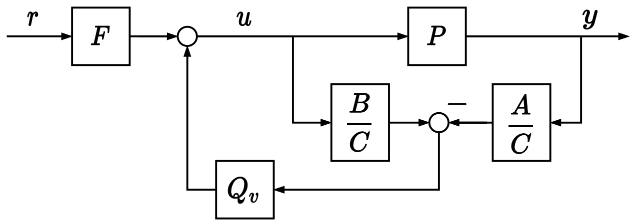

Fig. 10. To keep the factorized structure in implementation, a filter polynomial C can be introduced to ensure properness, without restricting design choices.

3.6 Introduce the concept of YKP using IMC

Signal a change in perspective by clarifying that the previous discussion was about choosing controller structures based on the intuition of the IMP, but that we will now use the same structures to describe all controllers that stabilize a given system. To do this we consider the purely theoretical case of a perfect model, $\hat{P} = P$.

Derive, or simply introduce, the transfer functions (2) characterizing the closed loop system when using an IMC controller with a perfect model. Note that if P is stable, the "Gang of Four" (2) will be stable for any stable Q.

Note, or prove (e.g. Morari and Zafiriou (1989)), that not only does every stable Q give a stable closed loop system for a stable P, but every stabilizing feedback controller C can be obtained by a stable Q and the expression (1).

3.7 Extend to the unstable case

Return to the transfer functions (2) and note that if P is unstable, it is sufficient to find a stable Q that stabilizes the critical transfer functions

$$PQ,\ P(1 - PQ) \tag{4}$$

to achieve internal stability for the closed loop system. Remark that we would like a parametrization without such constraints on the parameter.

Demonstrate how this can be achieved by choosing

$$Q = Q_0 + A^2 Q_1 \tag{5}$$

and inserting into the critical transfer functions (4) to obtain

$$PQ_0 + ABQ_1,\ P(1 - PQ_0) - B^2 Q_1, \tag{6}$$

using the fact that $P = \frac{B}{A}$. Note that if Q_0 is chosen to stabilize (4), we are free to choose any stable Q_1 since B and A are stable by definition. Compare to homogeneous and particular solutions of a differential equation.

Note, or prove (e.g. Morari and Zafiriou (1989)), that given a stabilizing Q_0 all stabilizing controllers are then parametrized by the choice of a stable Q_1.

3.8 Block-diagram representations and interpretations

Choose Q in the IMC controller as (5) and show, by inserting into Fig. 9, that the corresponding controller can be described as in Fig. 11, illustrating that the YKP can be interpreted as two IMC-loops.

Now remark, or if suitable show, that the block diagram in Fig. 11 can by some block diagram manipulation be turned into the one depicted in Fig. 12, thus illustrating another interpretation of the YKP as first applying an IMC-loop for disturbance rejection and then applying an outer, stabilizing, feedback loop.

Regarding the reciproval, Goodwin et al. (2001) notes that it is sometimes, but not always, possible to describe the YKP as an inner, stabilizing, feedback loop and an outer IMC-loop.

Since Fig. 12 does not seem to be common in textbooks a few remarks are in order. C is the feedback controller obtained from inserting Q_0 into (1), and Q_1^* and F^* can be chosen as stable transfer functions of sufficient relative degree, but are not the same as Q_1 and F in Fig. 11. In deriving Fig. 12 it is helpful to observe that the stability of (4) implies that Q_0 can always be factorized as $Q_0 = Q_0^* A$ where Q_0^* is stable, and that $(1 - Q_0 P)$ can be factorized in a similar manner.

4. PEDAGOGICAL MERITS

The concepts introduced above are standard fare in control theory, and we do not pretend that our way of presenting the material is really novel. We have however in our brief survey of textbooks not found quite the same way of linking the concepts together.

While we have not had the opportunity of trying this procedure in teaching, we would like to highlight what we think are the pedagogical strengths.

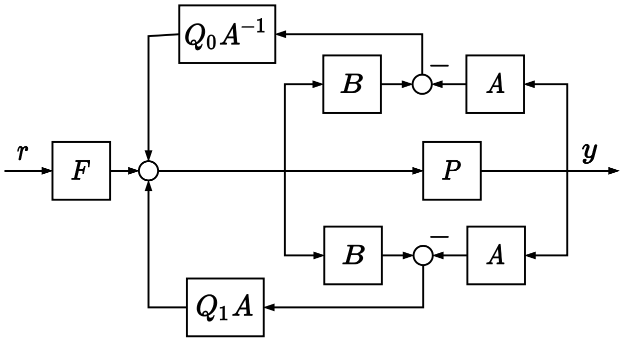

Fig. 11. The general YKP can be interpreted as two IMC-loops, where one is freely parametrized.

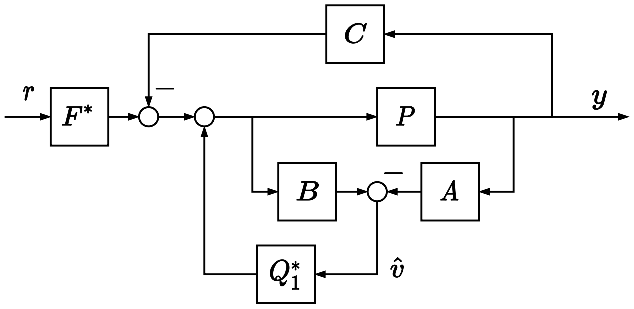

Fig. 12. An alternative interpretation of the YKP is as an inner disturbance rejection, or *"model following"*, loop combined with an outer stabilizing loop.

The IMP is quite an intuitive idea, and we believe that exposing students to it early and stressing its generality can help develop in understanding the fundamental connection between modelling, estimation and control.

Introducing the DOB structure alongside IMC can be a good opportunity to illustrate how block diagram manipulations reveal different interpretations of the same controller, and that control can be seen as estimation and compensation of disturbances.

Minimizing the differences between the block diagrams for the stable and unstable case YKP could aid students in connecting both cases, and the use of Fig. 9 and Fig. 12 could be advantageous in this regard compared to using Fig. 2 combined with e.g. Fig. 13 or Fig. 14.

Introducing factorization as a natural tool for investigating control structures SISO case can hopefully facilitate the transition to the MIMO case and the use of coprime transfer-function matrix factorization.

5. ADDITIONS

We would like to note two illustrative examples that can easily be integrated into the proposed procedure.

5.1 PID derivation from DOB

Using the DOB structure in with a second order model

$$\hat{P} = \frac{1}{s^2 + as + b}, \tag{7}$$

and choosing a first order low-pass filter

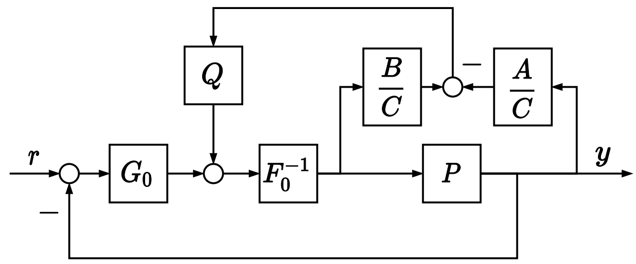

Fig. 13. Åström and Murray (2008) uses a similar block-diagram to illustrate the general YKP. The transfer functions G_0 and F_0 are obtained from a coprime factorization of a stabilizing controller. C is a polynomial with roots in the LHP.

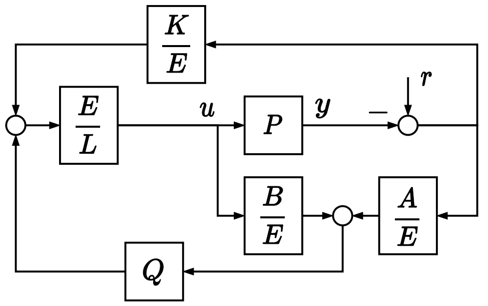

Fig. 14. Goodwin et al. (2001) illustrates the general YKP using this structure. The polynomials K and L constitute a polynomial factorization of a stabilizing controller, and the roots of the polynomial E is part of the closed-loop poles. Note that E can be eliminated from the block-diagram.

$$H = \frac{1}{1 + sT}, \tag{8}$$

as depicted in Fig. 15, is equivalent to using a controller

$$C = \frac{s^2 + as + b}{sT} \triangleq K_P + K_I \frac{1}{s} + K_D s. \tag{9}$$

for regulation, i.e. when $r = 0$. If the prefilter F is chosen as $F = \frac{s^2+as+b}{1+sT}$, then a regular unity feedback PID controller is obtained.

This example can serve as a good opportunity to elaborate on the fact that a simple model can often be enough to achieve acceptable control performance, and that this is one reason for the success of PID control.

5.2 Cascade control

The notion that IMC and DOB contain estimations of output/input disturbances can be taken further by not-

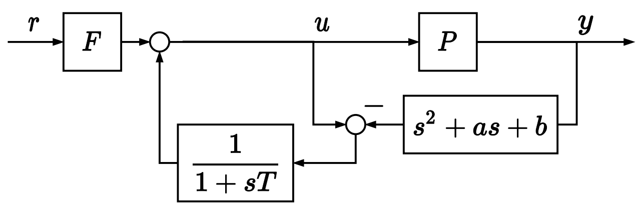

Fig. 15. A PID controller in disguise.

ing that every factorization N and M^{-1} of the system corresponds to viewing the system as composed of two subsystems in series, and to interpreting the signal v in Fig. 8 as the estimate of a disturbance entering between these two subsystems.

If the system is indeed composed of two physical subsystems, and we have access to a measurement of the intermediate signal, the IMC/DOB framework can easily incorporate this measurement, giving rise to a cascade-like IMC control structure, as illustrated by Fig. 16. A remark about this is made by Bequette (2003), in section 10.5.

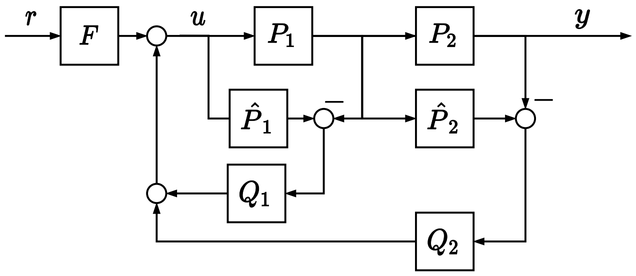

Fig. 16. IMC applied in a cascade control structure.

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